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Orthogonal Stability of Mixed Additive-Quadratic Jensen Type Functional Equation in Multi-Banach Spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of the following mixed additive-quadratic Jensen type functional equation: $2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y)$.

Keywords

Hyers-Ulam Stability, Additive-Quadratic Jensen Type Functional Equation, Multi-Banach Spaces, Fixed Point Method

1. Introduction

In 1940, Ulam [1] proposed the stability problem of functional equations concerning the stability of group homomorphisms. Suppose that $(G_1,\cdot)$ is a group and that $(G_2,*)$ is a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d\left(h(x\cdot y),h(x)*h(y)\right)<\delta$$

for all $x,y\in G_1$, then a homomorphism $H:G_1\to G_2$ exists with $d\left(h(x),H(x)\right)<\varepsilon$ for all $x\in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] proved a generalization of the Hyers theorem for additive mappings. The result of Rassias has provided a lot of influences during the past more than three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as Hyers-Ulam-Rassias

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stability of functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found in [4]-[11].

Pinsker [12] characterized orthogonal additive functional equation on an inner product space. The orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y), \quad x\perp y$$

in which $\perp$ is an orthogonality relation, is first investigated by Gudder and Strawther [13]. In 1985, Rätz [14] introduced a new definition of orthogonality by using more restrictive axioms than Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [15] investigated the problem in a rather more general framework.

In [16], Kenary and Cho proved the Hyers-Ulam-Rassias stability of mixed additive-quadratic Jensen type functional equation in non-Archimedean normed spaces and random normed spaces. In this paper, we prove the Hyers-Ulam stability of the following mixed additive-quadratic Jensen type functional equation:

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y) \tag{1}$$

in multi-Banach spaces.

The notion of multi-normed space is introduced by Dales and Polyakov [17]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [17]. Also, the stability problems in multi-Banach spaces are studied by Dales and Moslehian [18], Moslehian *et al*. ([19]-[21]) and Wang *et al*. [22].

Now, let us recall some concepts concerning multi-Banach space.

Let $(E,\|\cdot\|)$ be a complex normed space, and let $k\in\mathbb{N}$. We denote by E^k the linear space $E\oplus E\oplus\cdots\oplus E$ consisting of k-tuples $(x_1,\cdots,x_k)$, where $x_1,\cdots,x_k\in E$. The linear operations on E^k are defined coordinate wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1,2,\cdots,k\}$ and by Ω_k the group of permutations on k symbols.

Definition 1.1 ([17]) *A multi-norm on* $\{E^k:k\in\mathbb{N}\}$ *is a sequence*

$$(\|\cdot\|_k)=(\|\cdot\|_k:k\in\mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on E^k for each $k\in\mathbb{N}$, $\|x\|_1=\|x\|$ for each $x\in E$, and the following axioms are satisfied for each $k\in\mathbb{N}$ with $k\ge 2$:

(A1) $\left\|\left(x_{\sigma(1)},\cdots,x_{\sigma(k)}\right)\right\|_k=\left\|(x_1,\cdots,x_k)\right\|_k \quad (\sigma\in\Omega_k, x_1,\cdots,x_k\in E)$;

(A2) $\left\|(\alpha_1x_1,\cdots,\alpha_kx_k)\right\|_k\le\left(\max_{i\in\mathbb{N}_k}|\alpha_i|\right)\left\|(x_1,\cdots,x_k)\right\|_k \quad (\alpha_1,\cdots,\alpha_n\in\mathbb{C}, x_1,\cdots,x_k\in E)$;

(A3) $\left\|(x_1,\cdots,x_{k-1},0)\right\|_k=\left\|(x_1,\cdots,x_{k-1})\right\|_{k-1} \quad (x_1,\cdots,x_{k-1}\in E)$;

(A4) $\left\|(x_1,\cdots,x_{k-1},x_{k-1})\right\|_k=\left\|(x_1,\cdots,x_{k-1})\right\|_{k-1} \quad (x_1,\cdots,x_{k-1}\in E)$.

In this case, we say that $\left((E^k,\|\cdot\|_k):k\in\mathbb{N}\right)$ is a multi-normed space.

Suppose that $\left((E^k,\|\cdot\|_k):k\in\mathbb{N}\right)$ is a multi-normed space and take $k\in\mathbb{N}$. We need two properties of multi-norms which can be found in [17].

(*a*) $\left\|(x,\cdots,x)\right\|_k=\|x\| \quad (x\in E)$;

(*b*) $\max_{i\in\mathbb{N}_k}\|x_i\|\le\left\|(x_1,\cdots,x_k)\right\|_k\le\sum_{i=1}^{k}\|x_i\|\le k\max_{i\in\mathbb{N}_k}\|x_i\| \quad (x_1,\cdots,x_k\in E)$.

It follows from (b) that, if $(E,\|\cdot\|)$ is a Banach space, then $(E^k,\|\cdot\|_k)$ is a Banach space for each $k\in\mathbb{N}$; in this case, $\left((E^k,\|\cdot\|_k):k\in\mathbb{N}\right)$ is a multi-Banach space.

Now, we state two important examples of multi-norms for an arbitrary normed space E (see, for details, [17]).

Example 1.2 ([17]) *The sequence* $\left(\|\cdot\|_k : k\in\mathbb{N}\right)$ *on* $\left\{E^k : k\in\mathbb{N}\right\}$ *defined by*

$$\left\|x_1,\cdots,x_k\right\|_k := \max_{i\in\mathbb{N}_k}\left\|x_i\right\|,\quad \left(x_1,\cdots,x_k\in E\right)$$

is a multi-norm called the minimum multi-norm. The terminology "minimum" is justified by property (b).

Example 1.3 ([17]) *Let* $\left\{\left(\|\cdot\|_k^\alpha : k\in\mathbb{N}\right):\alpha\in A\right\}$ *be the (non-empty) family of all multi-norms on* $\left\{E^k : k\in\mathbb{N}\right\}$. *For* $k\in\mathbb{N}$, *set*

$$\left|\left\|x_1,\cdots,x_k\right\|\right|_k := \sup_{\alpha\in A}\left\|x_1,\cdots,x_k\right\|_k^\alpha,\quad \left(x_1,\cdots,x_k\in E\right).$$

Then $\left(|\|\cdot\||_k : k\in\mathbb{N}\right)$ is a multi-norm on $\left\{E^k : k\in\mathbb{N}\right\}$, which is called the maximum multi-norm.

We need the following observation which can be easily deduced from the triangle inequality for the norm $\|\cdot\|_k$ and the property (b) of multi-norms.

Lemma 1.4 [17] *Suppose that* $k\in\mathbb{N}$ *and* $\left(x_1,\cdots,x_k\right)\in E^k$. *For each* $j\in\{1,\cdots,k\}$, *let* $\left\{x_n^j\right\}_{n\in\mathbb{N}}$ *be a sequence in E such that* $\lim_{n\to\infty}x_n^j = x_j$. *Then for each* $\left(y_1,\cdots,y_k\right)\in E^k$, *we have*

$$\lim_{n\to\infty}\left(x_n^1-y_1,\cdots,x_n^k-y_k\right)=\left(x_1-y_1,\cdots,x_k-y_k\right).$$

Definition 1.5 [17] *Let* $\left(\left(E^k,\|\cdot\|_k\right):k\in\mathbb{N}\right)$ *be a multi-normed space. A sequence* $\{x_n\}$ *in E is a multi-null sequence if, for each* $\varepsilon>0$, *there exists* $n_0\in\mathbb{N}$ *such that*

$$\sup_{k\in\mathbb{N}}\left\|x_n,\cdots,x_{n+k-1}\right\|_k<\varepsilon,\quad \left(n\ge n_0\right).$$

Let $x\in E$. We say that the sequence $\{x_n\}$ is multi-convergent to x in E and write

$$\lim_{n\to\infty}x_n = x.$$

if $\{x_n - x\}$ is a multi-null sequence.

There are several orthogonality notations on a real normed space available. But here, we present the orthogonal concept introduced by Rätz [14]. This is given in the following definition.

Definition 1.6 *Suppose that X is a vector space (algebraic module) with dim* $X\ge 2$, *and* $\perp$ *is a binary relation on X with the following properties*:

1) Totality of $\perp$ for zero: $x\perp 0$, $0\perp x$ for all $x\in X$;

2) Independence: if $x,y\in X-\{0\}$ and $x\perp y$, then x and y are linearly independent;

3) Homogeneity: if $x,y\in X$ and $x\perp y$, then $\alpha x\perp\beta y$ for all $\alpha,\beta\in\mathbb{R}$;

4) Thalesian properity: if P is a 2-dimensional subspace of X, $x\in P$ and $\lambda\in\mathbb{R}_+$, which is the set of non-negative real numbers, then there exists $y_0\in P$ such that $x\perp y_0$ and $x+y_0\perp\lambda x-y_0$.

The pair $(X,\perp)$ is called an orthogonality space (resp., module). By an orthogonality normed space (normed module) we mean an orthogonality space (resp., module) having a normed (resp., normed module) structure.

Definition 1.7 *Let X be a set. A function* $d: X\times X\to[0,\infty]$ *is called a generalized metric on X if and only if d satisfies*

($M1$) $d(x,y)=0$ if and only if $x=y$;

($M2$) $d(x,y)=d(y,x)$ for all $x,y\in X$;

($M3$) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.8 ([23]) *Let* (X,d) *be a generalized complete metric space. Assume that* $J:X\to X$ *be a strictly contractive mapping with Lipschitz constant* $L<1$. *Then, for all* $x\in X$, *either*

$$d\left(J^n x, J^{n+1}x\right)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1) $d\left(J^n x, J^{n+1}x\right)<\infty$ for all $n\ge n_0$;

2) the sequence $\left\{J^n x\right\}$ converges to a fixed point x^* of J;

3) x^* is the unique fixed point of J in the set $X^* = \left\{ y \in X \mid d\left(J^{n_0}x, y\right) < \infty \right\}$;

4) $d\left(y, x^*\right) \le \frac{1}{1-L} d\left(Jy, y\right)$ for all $y \in X^*$.

2. Hyers-Ulam Stability of Mixed Additive-Quadratic Jensen Type Functional Equation

Throughout this section, let $\alpha > 0$, E be an orthogonality space and let $\left(\left(F^n, \|\cdot\|_n\right) : n \in \mathbb{N}\right)$ be a multi-Banach space. For convenience, we use the following abbreviation for a given mapping $f : E \to F$,

$$Df\left(x, y\right) = 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f\left(x\right) - f\left(y\right)$$

for all $x, y \in E$ with $x \perp y$.

2.1. Hyers-Ulam Stability of Functional Equation (1): An Odd Case

In this section, using direct method, we prove the Hyers-Ulam stability of the functional Equation (1) in multi-Banach space.

Definition 2.1 *An odd mapping* $f : E \to F$ *is called an orthogonally Jensen additive mapping if*

$$2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f\left(x\right) + f\left(y\right)$$

for all $x, y \in E$ with $x \perp y$.

Theorem 2.2 *Suppose that α is a nonnegative real number and* $f_a : E \to F$ *is an odd mapping satisfying*

$$\sup_{k \in \mathbb{N}} \left\| \left(Df_a\left(x_1, y_1\right), \cdots, Df_a\left(x_k, y_k\right)\right) \right\|_k \le \alpha \tag{2.1}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in E$ and $x_i \perp y_i \left(i = 1, \cdots, k\right)$. Then there exists a unique orthogonally Jensen additive mapping $A : E \to F$ such that

$$\sup_{k \in \mathbb{N}} \left\| \left(f_a\left(x_1\right) - A\left(x_1\right), \cdots, f_a\left(x_k\right) - A\left(x_k\right)\right) \right\|_k \le \alpha \tag{2.2}$$

for all $x_1, \cdots, x_k \in E$.

Proof. Replacing $y_1, \cdots, y_k$ by $0, \cdots, 0$ in (2.1), we get

$$\sup_{k \in \mathbb{N}} \left\| \left(2f_a\left(\frac{x_1}{2}\right) - f_a\left(x_1\right), \cdots, 2f_a\left(\frac{x_k}{2}\right) - f_a\left(x_k\right)\right) \right\|_k \le \alpha \tag{2.3}$$

for all $x_1, \cdots, x_k \in E$ since $0 \perp x_i \left(i = 1, \cdots, k\right)$. Replacing $x_1, \cdots, x_k$ by $2^n x_1, \cdots, 2^n x_k$ in (2.3) and dividing both sides by 2^n, we get

$$\sup_{k \in \mathbb{N}} \left\| \left(\frac{f_a\left(2^{n-1}x_1\right)}{2^{n-1}} - \frac{f_a\left(2^n x_1\right)}{2^n}, \cdots, \frac{f_a\left(2^{n-1}x_k\right)}{2^{n-1}} - \frac{f_a\left(2^n x_k\right)}{2^n}\right) \right\|_k \le 2^{-n}\alpha \tag{2.4}$$

for all $x_1, \cdots, x_k \in E$ since $0 \perp 2^n x_i \left(i = 1, \cdots, k\right)$. By using (2.4) and the principle of mathematical induction, we can easily get

$$\sup_{k \in \mathbb{N}} \left\| \left(\frac{f_a\left(2^{n+m}x_1\right)}{2^{n+m}} - \frac{f_a\left(2^n x_1\right)}{2^n}, \cdots, \frac{f_a\left(2^{n+m}x_k\right)}{2^{n+m}} - \frac{f_a\left(2^n x_k\right)}{2^n}\right) \right\|_k \le \alpha \sum_{i=n+1}^{n+m} 2^{-i} \tag{2.5}$$

for all $x_1, \cdots, x_k \in E$, $n, m \in \mathbb{N}$, $m \ge 1$.

We now fix $x \in E$. We have

$$\begin{aligned}
&\sup_{k\in\mathbb{N}}\left\|\left(\frac{f_a\left(2^{n+m}x\right)}{2^{n+m}}-\frac{f_a\left(2^n x\right)}{2^n},\cdots,\frac{f_a\left(2^{n+m+k-1}x\right)}{2^{n+m+k-1}}-\frac{f_a\left(2^{n+k-1}x\right)}{2^{n+k-1}}\right)\right\|_k \\
&=\sup_{k\in\mathbb{N}}\left\|\left(\frac{f_a\left(2^{n+m}x\right)}{2^{n+m}}-\frac{f_a\left(2^n x\right)}{2^n},\cdots,\frac{1}{2^{k-1}}\left(\frac{f_a\left(2^{n+m}\left(2^{k-1}x\right)\right)}{2^{n+m}}-\frac{f_a\left(2^n\left(2^{k-1}x\right)\right)}{2^n}\right)\right)\right\|_k \\
&\le\left\|\left(\frac{f_a\left(2^{n+m}x\right)}{2^{n+m}}-\frac{f_a\left(2^n x\right)}{2^n},\cdots,\frac{f_a\left(2^{n+m}\left(2^{k-1}x\right)\right)}{2^{n+m}}-\frac{f_a\left(2^n\left(2^{k-1}x\right)\right)}{2^n}\right)\right\|_k \\
&\le\alpha\sum_{i=n+1}^{n+m}2^{-i}.
\end{aligned}$$

where we have used the Definition 1.1 and also replaced $x_1,\cdots,x_k$ by $x,2x,\cdots,2^{k-1}x$ in (2.5). It follows that $\left\{\frac{f_a\left(2^n x\right)}{2^n}\right\}$ is a Cauchy sequence and so it is convergent in the multi-Banach spaces *F*. Set

$$A(x)=\lim_{n\to\infty}\frac{f_a\left(2^n x\right)}{2^n}$$

for all $x\in E$. Hence, for each $\varepsilon>0$, there exists n_0 such that

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f_a\left(2^n x\right)}{2^n}-A(x),\cdots,\frac{f_a\left(2^{n+k-1}x\right)}{2^{n+k-1}}-A(x)\right)\right\|_k<\varepsilon$$

for all $n\ge n_0$. In particular, by property (b) of multi-norms, we have

$$\lim_{n\to\infty}\left\|\frac{f_a\left(2^n x\right)}{2^n}-A(x)\right\|=0,\quad (x\in E). \tag{2.6}$$

We next put $n=0$ in (2.5) to get

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f_a\left(2^m x\right)}{2^m}-f_a\left(x_1\right),\cdots,\frac{f_a\left(2^m x\right)}{2^m}-f_a\left(x_k\right)\right)\right\|_k\le\alpha\sum_{i=1}^{m}2^{-i}.$$

Letting $m\to\infty$ and using Lemma 1.4 and (2.6), we obtain

$$\sup_{k\in\mathbb{N}}\left\|\left(A\left(x_1\right)-f_a\left(x_1\right),\cdots,A\left(x_k\right)-f_a\left(x_k\right)\right)\right\|_k\le\alpha.$$

Let $x,y\in E$ and $x\perp y$. Considering Definition 1.6, we have $2^n x\perp 2^n y$. Put $x_1=\cdots=x_k=2^n x$, $y_1=\cdots=y_k=2^n y$ in (2.1) and divide both sides by 2^n. Then, using property (a) of multi-norms, we obtain

$$\left\|\frac{f_a\left(2^n\cdot\frac{x+y}{2}\right)}{2^{n-1}}+\frac{f_a\left(2^n\cdot\frac{x-y}{2}\right)}{2^n}+\frac{f_a\left(2^n\cdot\frac{y-x}{2}\right)}{2^n}-\frac{f_a\left(2^n x\right)}{2^n}-\frac{f_a\left(2^n y\right)}{2^n}\right\|\le 2^{-n}\alpha$$

for all $x,y\in E$ and $x\perp y$. Taking $n\to\infty$, we get

$$2A\left(\frac{x+y}{2}\right)+A\left(\frac{x-y}{2}\right)+A\left(\frac{y-x}{2}\right)-A(x)-A(y)=0$$

for all $x, y \in E$ and $x \perp y$. Since *f* is an odd mapping, according to the definition of *A*, we know that *A* is an odd mapping. By Definition 2.1, the mapping *A* is an orthogonally additive mapping.

If A' is another orthogonally additive mapping satisfying (2.2), then

$$\begin{aligned}\|A(x)-A'(x)\| &= \frac{1}{2^n}\|A(2^n x)-A'(2^n x)\| \\ &\le \frac{1}{2^n}\|A(2^n x)-f_a(2^n x)\|+\frac{1}{2^n}\|f_a(2^n x)-A'(2^n x)\| \\ &\le \frac{1}{2^n}\cdot 2\alpha.\end{aligned}$$

Taking $n \to \infty$, we get $A = A'$. This completes the proof.

2.2. Hyers-Ulam Stability of Functional Equation (1): An Even Case

In this section, we prove the Hyers-Ulam stability of the functional Equation (1) in multi-Banach space with the fixed point method.

Definition 2.3 *An even mapping* $f: E \to F$ *is called an orthogonally Jensen quadratic mapping if*

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y)$$

for all $x, y \in E$ with $x \perp y$.

Theorem 2.4 *Suppose that α is a nonnegative real number and* $f_q: E \to F$ *is an even mapping satisfying*

$$\sup_{k\in\mathbb{N}}\left\|\left(Df_q(x_1,y_1),\cdots,Df_q(x_k,y_k)\right)\right\|_k \le \alpha \tag{2.7}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in E$ and $x_i \perp y_i\ (i=1,\cdots,k)$ and $f_q(0)=0$. Then there exists a unique orthogonally Jensen quadratic mapping $Q: E \to F$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(f_q(x_1)-Q(x_1),\cdots,f_q(x_k)-Q(x_k)\right)\right\|_k \le \frac{1}{3}\alpha \tag{2.8}$$

for all $x_1,\cdots,x_k \in E$.

Proof. Letting $y_1 = y_2 = \cdots = y_k = 0$ in (2.7), we get

$$\sup_{k\in\mathbb{N}}\left\|\left(4f_q\left(\frac{x_1}{2}\right)-f_q(x_1),\cdots,4f_q\left(\frac{x_k}{2}\right)-f_q(x_k)\right)\right\|_k \le \alpha \tag{2.9}$$

for all $x_1,\cdots,x_k \in E$ since $0 \perp x_i\ (i=1,\cdots,k)$. Replacing $\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_k}{2}$ by $x_1,\cdots,x_k$ and dividing both sides by 4, we get

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{1}{4}f_q(2x_1)-f_q(x_1),\cdots,\frac{1}{4}f_q(2x_k)-f_q(x_k)\right)\right\|_k \le \frac{1}{4}\alpha \tag{2.10}$$

Let $S=\{g: X\to Y \mid g(0)=0\}$ and introduce the generalized metric *d* defined on *S* by

$$d(g,h)=\inf\left\{c\in[0,\infty] \,\middle|\, \sup_{k\in\mathbb{N}}\left\|\left(g(x_1)-h(x_1),\cdots,g(x_k)-h(x_k)\right)\right\|_k \le c,\ \text{for } x_1,\cdots,x_k\in E\right\}$$

Then it is easy to show that (S,d) is a generalized complete metric space (see [5], Lemma 2.1).

We now define an operator $J: E \to E$ by

$$Jg(x)=\frac{1}{4}g(2x),\quad \forall x\in E.$$

we assert that J is a strictly contractive operator. Given $g,h\in S$, let $c\in[0,\infty]$ be an arbitrary constant with $d(g,h)\le c$. From the definition of d, it follows that

$$\sup_{k\in\mathbb{N}}\left\|\left(g(x_1)-h(x_1),\cdots,g(x_k)-h(x_k)\right)\right\|_k\le c$$

for all $x_1,\cdots,x_k\in E$. Therefore

$$\left\|\left(Jg(x_1)-Jh(x_1),\cdots,Jg(x_k)-Jh(x_k)\right)\right\|_k=\left\|\left(\frac{1}{4}g(2x_1)-\frac{1}{4}h(2x_1),\cdots,\frac{1}{4}g(2x_k)-\frac{1}{4}h(2x_k)\right)\right\|_k\le\frac{1}{4}c$$

for all $x_1,\cdots,x_k\in E$. Hence, it holds that $d(Jg,Jh)\le\frac{1}{4}c$, *i.e.*, $d(Jg,Jh)\le\frac{1}{4}d(g,h)$ for all $g,h\in S$. This means that J is a strictly contractive operator on S with the Lipschitz constant $L=\frac{1}{4}$.

By (2.10), we have $d(Jf_q,f_q)\le\frac{1}{4}\alpha<\infty$. According to Theorem 1.8, we deduce the existence of a fixed point of J, that is, the existence of a mapping $Q:X\to Y$ such that $Q(2x)=4Q(x)$ for all $x\in E$. Moreover, we have $d(J^n f_q,Q)\to 0$, which implies

$$Q(x)=\lim_{n\to\infty}J^n f_q(x)=\lim_{n\to\infty}\frac{f_q(2^n x)}{4^n}$$

for all $x\in E$. Also, $d(f_q,Q)\le\frac{1}{1-L}d(Jf_q,f_q)$ implies the inequality

$$d(f_q,Q)\le\frac{1}{1-\frac{1}{4}}d(Jf_q,f_q)\le\frac{1}{3}\alpha.$$

Let $x,y\in E$ and $x\perp y$. Considering Definition 1.6, we have $2^n x\perp 2^n y$. Set $x_1=\cdots=x_k=2^n x$, $y_1=\cdots=y_k=2^n y$ in (2.7) and divide both sides by 4^n. Then, using property (a) of multi-norms, we obtain

$$\left\|\frac{2f_a\left(2^n\cdot\frac{x+y}{2}\right)}{4^n}+\frac{f_a\left(2^n\cdot\frac{x-y}{2}\right)}{4^n}+\frac{f_a\left(2^n\cdot\frac{y-x}{2}\right)}{4^n}-\frac{f_a(2^n x)}{4^n}-\frac{f_a(2^n y)}{4^n}\right\|\le\frac{\alpha}{4^n}$$

for all $x,y\in E$ and $x\perp y$. Taking $n\to\infty$, we get

$$2Q\left(\frac{x+y}{2}\right)+Q\left(\frac{x-y}{2}\right)+Q\left(\frac{y-x}{2}\right)-Q(x)-Q(y)=0$$

for all $x,y\in E$ and $x\perp y$. Since f is an even mapping, Q is an even mapping. According to Definition 2.3, we know that Q is an orthogonally quadratic mapping.

The uniqueness of Q follows from the fact that Q is the unique fixed point of J with the property that there exists $l\in(0,\infty)$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(f_q(x_1)-Q(x_1),\cdots,f_q(x_k)-Q(x_k)\right)\right\|_k\le l$$

for all $x_1,\cdots,x_k\in E$. This completes the proof of the theorem.

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