

Equivalence of K -Functionals and Modulus of Smoothness Generated by a Generalized Dunkl Operator on the Real Line

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Abstract

This paper is intended to establish the equivalence between K -functionals and modulus of smoothness tied to a Dunkl type operator on the real line.

Keywords

Differential-Difference Operator, Generalized Fourier Transform, Generalized Translation Operators, K -Functionals, Modulus of Smoothness

1. Introduction

Consider the first-order singular differential-difference operator on the real line

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x},$$

where $\alpha > -1/2$ and $n = 0, 1, \dots$. For $n = 0$, we regain the differential-difference operator

$$D_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator with parameter $\alpha + 1/2$ associated with the reflection group \mathbb{Z}_2 on \mathbb{R} . Such operators have been introduced by Dunkl [1]-[3] in connection with a generalization of the classical

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theory of spherical harmonics. The one-dimensional Dunkl operator D_α plays a major role in the study of quantum harmonic oscillators governed by Wigner’s commutation rules [4]-[6].

The authors have developed in [7] [8] a new harmonic analysis on the real line related to the differential-difference operator Λ in which several classical analytic structures such as the Fourier transform, the translation operators, the convolution product, ..., were generalized. With the help of the translation operators tied to Λ , we construct in this paper generalized modulus of smoothness in the Hilbert space $L^2(\mathbb{R}, |x|^{2\alpha+1} dx)$. Next, we define Sobolev type spaces and K -functionals generated by Λ . Using essentially the properties of the Fourier transform associated to Λ , we establish the equivalence between K -functionals and modulus of smoothness.

In the classical theory of approximation of functions on \mathbb{R} , the modulus of smoothness are basically built by means of the translation operators $f \rightarrow f(x + y)$. As the notion of translation operators was extended to various contexts (see [9] [10] and the references therein), many generalized modulus of smoothness have been discovered. Such generalized modulus of smoothness are often more convenient than the usual ones for the study of the connection between the smoothness properties of a function and the best approximations of this function in weight functional spaces (see [11]-[13] and references therein).

In addition to modulus of smoothness, the K -functionals introduced by J. Peetre [14] have turned out to be a simple and efficient tool for the description of smoothness properties of functions. The study of the connection between these two quantities is one of the main problems in the theory of approximation of functions. In the classical setting, the equivalence of modulus of smoothness and K -functionals has been established in [15]. For various generalized modulus of smoothness these problems are studied, for example, in [16]-[19]. It is pointed out that the results obtained in [16] emerge as easy consequences of those stated in the present paper by simply taking $n = 0$.

2. Preliminaries

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ . Further details can be found in [7] [8]. In all what follows assume $\alpha > -1/2$ and n a non-negative integer.

The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz) \quad (z \in \mathbb{C}) \tag{1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C})$$

is the normalized spherical Bessel function of index α . It is well-known that the functions $e_\alpha(\lambda \cdot)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$D_\alpha u = \lambda u, \quad u(0) = 1. \tag{2}$$

Furthermore, we have the Laplace type integral representations:

$$e_\alpha(z) = a_\alpha \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) e^{zt} dt, \tag{3}$$

$$j_\alpha(z) = 2a_\alpha \int_0^1 (1-t^2)^{\alpha-1/2} \cos zt dt, \tag{4}$$

where

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+1/2)}. \tag{5}$$

The following properties will be useful for the sequel.

Lemma 1 1) For all $x \in \mathbb{R}$, $|e_\alpha(ix)| \leq 1$.

2) There is $c_\alpha > 0$ such that $|1 - e_\alpha(ix)| \geq c_\alpha$ for all $x \in \mathbb{R}$ with $|x| \geq 1$.

3) For all $x \in \mathbb{R} - \{0\}$, $e_\alpha(ix) \neq 1$.

4) For all $x \in \mathbb{R}$,

$$|1 - e_\alpha(ix)| \leq \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 3/2)} |x| \leq |x|.$$

Proof. Assertions (1) and (2) are proved in [16]. By (1), (4) and the fact that

$$1 = 2a_\alpha \int_0^1 (1-t^2)^{\alpha-1/2} dt$$

we have

$$|1 - e_\alpha(ix)| \geq 1 - j_\alpha(x) = 2a_\alpha \int_0^1 (1-t^2)^{\alpha-1/2} (1 - \cos xt) dt.$$

Clearly the integral above is null only for $x = 0$, which proves assertion (3). Let us check assertion (4). Using (3) and the fact that

$$1 = a_\alpha \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) dt \tag{6}$$

$$|1 - e^{iz}| \leq |z| \quad \text{for all } z \in \mathbb{R}$$

we get

$$\begin{aligned} |1 - e_\alpha(ix)| &= a_\alpha \left| \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) (1 - e^{ixt}) dt \right| \\ &\leq a_\alpha \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) |1 - e^{ixt}| dt \\ &\leq a_\alpha |x| \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) |t| dt \end{aligned}$$

By (6),

$$a_\alpha |x| \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) |t| dt \leq a_\alpha |x| \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) dt = |x|$$

Moreover,

$$\begin{aligned} &a_\alpha |x| \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1+t) |t| dt \\ &= 2a_\alpha |x| \int_0^1 (1-t^2)^{\alpha-1/2} t dt = \frac{a_\alpha |x|}{\alpha + 1/2} = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 3/2)} |x|, \end{aligned}$$

which concludes the proof.

Notation 1 Put

$$m_\alpha = \frac{1}{2^{2\alpha+2} (\Gamma(\alpha + 1))^2}.$$

We denote by

- $L_\alpha^2(\mathbb{R})$ the class of measurable functions f on \mathbb{R} for which

$$\|f\|_{2,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx \right)^{1/2} < \infty.$$

- $\mathcal{S}(\mathbb{R})$ the space of C^∞ functions f on \mathbb{R} , which are rapidly decreasing together with their derivatives, i.e., such that for all $m, n = 0, 1, \dots$,

$$q_{m,n}(f) = \sup_{x \in \mathbb{R}} (1 + |x|)^m \left| \frac{d^n}{dx^n} f(x) \right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $q_{m,n}$, $m, n = 0, 1, \dots$.

- $\mathcal{S}_n(\mathbb{R})$ the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions f such that

$$f(0) = \dots = f^{(2n-1)}(0) = 0.$$

- $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions on \mathbb{R} .
- $\mathcal{S}'_n(\mathbb{R})$ the topological dual of $\mathcal{S}_n(\mathbb{R})$.

Clearly Λ is a linear bounded operator from $\mathcal{S}_n(\mathbb{R})$ into itself. Accordingly, if $S \in \mathcal{S}'_n(\mathbb{R})$ define $\Lambda S \in \mathcal{S}'_n(\mathbb{R})$ by

$$\langle \Lambda S, \psi \rangle = -\langle S, \Lambda \psi \rangle, \quad \psi \in \mathcal{S}_n(\mathbb{R})$$

For $k = 0, 1, \dots$ and $S \in \mathcal{S}'(\mathbb{R})$, let $x^k S \in \mathcal{S}'(\mathbb{R})$ be defined by

$$\langle x^k S, \psi \rangle = \langle S, x^k \psi \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}).$$

Definition 1 The generalized Fourier transform of a function $f \in \mathcal{S}_n(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x) e_{\alpha+2n}(-i\lambda x) |x|^{2\alpha+2n+1} dx, \quad \lambda \in \mathbb{R}.$$

Remark 1 If $n=0$ then \mathcal{F} reduces to the Dunkl transform with parameter $\alpha+1/2$ associated with the reflection group \mathbb{Z}_2 on \mathbb{R} (see [3]).

Theorem 1 The generalized Fourier transform \mathcal{F} is a topological isomorphism from $\mathcal{S}_n(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$. The inverse transform is given by

$$\mathcal{F}^{-1}(g)(x) = x^{2n} \int_{\mathbb{R}} g(\lambda) e_{\alpha+2n}(i\lambda x) d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = m_{\alpha+2n} |\lambda|^{2\alpha+4n+1} d\lambda.$$

Theorem 2 1) For every $f \in \mathcal{S}_n(\mathbb{R})$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda).$$

2) The generalized Fourier transform \mathcal{F} extends uniquely to an isometric isomorphism from $L^2_{\alpha}(\mathbb{R})$ onto $L^2(\mathbb{R}, \sigma)$.

Definition 2 The generalized Fourier transform of a distribution $S \in \mathcal{S}'_n(\mathbb{R})$ is defined by

$$\langle \mathcal{F}(S), \psi \rangle = \langle S, \mathcal{F}^{-1}(\psi) \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}).$$

Theorem 3 The generalized Fourier transform \mathcal{F} is one-to-one from $\mathcal{S}'_n(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$.

Lemma 2 If $f \in L^2_{\alpha}(\mathbb{R})$ then the functional

$$\langle T_f^{\alpha}, \psi \rangle = \int_{\mathbb{R}} f(x) \psi(x) |x|^{2\alpha+1} dx, \quad \psi \in \mathcal{S}(\mathbb{R}),$$

is a tempered distribution \mathbb{R} . Moreover,

$$\mathcal{F}(T_f^{\alpha}) = T_g^{\alpha+2n} \tag{7}$$

with $g(\lambda) = m_{\alpha+2n} \mathcal{F}(f)(-\lambda)$.

Proof. The fact that $T_f^{\alpha} \in \mathcal{S}'(\mathbb{R})$ follows readily by Schwarz inequality. Let $\psi \in \mathcal{S}(\mathbb{R})$. It is easily checked that

$$\overline{\mathcal{F}^{-1}(\psi)} = \mathcal{F}^{-1}(\tilde{\psi})$$

where $\tilde{\psi}(\lambda) = \overline{\psi(-\lambda)}$. So using Theorem 2 we get

$$\begin{aligned} \langle \mathcal{F}(T_f^\alpha), \psi \rangle &= \int_{\mathbb{R}} f(x) \mathcal{F}^{-1}(\psi)(x) |x|^{2\alpha+1} dx = \int_{\mathbb{R}} f(x) \overline{\mathcal{F}^{-1}(\tilde{\psi})(x)} |x|^{2\alpha+1} dx \\ &= m_{\alpha+2n} \int_{\mathbb{R}} \mathcal{F}(f)(\lambda) \psi(-\lambda) |\lambda|^{2\alpha+4n+1} d\lambda \\ &= m_{\alpha+2n} \int_{\mathbb{R}} \mathcal{F}(f)(-\lambda) \psi(\lambda) |\lambda|^{2\alpha+4n+1} d\lambda = \langle T_g^{\alpha+2n}, \psi \rangle, \end{aligned}$$

which completes the proof.

Lemma 3 Let $f \in \mathcal{S}_n(\mathbb{R})$ and $S \in \mathcal{S}'_n(\mathbb{R})$. Then for $k = 1, 2, \dots$ we have

$$\mathcal{F}(\Lambda^k f)(\lambda) = (i\lambda)^k \mathcal{F}(f)(\lambda) \tag{8}$$

$$\mathcal{F}(\Lambda^k S) = (-i\lambda)^k \mathcal{F}(S) \tag{9}$$

Proof. Identity (8) may be found in [7]. If $\psi \in \mathcal{S}(\mathbb{R})$ then

$$\langle \mathcal{F}(\Lambda^k S), \psi \rangle = \langle \Lambda^k S, \mathcal{F}^{-1}(\psi) \rangle = (-1)^k \langle S, \Lambda^k \mathcal{F}^{-1}(\psi) \rangle$$

But by (8),

$$\Lambda^k \mathcal{F}^{-1}(\psi) = \mathcal{F}^{-1}((i\lambda)^k \psi)$$

So

$$\langle \mathcal{F}(\Lambda^k S), \psi \rangle = (-1)^k \langle S, \mathcal{F}^{-1}((i\lambda)^k \psi) \rangle = (-1)^k \langle \mathcal{F}(S), (i\lambda)^k \psi \rangle = (-1)^k \langle (i\lambda)^k \mathcal{F}(S), \psi \rangle,$$

which ends the proof.

Notation 2 From now on assume $m = 1, 2, \dots$. Let $\mathcal{W}_{2,\alpha}^m$ be the Sobolev type space constructed by the differential-difference operator Λ , i.e.,

$$\mathcal{W}_{2,\alpha}^m = \{f \in L_\alpha^2(\mathbb{R}) : \Lambda^j f \in L_\alpha^2(\mathbb{R}), j = 1, 2, \dots, m\}.$$

More explicitly, $f \in \mathcal{W}_{2,\alpha}^m$ if and only if for each $j = 1, 2, \dots, m$, there is a function in $L_\alpha^2(\mathbb{R})$ abusively denoted by $\Lambda^j f$, such that $\Lambda^j T_f^\alpha = T_{\Lambda^j f}^\alpha$.

Proposition 1 For $f \in \mathcal{W}_{2,\alpha}^m$ we have

$$\mathcal{F}(\Lambda^m f)(\lambda) = (i\lambda)^m \mathcal{F}(f)(\lambda). \tag{10}$$

Proof. From the definition of $\mathcal{W}_{2,\alpha}^m$ we have

$$\Lambda^m T_f^\alpha = T_{\Lambda^m f}^\alpha$$

By (7) and (9),

$$\mathcal{F}(\Lambda^m T_f^\alpha) = (-i\lambda)^m \mathcal{F}(T_f^\alpha) = T_g^{\alpha+2n}$$

with $g(\lambda) = m_{\alpha+2n} (-i\lambda)^m \mathcal{F}(f)(-\lambda)$. Again by (7),

$$\mathcal{F}(T_{\Lambda^m f}^\alpha) = T_h^{\alpha+2n}$$

with $h(\lambda) = m_{\alpha+2n} \mathcal{F}(\Lambda^m f)(-\lambda)$. Identity (10) is now immediate.

Definition 3 The generalized translation operators τ^x , $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned} \tau^x f(y) &= \frac{(xy)^{2n}}{2} \int_{-1}^1 \frac{f(\sqrt{x^2 + y^2 - 2xyt})}{(x^2 + y^2 - 2xyt)^n} \left(1 + \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}\right) A(t) dt \\ &+ \frac{(xy)^{2n}}{2} \int_{-1}^1 \frac{f(-\sqrt{x^2 + y^2 - 2xyt})}{(x^2 + y^2 - 2xyt)^n} \left(1 - \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}\right) A(t) dt \end{aligned}$$

where

$$A(t) = a_{\alpha+2n} (1+t)(1-t^2)^{\alpha+2n-1/2}$$

with $a_{\alpha+2n}$ given by (5).

Proposition 2 Let $x \in \mathbb{R}$ and $f \in L^2_\alpha(\mathbb{R})$. Then $\tau^x f \in L^2_\alpha(\mathbb{R})$ and

$$\|\tau^x f\|_{2,\alpha} \leq 2x^{2n} \|f\|_{2,\alpha} \tag{11}$$

Furthermore,

$$\mathcal{F}(\tau^x f)(\lambda) = x^{2n} e_{\alpha+2n}(i\lambda x) \mathcal{F}(f)(\lambda) \tag{12}$$

3. Equivalence of K -Functionals and Modulus of Smoothness

Definition 4 Let $f \in L^2_\alpha(\mathbb{R})$ and $r > 0$. Then

- The generalized modulus of smoothness is defined by

$$\omega_m(f, r)_{2,\alpha} = \sup_{0 < h \leq r} \|\Delta_h^m f\|_{2,\alpha}$$

where

$$\Delta_h^m f = (\tau^h - h^{2n} I)^m f,$$

I being the unit operator.

- The generalized K -functional is defined by

$$K_m(f, r)_{2,\alpha} = \inf \left\{ \|f - g\|_{2,\alpha} + r \|\Lambda^m g\|_{2,\alpha} : g \in \mathcal{W}_{2,\alpha}^m \right\}.$$

The next theorem, which is the main result of this paper, establishes the equivalence between the generalized modulus of smoothness and the generalized K -functional:

Theorem 4 There are two positive constants $c_1 = c_1(m, n, \alpha)$ and $c_2 = c_2(m, n, \alpha)$ such that

$$c_1 \omega_m(f, r)_{2,\alpha} \leq r^{2mn} K_m(f, r^m)_{2,\alpha} \leq c_2 \omega_m(f, r)_{2,\alpha}$$

for all $f \in L^2_\alpha(\mathbb{R})$ and $r > 0$.

In order to prove Theorem 4, we shall need some preliminary results.

Lemma 4 Let $f \in L^2_\alpha(\mathbb{R})$ and $h > 0$. Then

$$\|\Delta_h^m f\|_{2,\alpha} \leq 3^m h^{2mn} \|f\|_{2,\alpha} \tag{13}$$

$$\mathcal{F}(\Delta_h^m f)(\lambda) = h^{2mn} (e_{\alpha+2n}(i\lambda h) - 1)^m \mathcal{F}(f)(\lambda) \tag{14}$$

Proof. The result follows easily by using (11), (12) and an induction on m .

Lemma 5 For all $f \in \mathcal{W}_{2,\alpha}^m$ and $h > 0$ we have

$$\|\Delta_h^m f\|_{2,\alpha} \leq h^{m(2n+1)} \|\Lambda^m f\|_{2,\alpha} \tag{15}$$

Proof. By (10), (14), Lemma 1 (4) and Theorem 2 we have

$$\begin{aligned} \|\Delta_h^m f\|_{2,\alpha}^2 &= \int_{\mathbb{R}} |\mathcal{F}(\Delta_h^m f)(\lambda)|^2 d\sigma(\lambda) \\ &= h^{4mn} \int_{\mathbb{R}} |1 - e_{\alpha+2n}(i\lambda h)|^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &\leq h^{2m(2n+1)} \int_{\mathbb{R}} |\lambda|^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= h^{2m(2n+1)} \int_{\mathbb{R}} |\mathcal{F}(\Lambda^m f)(\lambda)|^2 d\sigma(\lambda) \\ &= h^{2m(2n+1)} \|\Lambda^m f\|_{2,\alpha}^2 \end{aligned}$$

which is the desired result.

Notation 3 For $f \in L_{\alpha}^2(\mathbb{R})$ and $\nu > 0$ define the function

$$P_{\nu}(f)(x) = x^{2n} \int_{-\nu}^{\nu} \mathcal{F}(f)(\lambda) e_{\alpha+2n}(i\lambda x) d\sigma(\lambda)$$

Proposition 3 Let $f \in L_{\alpha}^2(\mathbb{R})$ and $\nu > 0$. Then

1) The function $P_{\nu}(f)$ is infinitely differentiable on \mathbb{R} and

$$\Lambda^k P_{\nu}(f)(x) = x^{2n} \int_{-\nu}^{\nu} \mathcal{F}(f)(\lambda) (i\lambda)^k e_{\alpha+2n}(i\lambda x) d\sigma(\lambda) \tag{16}$$

for all $k = 0, 1, \dots$.

2) For all $k = 0, 1, \dots$, $\Lambda^k P_{\nu}(f) \in L_{\alpha}^2$ and

$$\mathcal{F}(\Lambda^k P_{\nu}(f))(\lambda) = (i\lambda)^k \mathcal{F}(f)(\lambda) \chi_{\nu}(\lambda), \tag{17}$$

where

$$\chi_{\nu}(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq \nu, \\ 0 & \text{if } |\lambda| > \nu. \end{cases}$$

Proof. The fact that $P_{\nu}(f) \in C^{\infty}(\mathbb{R})$ follows from the derivation theorem under the integral sign. Identity (16) follows readily from (2) and the relationship

$$\Lambda(x^{2n} f) = x^{2n} D_{\alpha+2n} f$$

which is proved in [7]. Assertion (2) is a consequence of (16) and Theorem 2.

Lemma 6 There is a positive constant $c = c(\alpha, n)$ such that

$$\|f - P_{\nu}(f)\|_{2,\alpha} \leq c^{-m} \nu^{2mn} \|\Delta_{1/\nu}^m f\|_{2,\alpha}$$

for any $f \in L_{\alpha}^2(\mathbb{R})$ and $\nu > 0$.

Proof. By (17) and Theorem 2, we have

$$\|f - P_{\nu}(f)\|_{2,\alpha}^2 = \int_{\mathbb{R}} |1 - \chi_{\nu}(\lambda)|^2 |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = \int_{|\lambda| \geq \nu} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda)$$

By Lemma 1 (2) there is a constant $c > 0$ which depends only on α and n such that

$$|1 - e_{\alpha+2n}(i\lambda/\nu)| \geq c$$

for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq \nu$. From this, (14) and Theorem 2 we get

$$\begin{aligned} \|f - P_{\nu}(f)\|_{2,\alpha}^2 &\leq c^{-2m} \int_{|\lambda| \geq \nu} |1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= c^{-2m} \nu^{4mn} \int_{\mathbb{R}} |\mathcal{F}(\Delta_{1/\nu}^m f)(\lambda)|^2 d\sigma(\lambda) = c^{-2m} \nu^{4mn} \|\Delta_{1/\nu}^m f\|_{2,\alpha}^2 \end{aligned}$$

which achieves the proof.

Corollary 1 For all $f \in L^2_\alpha(\mathbb{R})$ and $\nu > 0$ we have

$$\|f - P_\nu(f)\|_{2,\alpha} \leq c^{-m} \nu^{2mn} \omega_m(f, 1/\nu)_{2,\alpha}$$

where c is as in Lemma 6.

Lemma 7 There is a positive constant $C = C(\alpha, n)$ such that

$$\|\Lambda^m P_\nu(f)\|_{2,\alpha} \leq C^m \nu^{m(2n+1)} \|\Delta_{1/\nu}^m f\|_{2,\alpha}$$

for every $f \in L^2_\alpha(\mathbb{R})$ and $\nu > 0$.

Proof. By (17) and Theorem 2 we have

$$\begin{aligned} \|\Lambda^m P_\nu(f)\|_{2,\alpha}^2 &= \int_{-\nu}^{\nu} \lambda^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= \int_{-\nu}^{\nu} \frac{\lambda^{2m}}{|1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m}} |1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \end{aligned}$$

Put

$$C = \sup_{|t| \leq 1} \frac{|t|}{|1 - e_{\alpha+2n}(it)|}.$$

By L'Hôpital's rule,

$$\lim_{t \rightarrow 0} \frac{|t|}{|1 - e_{\alpha+2n}(it)|} = 2(\alpha + 2n + 1).$$

This when combined with Lemma 1 (3) entails $0 < C < \infty$. Moreover,

$$\begin{aligned} \sup_{|\lambda| \leq \nu} \frac{\lambda^{2m}}{|1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m}} &= \nu^{2m} \sup_{|\lambda| \leq \nu} \frac{(\lambda/\nu)^{2m}}{|1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m}} \\ &= \nu^{2m} \sup_{|t| \leq 1} \frac{t^{2m}}{|1 - e_{\alpha+2n}(it)|^{2m}} = (C\nu)^{2m} \end{aligned}$$

Therefore

$$\begin{aligned} \|\Lambda^m P_\nu(f)\|_{2,\alpha}^2 &\leq (C\nu)^{2m} \int_{-\nu}^{\nu} |1 - e_{\alpha+2n}(i\lambda/\nu)|^{2m} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= C^{2m} \nu^{2m(2n+1)} \int_{-\nu}^{\nu} |\mathcal{F}(\Delta_{1/\nu}^m f)(\lambda)|^2 d\sigma(\lambda) \\ &\leq C^{2m} \nu^{2m(2n+1)} \int_{\mathbb{R}} |\mathcal{F}(\Delta_{1/\nu}^m f)(\lambda)|^2 d\sigma(\lambda) \\ &= C^{2m} \nu^{2m(2n+1)} \|\Delta_{1/\nu}^m f\|_{2,\alpha}^2 \end{aligned}$$

by virtue of (14) and Theorem 2.

Corollary 2 For any $f \in L^2_\alpha(\mathbb{R})$ and $\nu > 0$ we have

$$\|\Lambda^m P_\nu(f)\|_{2,\alpha} \leq C^m \nu^{m(2n+1)} \omega_m(f, 1/\nu)_{2,\alpha}$$

where C is as in Lemma 7.

Proof of Theorem 4. 1) Let $h \in]0, r]$ and $g \in \mathcal{W}_{2,\alpha}^m$. By (13) and (15), we have

$$\begin{aligned}
\|\Delta_h^m f\|_{2,\alpha} &\leq \|\Delta_h^m (f - g)\|_{2,\alpha} + \|\Delta_h^m g\|_{2,\alpha} \\
&\leq 3^m h^{2mn} \|f - g\|_{2,\alpha} + h^{m(2n+1)} \|\Lambda^m g\|_{2,\alpha} \\
&\leq 3^m r^{2mn} \|f - g\|_{2,\alpha} + r^{m(2n+1)} \|\Lambda^m g\|_{2,\alpha} \\
&\leq 3^m r^{2mn} \left(\|f - g\|_{2,\alpha} + r^m \|\Lambda^m g\|_{2,\alpha} \right)
\end{aligned}$$

Calculating the supremum with respect to $h \in]0, r]$ and the infimum with respect to all possible functions $g \in \mathcal{W}_{2,\alpha}^m$ we obtain

$$c_1 \omega_m(f, r)_{2,\alpha} \leq r^{2mn} K_m(f, r^m)_{2,\alpha}$$

with $c_1 = 3^{-m}$.

2) Let ν be a positive real number. As $P_\nu(f) \in \mathcal{W}_{2,\alpha}^m$ it follows from the definition of the K -functional and Corollaries 1 and 2 that

$$\begin{aligned}
K_m(f, r^m)_{2,\alpha} &\leq \|f - P_\nu(f)\|_{2,\alpha} + r^m \|\Lambda^m P_\nu(f)\|_{2,\alpha} \\
&\leq c^{-m} \nu^{2mn} \omega_m(f, 1/\nu)_{2,\alpha} + C^m r^m \nu^{m(2n+1)} \omega_m(f, 1/\nu)_{2,\alpha} \\
&\leq \nu^{2mn} \left(c^{-m} \omega_m(f, 1/\nu)_{2,\alpha} + C^m (\nu r)^m \omega_m(f, 1/\nu)_{2,\alpha} \right).
\end{aligned}$$

Since ν is arbitrary, by choosing $\nu = 1/r$ we get

$$r^{2mn} K_m(f, r^m)_{2,\alpha} \leq c_2 \omega_m(f, r)_{2,\alpha}$$

with $c_2 = c^{-m} + C^m$. This concludes the proof.

References

- [1] Dunkl, C.F. (1989) Differential-Difference Operators Associated to Reflection Groups. *Transactions of the American Mathematical Society*, **311**, 167-183. <http://dx.doi.org/10.1090/S0002-9947-1989-0951883-8>
- [2] Dunkl, C.F. (1991) Integral Kernels with Reflection Group Invariance. *Canadian Journal of Mathematics*, **43**, 1213-1227. <http://dx.doi.org/10.4153/CJM-1991-069-8>
- [3] Dunkl, C.F. (1992) Hankel Transforms Associated to Finite Reflection Groups. *Contemporary Mathematics*, **138**, 128-138. <http://dx.doi.org/10.1090/conm/138/1199124>
- [4] Kamefuchi, S. and Ohnuki, Y. (1982) Quantum Field Theory and Parastatistics. University of Tokyo Press, Springer-Verlag, Tokyo, Berlin.
- [5] Rosenblum, M. (1994) Generalized Hermite Polynomials and the Bose-Like Oscillator Calculus. *Operator Theory: Advances and Applications*, **73**, 369-396. http://dx.doi.org/10.1007/978-3-0348-8522-5_15
- [6] Yang, L.M. (1951) A Note on the Quantum Rule of the Harmonic Oscillator. *Physical Review Letters*, **84**, 788-790. <http://dx.doi.org/10.1103/PhysRev.84.788>
- [7] Al Sadhan, S.A., Al Subaie, R.F. and Mourou, M.A. (2014) Harmonic Analysis Associated with A First-Order Singular Differential-Difference Operator on the Real Line. *Current Advances in Mathematics Research*, **1**, 23-34.
- [8] Al Subaie, R.F. and Mourou, M.A. (2014) Inversion of Two Dunkl Type Intertwining Operators on \mathbf{R} Using Generalized Wavelets. *Far East Journal of Applied Mathematics*, **88**, 91-120.
- [9] Mourou, M.A. and Trimèche, K. (2003) Transmutation Operators and Paley-Wiener Theorem Associated with a Singular Differential-Difference Operator on the Real Line. *Analysis and Applications*, **1**, 43-69. <http://dx.doi.org/10.1142/S0219530503000090>
- [10] Trimèche, K. (1981) Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentiel singulier sur $[0, +\infty[$. *Journal de Mathématiques Pures et Appliquées*, **60**, 51-98.
- [11] Platonov, S.S. (2000) Generalized Bessel Translations and Certain Problems of the Theory of Approximation of Functions in the Metrics of $L_{2,\alpha}$. I. *Trudy Petrozavodskogo Gosudarstvennogo Universiteta, Seriya Matematika*, **7**, 70-82.
- [12] Platonov, S.S. (2001) Generalized Bessel Translations and Certain Problems of the Theory of Approximation of Func-

- tions in the Metrics of $L_{2,\alpha}$. II. *Trudy Petrozavodskogo Gosudarstvennogo Universiteta, Seriya Matematika*, **8**, 1-17.
- [13] Potapov, M.K. (1998) Application of the Operator of Generalized Translation in Approximation Theory. *Vestnik Moskovskogo Universiteta, Seriya Matematika, Mekhanika*, **3**, 38-48.
- [14] Peetre, J. (1963) A Theory of Interpolation of Normed Spaces. Notes de Universidade de Brasilia, Brasilia.
- [15] Berens, H. and Buter, P.L. (1967) Semi-Groups of Operators and Approximation. Grundlehren der mathematischen Wissenschaften, **145**, Springer, Berlin.
- [16] Belkina, E.S. and Platonov, S.S. (2008) Equivalence of K -Functionals and Modulus of Smoothness Constructed by Generalized Dunkl Translations. *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika*, **8**, 3-15.
- [17] Dai, F. (2003) Some Equivalence Theorems with K -Functionals. *Journal of Approximation Theory*, **121**, 143-157. [http://dx.doi.org/10.1016/S0021-9045\(02\)00059-X](http://dx.doi.org/10.1016/S0021-9045(02)00059-X)
- [18] Ditzian, Z. and Totik, V. (1987) Moduli of Smoothness. Springer-Verlag, New York. <http://dx.doi.org/10.1007/978-1-4612-4778-4>
- [19] Löfström, J. and Peetre, J. (1969) Approximation Theorems Connected with Generalized Translations. *Mathematische Annalen*, **181**, 255-268. <http://dx.doi.org/10.1007/BF01350664>