Advances in Pure Mathematics, 2015, 5, 314-323
Published Online April 2015 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2015.55030

Scientific Research Publishing

On Simple Completely Reducible Binary-Lie Superalgebras over $\mathfrak{sl}_2(\mathbb{F})$

Manuel Arenas

Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Santiago, Chile
Email: mcarenascl@yahoo.com

Received 9 March 2015; accepted 12 April 2015; published 20 April 2015

Abstract

In this article, we prove that if $\mathfrak{B}$ is a simple binary-Lie superalgebra whose even part is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ and whose odd part is a completely reducible binary-Lie-module over the even part, then $\mathfrak{B}$ is a Lie superalgebra. We introduce also a binary-Lie module over $\mathfrak{sl}_2(\mathbb{F})$ which is not completely reducible.

Keywords

Binary-Lie, Super-Algebras

1. Introduction

All algebras mentioned in this article are algebras over a fixed arbitrary field $\mathbb{F}$ of characteristic zero.

A superalgebra is a $\mathbb{Z}_2$-graded algebra, *i.e.*, an algebra $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$ such that $\mathfrak{A}_{\bar{i}}\mathfrak{A}_{\bar{j}} \subseteq \mathfrak{A}_{\bar{i}+\bar{j}}$ for every $\bar{i}, \bar{j} \in \{\bar{0},\bar{1}\} = \mathbb{Z}_2$. The elements of $\mathfrak{A}_{\bar{0}} \cup \mathfrak{A}_{\bar{1}}$ are called homogeneous. Given a homogeneous element x we define $\rho(x) = i$ if $x \in \mathfrak{A}_{\bar{i}}$, $i \in \{0,1\}$ If x is homogeneous we say that x is even if $\rho(x) = 0$, and that x is odd if $\rho(x) = 1$. A superalgebra $\mathfrak{A}$ is said to be anti-commutative if

$$xy = -(-1)^{\rho(x)\rho(y)} yx, \quad \forall x, y \in \mathfrak{A}_{\bar{0}} \cup \mathfrak{A}_{\bar{1}}.$$

Let us remember that for any anti-commutative algebra we define the Jacobian $J : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ by the formula $J(x, y, z) = (xy)z + (yz)x + (zx)y = (x, y, z) - (xz)y$, and a Lie algebra is an anti-commutative algebra whose Jacobian is the null function (see [1] or [2] for properties of Lie algebras). The super-analog of the Jacobian, *i.e.* the analog of the Jacobian for anti-commutative superalgebras is the function defined by

$$J_s(x,y,z)=(x,y,z)-(-1)^{\rho(y)\rho(z)}(xz)y,\quad \forall x,\ y,\ z\in\mathfrak{A}_{\bar{0}}\cup\mathfrak{A}_{\bar{1}}.$$

Since the Jacobian is an 3-linear alternating function, its super-analog J_s satisfies the identities

$$-(-1)^{\rho(x)\rho(y)}J_s(y,x,z)=J_s(x,y,z)=-(-1)^{\rho(y)\rho(z)}J_s(x,z,y) \tag{1}$$

for every $x,\ y,\ z$ homogeneous. A superalgebra $\mathfrak{L}=\mathfrak{L}_{\bar{0}}\oplus\mathfrak{L}_{\bar{1}}$ is a Lie superalgebra if and only if $J_s(x,y,z)=0$ for every $x,y,z\in\mathfrak{L}_{\bar{0}}\cup\mathfrak{L}_{\bar{1}}$, (see [3] for information about Lie superalgebras).

Lie algebras are a particular case of Malcev algebras (see [4]-[7] for definition and properties of Malcev algebras). Analogously Lie superalgebras are a particular case of Malcev superalgebras, (see [8]-[10] for information about Malcev superalgebras).

An algebra is called binary-Lie if every pair of elements generates a Lie algebra. This class of algebras contains properly the class of Malcev algebras and it was characterized by A. T. Gainov (see Id. (2) in [11], Section 2, p. 142). Gainov proved that an anti-commutative algebra $\mathfrak{B}$ was binary Lie if and only if

$$((xy)y)x+((yx)x)y=0 \tag{2}$$

or equivalently

$$J(xy,x,y)=0 \tag{3}$$

for every $x,y\in\mathfrak{B}$. If we define the function

$$\phi(a,b,c,d)=J(ab,c,d)+J(a,bc,d)+J(a,b,cd)-J(ad,b,c) \tag{4}$$

we have that the identity $\phi(a,b,c,d)=0$ is the complete linearization of (3), so it is satisfied in every binary-Lie algebra (see also [12]-[18] for information about binary-Lie algebras).

In consequence, we say that an anti-commutative superalgebra $\mathfrak{B}$ is a binary-Lie superalgebra, if it satisfies $\phi_s(a,b,c,d)=0$ for every a,b,c,d homogeneous in $\mathfrak{B}$, where ϕ_s is the super-analog of the function ϕ, *i.e.*

$$\phi_s(a,b,c,d)=J_s(ab,c,d)+J_s(a,bc,d)+J_s(a,b,cd)_s-(-1)^{\rho(d)(\rho(b)+\rho(c))}J_s(ad,b,c). \tag{5}$$

A subset $\mathfrak{I}$ of a superalgebra $\mathfrak{A}$ is said to be a super-ideal of $\mathfrak{A}$, if and only if $\mathfrak{I}$ is an ideal of $\mathfrak{A}$ and $\mathfrak{I}=\mathfrak{I}_{\bar{0}}\oplus\mathfrak{I}_{\bar{1}}$ where $\mathfrak{I}_{\bar{i}}\subseteq\mathfrak{A}_{\bar{i}}$ for every $i\in\{0,1\}$. We say that a superalgebra is simple if its unique super-ideals are $\{0\}$ and the superalgebra itself. Simple Lie superalgebras have been classified by V. G. Kac in [3] and simple Malcev superalgebras have been classified by I. P. Shestakov in [19].

For every superalgebra $\mathfrak{A}$, the space $\mathfrak{A}_{\bar{0}}$ is an algebra and $\mathfrak{A}_{\bar{1}}$ is a module over $\mathfrak{A}_{\bar{0}}$. If $\mathfrak{L}$ is a Lie superalgebra then $\mathfrak{L}_{\bar{0}}$ is a Lie algebra and $\mathfrak{L}_{\bar{1}}$ is a Lie module. The same is true for binary-Lie superalgebras, *i.e.*, for any binary-Lie superalgebra $\mathfrak{B}$ the algebra $\mathfrak{B}_{\bar{0}}$ is a binary-Lie algebra and $\mathfrak{B}_{\bar{1}}$ is a binary-Lie module over $\mathfrak{B}_{\bar{0}}$.

As usual we call $\mathfrak{sl}_2(\mathbb{F})$ the Lie algebra consisting in all two by two matrices with coficients in $\mathbb{F}$ and null trace. This algebra is a simple Lie algebra of dimension three, moreover if $\mathbb{F}$ is algebraically closed, $\mathfrak{sl}_2(\mathbb{F})$ is the unique 3-dimensional simple Lie algebra over $\mathbb{F}$. Our aim is to characterize binary-Lie superalgebras whose even part is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ and whose odd part is a completely reducible module over the even part. In particular we want to prove the following theorem.

Main Theorem. *Let* $\mathfrak{B}=\mathfrak{B}_{\bar{0}}\oplus\mathfrak{B}_{\bar{1}}$ *be a simple binary-Lie superalgebra, such that* $\mathfrak{B}_{\bar{0}}$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{F})$ *and* $\mathfrak{B}_{\bar{1}}$ *is a completely reducible module over* $\mathfrak{B}_{\bar{0}}$. *Then* $\mathfrak{B}$ *is a Lie superalgebra*.

In Section 2, we explain some basics facts about binary-Lie superalgebras and irreducible binary-Lie modules over $\mathfrak{sl}_2(\mathbb{F})$. We also give an example of a non-completely reducible binary-Lie module over $\mathfrak{sl}_2(\mathbb{F})$. In Section 3, we prove that, under the conditions described in the last parragraph the odd part is a Lie module over the even part. Finally in Section 4, we prove the main theorem.

2. Modules and Superalgebras

According to our main purpose, we must pay attention to the theory of modules over $\mathfrak{sl}_2(\mathbb{F})$. We know that $\mathfrak{sl}_2(\mathbb{F})$ has a basis $\{A,X,H\}$ whose products are given by $AX=H$, $AH=2A$, $XH=-2X$. If $\mathfrak{N}$ is an irreducible Lie module over $\mathfrak{sl}_2(\mathbb{F})$ of dimension n, then $\mathfrak{N}$ has a basis $\{u_0,\cdots,u_n\}$ whose products are defined by

$$\begin{array}{lll} u_0 A = 0, & u_i A = \left(\left(i-1\right)i - ni\right)u_{i-1}, & \forall i \neq 0 \\ u_n X = 0, & u_i X = u_{i+1}, & \forall i \neq n \\ u_i H = \left(n-2i\right)u_i, & & \forall i \in \left\{0,\cdots,n\right\}. \end{array} \tag{6}$$

We call this module, the irreducible Lie module of type n. Besides those modules, there is a non-Lie, binary-Lie module over $\mathfrak{sl}_2\left(\mathbb{F}\right)$ (in fact it is Malcev) called the irreducible module of type M_2 (see Id. (5) in [16], Section 1, p. 245). This module have a basis $\left\{v_2, v_{-2}\right\}$ with products given by:

$$\begin{array}{lll} v_2 A = -2v_{-2}, & v_2 X = 0, & v_2 H = 2v_2, \\ v_{-2} A = 0, & v_{-2} X = 2v_2, & v_{-2} H = -2v_{-2}. \end{array} \tag{7}$$

The following result of A. N. Grishkov (see Lemma 3 in [16], Section 1, p. 247) implies that there is no other irreducible $\mathfrak{sl}_2\left(\mathbb{F}\right)$-module:

Let $\mathfrak{M}$ be a binary-Lie module over $\mathfrak{sl}_2\left(\mathbb{F}\right)$ Then $\mathfrak{M}$ has a Lie sub-module $\mathfrak{N}$ such that $\mathfrak{M}/\mathfrak{N}$ can be decomposed as the direct sum of $\mathfrak{sl}_2\left(\mathbb{F}\right)$-modules of type M_2.

We conclude that if an irreducible module over $\mathfrak{sl}_2\left(\mathbb{F}\right)$ is not Lie it has to be isomorphic to the irreducible binary-Lie module of type M_2.

Remark 1. *For every $y \in \mathfrak{sl}_2\left(\mathbb{F}\right)$ let ad_y be the adjoint operator in $\mathfrak{sl}_2\left(\mathbb{F}\right)$, i.e., $z \cdot ad_y = zy$. Since ad_A and ad_X are nilpotent operators, both have* 0 *as its only eigenvalue. Therefore, if $y \in \mathfrak{sl}_2\left(\mathbb{F}\right)$ satisfies that $y \cdot ad_A = \lambda y$ or $y \cdot ad_X = \lambda y$ for some $\lambda \neq 0$, then $y = 0$. The set of eigenvalues of ad_H is $\left\{-2,0,2\right\}$. Therefore, if $y \in \mathfrak{sl}_2\left(\mathbb{F}\right)$ satisfies that $y \cdot ad_H = \lambda y$ for some $\lambda \notin \left\{-2,0,2\right\}$, then $y = 0$.*

Remark 2. *For every $y \in \mathfrak{sl}_2\left(\mathbb{F}\right)$ we have that $yA = 0$ implies $y = \alpha A$ for some $\alpha \in \mathbb{F}$, also $yX = 0$ implies $y = \beta X$ for some $\beta \in \mathbb{F}$, and finally $yH = 0$ implies $y = \gamma H$ for some $\gamma \in \mathbb{F}$.*

We notice that not every binary Lie module over $\mathfrak{sl}_2\left(\mathbb{F}\right)$ is completely reducible as we can see in the following example:

Example 1. *Let $\mathfrak{M} = \mathbb{F}u \oplus \mathbb{F}v \oplus \mathbb{F}w$ be the $\mathfrak{sl}_2\left(\mathbb{F}\right)$-module where the products are given by*

$$\begin{array}{lll} uA = 0, & uX = 0, & uH = 0, \\ vA = u - 2w, & vX = u, & vH = u + 2v, \\ wA = u, & wX = u + 2v, & wH = u - 2w. \end{array} \tag{8}$$

We observe that $\mathbb{F}u$ is an irreducible Lie module of type 0 and the quotient $\mathfrak{M}/Fu$ is an irreducible module of type M_2. Let $\mathfrak{N}$ be a non trivial sub-module of $\mathfrak{M}$, then we have that $y = \alpha u + \beta v + \gamma w \in \mathfrak{N}$ for some scalars $\alpha, \beta, \gamma \in \mathbb{F}$ and at least one of them is different from zero. We have that $\left(yX\right)X = 2\gamma u \in \mathfrak{N}$ and $\left(yA\right)A = -2\beta u \in \mathfrak{N}$. Thus, if $\beta \neq 0$ or $\gamma \neq 0$, then $\mathbb{F}u \subseteq \mathfrak{N}$. If $\beta = \gamma = 0$, then $\alpha u \in \mathfrak{M}$ with $\alpha \neq 0$. We conclude that in any case $Fu \subseteq \mathfrak{N}$ therefore $\mathfrak{N}/Fu$ is a sub-module of $\mathfrak{M}/Fu$, but $\mathfrak{M}/Fu$ is irreducible. This implies that either $\mathfrak{N} = Fu$ or $\mathfrak{N} = \mathfrak{M}$. We conclude that Fu is the only irreducible sub-module of $\mathfrak{M}$ which is not completely reducible.

Therefore, it only remains to prove that the split null extension $\mathfrak{B} = \mathfrak{sl}_2\left(\mathbb{F}\right) \oplus \mathfrak{M}$ is a binary-Lie algebra. To do that we define $m : \mathbb{F}^{12} \to \mathbb{F}^{12}$ by:

$$m\left(a,b,c,d,e,f,g,h,i,j,k,l\right) = \begin{bmatrix} 2^*\left(a^*i - c^*g\right) \\ 2^*\left(c^*h - b^*i\right) \\ a^*h - b^*g \\ \left(\left(e+f\right)^*\left(g+h+i\right)\right) - \left(\left(a+b+c\right)^*\left(k+l\right)\right) \\ 2^*\left(e^*i + f^*h - c^*k - b^*l\right) \\ 2^*\left(a^*k + c^*l - e^*g - f^*i\right) \end{bmatrix}.$$

Note that the vector on the right hand side is written as a column to fit the equation in one line. This function gives the coordinates of the product xy in the ordered basis $\left(A, X, H, u, v, w\right)$, where

$x = aA + bX + cH + du + ev + fw$ and $y = gA + hX + iH + ju + kv + lw$ hence the expression $((xy)y)x + ((yx)x)y$ is checked to be identically 0.

In what follows $\mathfrak{B}$ is a binary-Lie superalgebra whose even part $\mathfrak{B}_{\bar{0}}$ is $\mathfrak{sl}_2(\mathbb{F})$ and whose odd part $\mathfrak{B}_{\bar{1}}$ is a completely reducible module over $\mathfrak{B}_{\bar{0}}$, Under such conditions we have that

$$\mathfrak{B} = \mathfrak{sl}_2(\mathbb{F}) \oplus \left(\bigoplus_{i=1}^{p} \mathfrak{U}_i \right) \oplus \left(\bigoplus_{j=1}^{q} \mathfrak{V}_j \right) \tag{9}$$

for some pair (p,q), where $\mathfrak{U}_i$ is a Lie irreducible module of type n_i for every $i = 1,\cdots,p$, and $\mathfrak{V}_j$ is a module of type M_2 for every $j = 1,\cdots,q$.

We finish this section with the following lemma.

Lemma 1. *Let $\mathfrak{B} = \mathfrak{B}_{\bar{0}} \oplus \mathfrak{B}_{\bar{1}}$ be a simple binary-Lie superalgebra with a direct decomposition as in* (9). *Then $n_i \neq 0$ for any $i \in \{1,\cdots,p\}$.*

Proof. Assume $n_i = 0$ for some i. Without loss of generality we can suppose that $i = p$. It follows that $\mathfrak{U}_p = \mathbb{F}u$, where $uX = uA = uH = 0$. We conclude $\mathfrak{U}_p \mathfrak{B}_{\bar{0}} = 0$. Define

$$\mathfrak{B}'_{\bar{1}} = \left(\bigoplus_{i=1}^{p-1} \mathfrak{U}_i \right) \oplus \left(\bigoplus_{j=1}^{q} \mathfrak{V}_j \right).$$

We have that $\mathfrak{B}'_{\bar{1}}$ is a sub-module of $\mathfrak{B}_{\bar{1}}$ over $\mathfrak{B}_{\bar{0}}$. Next define $\mathfrak{J} = \mathfrak{B}_{\bar{0}} + \mathfrak{B}'_{\bar{1}}$ and note that $\mathfrak{B} = \mathfrak{J} + \mathfrak{U}_p$. We can easily see that $\mathfrak{J}\mathfrak{J} \subseteq \mathfrak{J}$, and $\mathfrak{J}\mathfrak{U}_p \subseteq \mathfrak{B}_{\bar{0}} \subseteq \mathfrak{J}$. It follows that $\mathfrak{J}\mathfrak{B} = \mathfrak{J}(\mathfrak{J} + \mathfrak{U}_p) \subseteq \mathfrak{J}$, whence $\mathfrak{J}$ is a super-ideal of $\mathfrak{B}$ with $\mathfrak{J}_{\bar{0}} = \mathfrak{B}_{\bar{0}}$ and $\mathfrak{J}_{\bar{1}} = \mathfrak{B}'_{\bar{1}}$. It is not zero because $\mathfrak{B}_{\bar{0}} \subseteq \mathfrak{J}$, and it is not $\mathfrak{B}$ since $\mathfrak{U}_p \not\subseteq \mathfrak{J}$. We obtain a contradiction with the simplicity of $\mathfrak{B}$. □

3. Sub-Modules of Type M_2

The aim of this section is to prove that none element of type M_2 can be found in the decomposition given by (9). This implies that $\mathfrak{B}_{\bar{1}}$ is a Lie module over $\mathfrak{B}_{\bar{0}}$. Since $\mathfrak{sl}_2(\mathbb{F})$ is a Lie algebra, $J_s(r,s,t)$ is zero for every $r,s,t \in \mathfrak{B}_{\bar{0}}$. So, if we set a,b even and c,d odd in (5), we get

$$0 = \phi_s(a,b,c,d) = J_s(ab,c,d) + J_s(a,bc,d) + J_s(ad,b,c). \tag{10}$$

On the other hand, if we set $a \in \mathfrak{B}_{\bar{0}}$, and $b,c,d \in \mathfrak{B}_{\bar{1}}$ in (5) we get

$$0 = \phi_s(a,b,c,d) = J_s(ab,c,d) - J_s(ad,b,c) + J_s(cd,a,b) - J_s(bc,a,d). \tag{11}$$

Lemma 2. *Let $\mathfrak{B}$ be a binary-Lie superalgebra with a decomposition given by* (9). *Let $\{v_{2,i}, v_{-2,i}\}$ be the basis of $\mathfrak{V}_i$ that satisfies* (7). *Then, there are three functions $\alpha,\beta,\gamma : \{1,2,\cdots,q\} \times \{1,2,\cdots,q\} \to \mathbb{F}$ such that, for every $(i,j) \in \{1,2,\cdots,q\} \times \{1,2,\cdots,q\}$, we have $v_{2,i}v_{2,j} = \alpha(i,j)A$, $v_{-2,i}v_{-2,j} = \beta(i,j)X$ and $v_{2,i}v_{-2,j} = \gamma(i,j)A$.*

Remark 3. *Observe that, since odd elements commute, the functions α and β are always symmetric while γ might fail to be.*

Proof. Using (10), a straightforward computation gives

$$0 = \phi_s(A, X, v_{2,i}, v_{-2,j}) = J_s(H, v_{2,i}, v_{-2,j}) = (v_{2,i}v_{-2,j})H.$$

Therefore, Remark 2 implies that $v_{2,i}v_{-2,j} = \gamma(i,j)H$ for some $\gamma(i,j) \in \mathbb{F}$. In the same way we obtain

$$0 = \phi_s(H, X, v_{2,i}, v_{2,j}) = 4J_s(X, v_{2,i}, v_{2,j}) = 4(v_{2,i}v_{2,j})X,$$

and

$$\phi_s(H, A, v_{-2,i}, v_{-2,j}) = -4J_s(A, v_{-2,i}, v_{-2,j}) = -4(v_{-2,i}v_{-2,j})A.$$

It follows that $v_{2,i}v_{2,j} = \beta(i,j)X$, for some $\beta(i,j) \in \mathbb{F}$, and $v_{-2,i}v_{-2,j} = \alpha(i,j)A$, for some $\alpha(i,j) \in \mathbb{F}$. □

Lemma 3. *Let $\mathfrak{B}$ be a binary-Lie superalgebra with a decomposition given by* (9). *Let $\mathfrak{V} = \mathfrak{V}_i$ for some $1 \le i \le q$, be a sub-module of type M_2. Then $\mathfrak{V}^2 = 0$.*

Proof. Let $\{v_2, v_{-2}\}$ be the basis of $\mathfrak{V}$ satisfying (7), while we write $\alpha = \alpha(i,i)$, $\beta = \beta(i,i)$ and $\gamma = \gamma(i,i)$ for simplicity. It suffices to prove that $\alpha = \beta = \gamma = 0$. Using (11), straightforward computations give both

$$0=\phi_s\left(H,v_{-2},v_2,v_2\right)=4\left(\beta-4\gamma\right)v_2,$$

and

$$0=\phi_s\left(H,v_2,v_{-2},v_{-2}\right)=4\left(\alpha-4\gamma\right)v_{-2}.$$

It follows that $\alpha=\beta=4\gamma$. Assuming this result, and using (11) again, we obtain

$$0=\phi_s\left(X,v_2,v_{-2},v_{-2}\right)=-36\gamma v_2,$$

we conclude $\gamma=\beta=\alpha=0$ as claimed. □

Lemma 4. *Let* $\mathfrak{B}$ *be a simple binary-Lie superalgebra with decomposition given by* (9). *Let be* $\mathfrak{V}_i$ *and* $\mathfrak{V}_j$ *for some pair* (i,j) *with* $1\le i\le q$, *and* $1\le j\le q$, *two sub-modules of type* M_2. *Then* $\mathfrak{V}_i\mathfrak{V}_j=0$.

Proof. If $i=j$ the result follows from lemma 3. Fix a pair (i,j) with $i\ne j$. Denote by $\{v_{2,i},v_{-2,i}\}$ and $\{v_{2,j},v_{-2,j}\}$ the basis of $\mathfrak{V}_i$ and $\mathfrak{V}_j$ respectively satisfying (7). Because of remark 3, it suffices to prove that $\alpha(i,j)=\beta(i,j)=\gamma(i,j)=\gamma(j,i)=0$. Using (11) and Lemma 3, straightforward computations give us the following results:

$$0=\phi_s\left(H,v_{2,i},v_{-2,j},v_{-2,j}\right)=-16\gamma(i,j)v_{-2,j}$$

$$0=\phi_s\left(H,v_{2,j},v_{-2,i},v_{-2,i}\right)=-16\gamma(j,i)v_{-2,i}.$$

It follows that $\gamma(i,j)=\gamma(j,i)=0$. Using this last result and (11) we get that

$$0=\phi_s\left(H,v_{2,j},v_{-2,i},v_{-2,j}\right)=4\alpha(i,j)v_{-2,j}$$

and

$$0=\phi_s\left(H,v_{-2,j},v_{2,i},v_{2,j}\right)=4\beta(i,j)v_{2,j}.$$

We conclude $\alpha(i,j)=\beta(i,j)=0$ as claimed. □

Lemma 5. *Let* $\mathfrak{B}$ *be a superalgebra with a direct decomposition given by* (9). *Let* $\mathfrak{U}=\mathfrak{U}_i$ *for some* $1\le i\le p$ *be a Lie sub-module of type* $n_i=n$ *and let* $\mathfrak{V}=\mathfrak{V}_j$ *for some* $1\le j\le q$, *be a sub-module of type* M_2. *Then* $\mathfrak{U}\mathfrak{V}=0$.

Proof. Call u_i, $0\le i\le n$, the elements of the basis of $\mathfrak{U}$ satisfying (6) and v_2, v_{-2} the elements of the basis of $\mathfrak{V}$ satisfying (7). Let $i=1,\cdots,n$. Using (10) we obtain $\phi_s\left(X,X,u_{i-1},v_2\right)=-J_s\left(X,u_{i-1}X,v_2\right)$, whence it follows

$$J_s\left(X,u_i,v_2\right)=0, \tag{12}$$

for every $i\in\{1,\cdots,n\}$. On the other hand, setting $a=H$, $b=X$, $c=v_2$, $d=u_0$ in (10), we obtain $\phi_s\left(H,X,v_2,u_0\right)=(n+2)J_s\left(X,u_0,v_2\right)=0$. So (12) is satisfied by every $i=1,\cdots,n$. Computing the left hand side of (12), we find that, for every $i\in\{1,\cdots,n\}$, identity $u_iv_2=\left(u_{i-1}v_2\right)X$ holds. At this point, a simple induction proves that

$$u_iv_2=\left(u_0v_2\right)\left(ad_X\right)^i \tag{13}$$

for every $i=1,\cdots,n$ (the case $i=0$ is trivial). Now using (10), we obtain

$$\phi_s\left(X,A,u_0,v_2\right)=J_s\left(H,u_0,v_2\right)=-(n+2)\left(u_0v_2\right)+\left(u_0v_2\right)H=0.$$

Since, $n\ne 0$ by Lemma 1, we have that $n+2\notin\{-2,0,2\}$. Thus Remark 1 implies that $u_0v_2=0$, and (13) implies that $u_iv_2=0$ for every $i\in\{0,\cdots,n\}$.

Let be $i=0,\cdots,n-1$. Using (10) again we conclude $\phi_s\left(A,A,u_{i+1},v_{-2}\right)=-J_s\left(A,u_{i+1}A,v_{-2}\right)=(i-n)(i+1)J_s\left(A,u_i,v_{-2}\right)$ and since $i\ne n$ it follows that

$$J_s\left(A,u_i,v_{-2}\right)=0, \tag{14}$$

for every $i\in\{0,\cdots,n-1\}$. On the other hand, setting $a=A$, $b=H$, $c=u_n$, $d=v_{-2}$ in (10), we obtain $\phi_s\left(A,H,u_n,v_{-2}\right)=(n+2)J_s\left(A,u_n,v_{-2}\right)=0$. So (14) is satisfied by every $i\in\{0,\cdots,n\}$. Computing the left side of (14) when $i \in \{0, \cdots, n-1\}$, we have

$$u_{i-1}v_{-2} = \frac{-1}{(n+1-i)i}(u_i v_{-2})A. \tag{15}$$

Form here a simple induction proves

$$u_{n-i}v_{-2} = \zeta_i (u_n v_{-2})(ad_A)^i \tag{16}$$

for every $i \in \{0, \cdots, n\}$, where

$$\zeta_i = (-1)^i \frac{(n-i)!}{i!n!}, \quad \forall 0 \le i \le n.$$

Now using (10) we obtain $\phi_s(A, X, u_n, v_{-2}) = J_s(H, u_n, v_{-2}) = (n+2)(u_n v_{-2}) + (u_n v_{-2})H$. Since $n \ne 0$ we have $-(n+2) \notin \{-2, 0, 2\}$, whence remark 1 implies that $u_n v_{-2} = 0$, and therefore (16) implies $u_i v_{-2} = 0$ for every $i = 0, \cdots, n$. □

Now we can prove the following:

Theorem 1. *Let $\mathfrak{B} = \mathfrak{B}_{\bar{0}} \oplus \mathfrak{B}_{\bar{1}}$ be a simple binary-Lie superalgebra, such that $\mathfrak{B}_{\bar{0}}$ is $\mathfrak{sl}_2(\mathbb{F})$ and $\mathfrak{B}_{\bar{1}}$ has a decomposition given by* (9). *Then $q = 0$, i.e there is no sub-module of type M_2 in the decomposition. In other words $\mathfrak{B}_{\bar{1}}$ is a Lie module over $\mathfrak{B}_{\bar{0}}$.*

Proof. Let $\mathfrak{J} = \mathfrak{V}_i$ be a space of type M_2 in the direct decomposition given by (9). Thanks to Lemma 4 we have $\mathfrak{V}_j \mathfrak{J} = 0$ for every $1 \le j \le q$. As a consequence of Lemma 5 we get $\mathfrak{U}_i \mathfrak{J} = 0$ for every $1 \le i \le p$. Thus $\mathfrak{B}_{\bar{1}} \mathfrak{J} = 0$, and since $\mathfrak{B}_{\bar{0}} \mathfrak{J} \subseteq \mathfrak{J}$, we conclude that $\mathfrak{J}$ is a super-ideal of $\mathfrak{B}$ with $\mathfrak{J}_{\bar{1}} = \mathfrak{J}$ and $\mathfrak{J}_{\bar{0}} = \{0\}$ Since $\mathfrak{J} \ne \mathfrak{B}$ and $\mathfrak{B}$ is simple, necessarily $\mathfrak{J} = 0$.

We have prove that the decomposition of $\mathfrak{B}$ given by (9) reduces to

$$\mathfrak{B} = \mathfrak{sl}_2(\mathbb{F}) \oplus \left(\bigoplus_{i=1}^{p} \mathfrak{U}_i \right) \tag{17}$$

where $\mathfrak{U}_i$ is a Lie irreducible sub-module of type n_i, for every $1 \le i \le p$. Therefore $\mathfrak{B}_{\bar{1}}$ is a Lie module over $\mathfrak{B}_{\bar{0}}$. □

4. The Main Theorem

In this section we are going to prove that A is a Lie superalgebra. We need two previous lemmas.

Lemma 6. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two sub-modules of type n and m respectively in the decomposition given by* (17). *Then $J_s(\mathfrak{B}_{\bar{1}}, \mathfrak{U}, \mathfrak{V}) = 0$, which clearly implies that $J_s(B_{\bar{1}}, B_{\bar{1}}, B_{\bar{1}}) = 0$.*

Proof. Since $\mathfrak{B}_{\bar{1}}$ is a Lie module if $a \in \mathfrak{B}_{\bar{0}}$ and $b, c, d \in \mathfrak{B}_{\bar{1}}$ we have that $J_s(bc, a, d) = J_s(cd, a, b) = 0$, whence (11) becomes

$$\phi_s(a, b, c, d) = J_s(ab, c, d) - J_s(ad, b, c) = 0. \tag{18}$$

Without loss of generality, we assume that $n \le m$. Let *w* be an arbitrary element of $\mathfrak{B}_{\bar{1}}$. Set $a = H$, $b = u_i$, $c = w$, and $d = v_j$ in (18). We have

$$((m-n) + 2(i-j)) J_s(u_i, w, v_j) = 0.$$

Therefore, if i, j satisfy $(m-n) + 2(i-j) \ne 0$ then $J_s(u_i, v_j, w) = 0$. In particular, if $m-n$ is odd, then $J_s(u_i, v_j, w) = 0$ for every $0 \le i \le n$ and $0 \le j \le m$. Assume that $m-n$ is an even integer (zero included). If we set $a = X$, $b = u_i$, $c = w$, and $d = v_j$ in (18) we get

$$J_s(u_i X, w, v_j) - J_s(v_j X, u_i, w) = 0. \tag{19}$$

If i, j are two indices such that $(n-m) + 2(j-i) = 0$ and $i \ne 0$, then $n - m + 2((j+1) - (i-1)) = n - m + 2(j-i) + 4 = 4 \ne 0$. Thus Equation (18) implies

$$J_s(u_i, w, v_j) = J_s(u_{i-1}X, w, v_j) = J_s(v_j X, w, u_{i-1}) = J_s(v_{j+1}, w, u_{i-1}) = 0.$$

for every $j \ne m$ and every $i \ne 0$. On the other hand (19) implies

$$J_s(u_i, w, v_m) = J_s(u_{i-1}X, w, v_m) = J_s(v_m X, w, u_i) = J_s(0, w, u_i) = 0$$

for every $i \neq 0$. Observe that $(n-m)+2(j-i)=0$ implies $j \leq i$. Then $J_s(u_i, v_j, w) = 0$ for every $(i, j) \neq (0,0)$.

Next, set $a = A$, $b = u_1$, $c = w$, and $d = v_0$ in (18). It follows that

$$-nJ_s(u_0, w, v_0) = 0.$$

Since $n \neq 0$ by lemma (1), necessarily $J_s(u_0, w, v_0) = 0$. We conclude that $J_s(u_i, v_j, w) = 0$ for every $i, j \in \{1, \cdots, p\}$. □

Lemma 7. *Let* $\mathfrak{U} = \mathfrak{U}_i$, $\mathfrak{V} = \mathfrak{U}_j$ *be two submodules of type* $n = n_i$ *and* $m = n_j$ *respectively in the decomposition given by* (17). *Then* $J_s(\mathfrak{B}_{\bar{0}}, \mathfrak{U}, \mathfrak{V}) = 0$, *which clearly implies that* $J_s(B_{\bar{0}}, B_{\bar{1}}, B_{\bar{1}}) = 0$.

Proof. Let be $\{u_0, \cdots, u_n\}$ and $\{v_0, \cdots, v_m\}$ the basis of $\mathfrak{U}$ and $\mathfrak{V}$ respectively satisfying (6). It is enough to prove that every pair of indices $(i, j) \in \{0, \cdots, n\} \times \{0, \cdots, m\}$ satisfies $J_s(H, u_i, v_j) = J_s(X, u_i, v_j) = J_s(A, u_i, v_j) = 0$. For simplicity we use the notation $H_{i,j} = J_s(H, u_i, v_j)$, $X_{i,j} = J_s(X, u_i, v_j)$ and $A_{i,j} = J_s(A, u_i, v_j)$. We call H-matrix to the $n+1$ by $m+1$ matrix with entrances in $\mathfrak{B}_{\bar{0}}$ whose (i, j) entrence is $H_{i,j}$ (called the H-coefficients). Similarly we define the X-matrix and the A-matrix. We need to prove that these three matrices are the null matrix. **Table 1** shows us some identities obtained using (10) by evaluating ϕ_s in different 4-tuples in $\mathfrak{B}_{\bar{0}} \times \mathfrak{B}_{\bar{0}} \times \mathfrak{B}_{\bar{1}} \times \mathfrak{B}_{\bar{1}}$.

We claim that the eight identities of **Table 1** suffice to prove that the X-coefficient and the H-coefficient are zero, with the only possible exception of $H_{0,0}$ when $n = m$.

To explain how this implication works we notice that identity (1) of **Table 1** implies that the H-matrix is zero if $m-n$ is odd and has the form in **Figure 1** if $m-n$ is even. We also see that identities (4), (5), (7) and (8) imply that the X-matrix is as in **Figure 2**, note that, in every position where neither (4) nor (5) implies that X-coefficient is zero, either (7) or (8) does.

We introduce now a diagram notation to keep track of the information involved by the other identities. First we write down two matrices in the same diagram as follows: If M and N are two matrices of the same size we put both in a double matrix diagram as in the left side of **Figure 3**.

Table 1. Some identities involving *H*-coefficients and *X*-coefficients implied by (10).

		Identity	Restriction
(1)	$\phi_s(H, H, u_i, v_j)$	$((m-n)-2(j-i))H_{i,j} = 0$	
(2)	$\phi_s(H, X, u_i, v_j)$	$(m+2-2j)X_{i,j} = H_{i+1,j}$	$(i \neq n)$
(3)	$\phi_s(H, X, v_j, u_i)$	$(n+2-2i)X_{i,j} = H_{i,j+1}$	$(j \neq m)$
(4)	$\phi_s(H, X, u_n, v_j)$	$(m+2-2j)X_{n,i} = 0$	
(5)	$\phi_s(X, H, u_i, v_m)$	$(n+2-2i)X_{i,m} = 0$	
(6)	$\phi_s(X, X, u_i, v_j)$	$X_{i+1,j} = X_{i,j+1}$	$(i \neq n, j \neq m)$
(7)	$\phi_s(X, X, u_n, v_j)$	$X_{n,j+1} = 0$	$(j \neq m)$
(8)	$\phi_s(X, X, u_i, v_m)$	$X_{i+1,m} = 0$	$(i \neq n)$

$$\left(\begin{array}{ccc|ccc|ccc} 0 & \cdots & 0 & \bullet & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{\frac{m-n}{2}} & & & \underbrace{0 \quad \cdots \quad \bullet}_{n+1} & & & \underbrace{0 \quad \cdots \quad 0}_{\frac{m-n}{2}} & & \end{array}\right)$$

Figure 1. The *H*-matrix with *m* − *n* even.

$$\begin{pmatrix} \bullet & \cdots & \bullet & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \bullet & \cdots & \bullet & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Figure 2. The *X*-matrix with *m* and *n* arbitrary.

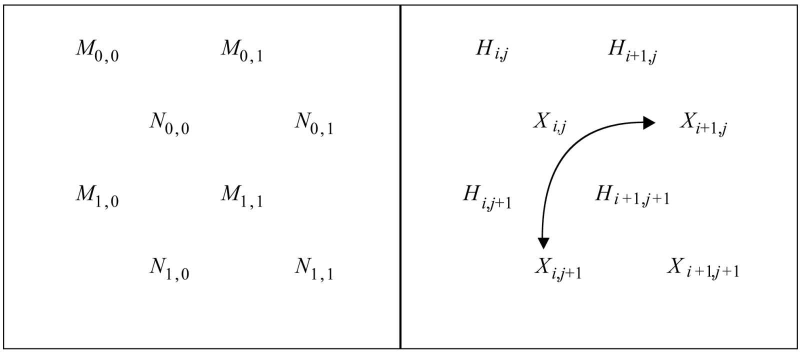

Figure 3. Left: Example of a 2 × 2 double matrix. Right: Digram of identity (6) in **Table 1**.

In those diagrams we draw an arrow from a coefficient to another one if the nullity of the second one can be obtained from the nullity of the first one. We use a full triangle on the tip of the arrow if the implication works without any restriction and an empty triangle on the tip of the arrow if the implication depend of some restriction explained in the legend of the figure. With this notation, the information from identities (2) and (3) of **Table 1** is encoded in **Figure 4**, and information from identity (6) of **Table 1** is encoded in the right side of **Figure 3**. If a coefficient of the X-matrix is in the last column or in the last row it is zero as we see in **Figure 2**. Otherwise we can see in **Figure 4** that we can apply identity (2) or (3) in **Table 1** and conclude that it is zero except in the following situations.

1) Neither identity (2) nor identity (3) can be applied. This occurs only when $2i = n+2$ and $2j = m+2$.

2) Identity (2) cannot be applied and the H-coefficient that is pointed by the empty triangle in the right side of **Figure 4** corresponds to a bullet in **Figure 1**. This occurs only when $2j = m+2$ and $m-n = 2\left(\left(j+1\right)-i\right)$.

3) Identity (3) cannot be applied and the H-coefficient that is pointed by the empty triangle in the left side of **Figure 4** corresponds to a bullet in **Figure 1**. This occurs only when $2i = n+2$ and $m-n = 2\left(j-\left(i+1\right)\right)$.

Therefore, the only coefficients that might be different from zero are $X_{\frac{n}{2}+1,\frac{m}{2}+1}$, $X_{\frac{n}{2}+2,\frac{m}{2}+1}$ or $X_{\frac{n}{2}+1,\frac{m}{2}+2}$. We call them the exceptional X-coefficients.

As we see in **Figure 2**, if any of this three coefficients is in the last row or the last column, then it is zero. Otherwise this X-coefficient can be put in the right side of **Figure 3** with a non-exceptional X-coefficient in the other side of the arrow. Therefore, it is zero (notice that $X_{\frac{n}{2}+2,\frac{m}{2}+1}$ and $X_{\frac{n}{2}+1,\frac{m}{2}+2}$ can be put in the left side of **Figure 3** in each tip of the arrow, but if one them is not in the last row or column, then either $X_{\frac{n}{2}+3,\frac{m}{2}}$ or $X_{\frac{n}{2},\frac{m}{2}+3}$ is available).

We have proved that the X-matrix is the matrix zero. Now we can see that every H-coefficient different than $H_{0,0}$ can be put either in the left side or in the right side of **Figure 4** with an X-coefficient on the other side of the arrow. We conclude that $H_{i,j} = 0$, for every $\left(i, j\right) \neq \left(0,0\right)$.

Now we have to look at **Table 2**. Information from identities (1) and (2) is encoded in **Figure 5**. We can see that the A-coefficients are 0 except in three cases.

1) Neither identity (1) nor identity (2) can be applied. This occurs only for $A_{0,0}$.

2) Identity (1) cannot be applied, and the H-coefficient pointed by the triangle in the right side of **Figure 5** is $H_{0,0} \neq 0$. This occurs only for $A_{1,0}$ when $n = m$.

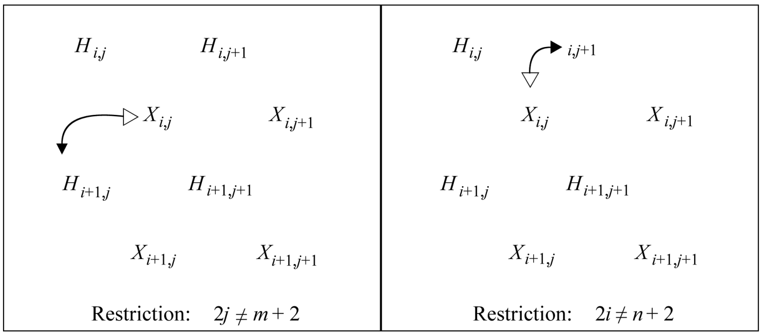

Figure 4. Diagram of identities (2) and (3) in **Table 1** with their restrictions.

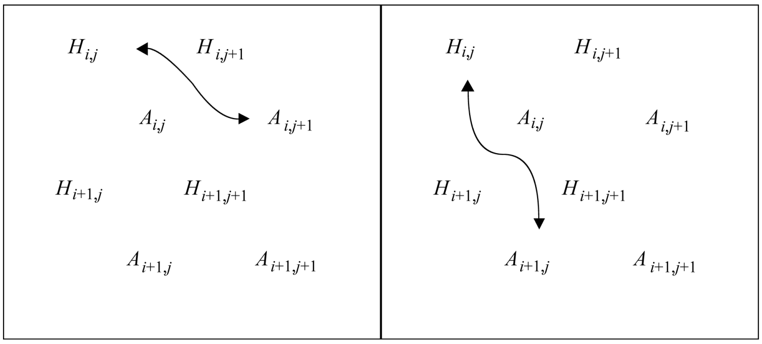

Figure 5. Diagram of identities (1) and (2) in **Table 2**.

Table 2. Identities involving the *H*-coefficients and *A*-coefficients implied by (10) and the nullity of the *X*-matrix.

		Identity	Restriction
(1)	$\phi_s\left(A, X, u_i, v_j\right)$	$H_{i,j} = -A_{i+1,j}$	$i \neq n$
(2)	$\phi_s\left(X, A, u_i, v_j\right)$	$H_{i,j} = A_{i,j+1}$	$j \neq m$
(3)	$\phi_s\left(A, A, u_i, v_0\right)$	$\left(n+1-i\right) iA_{i-1,0} = 0$	$i \neq 0$
(4)	$\phi_s\left(A, H, u_1, v_0\right)$	$A_{1,0} = 0$	$n = 1$

3) Identity (2) cannot be applied, and the H-coefficient pointed by the triangle in the left side of **Figure 5** is $H_{0,0} \neq 0$. This occurs only for $A_{0,1}$ when $n = m$.

To prove that $A_{0,0} = 0$ we set $i = 1$ in identity (3) of **Table 2**. This always can be done since $n \neq 0$. The second and the third cases occur only if $n = m$, and in this case, it is enough to prove that just one among $H_{0,0}$, $A_{1,0}$, $A_{0,1}$ is zero. In this case the desired result can be proved using identity (3) of **Table 2** with $i = 2$, but we are able do that provided $n \geq 2$. Thus, the only coefficients we have not proved yet to be zero, are $H_{0,0} = -A_{1,0} = A_{0,1}$ when $n = m = 1$. This is the reason why we included identity (4) in **Table 2**. At this point we have proved that every H-coefficient is zero, every X-coefficient is zero and every A-coefficient is zero. This means that

$$J_s\left(H, u_i, v_j\right) = J_s\left(X, u_i, v_j\right) = J_s\left(A, u_i, v_j\right) = 0$$

for every pair i, j finishing thus the proof of the lemma. □

Proof of the main theorem: We need to prove that $J_s\left(w, u, v\right) = 0$ for every *w*, *u*, *v* homogeneous. Since $\mathfrak{B}_0$ is a Lie algebra $J_s\left(w, u, v\right) = J\left(w, u, v\right) = 0$ whenever these three elements are even. Theorem 1 implies that the identity holds whenever two of these elements are even and the other one is odd. So we only have to prove that $J_s\left(w, u, v\right) = 0$ when at least two of this three elements are odd. Thanks to the simmetries described in identity (1) it suffices to prove that $J_s\left(w, u, v\right) = 0$ when *u* and *v* are odd. Because of the decomposition given by (17) we only have to prove that the identity holds when $u \in \mathfrak{U}_i$ for some $i \in \{1, \cdots, p\}$ and $v \in \mathfrak{V}_j$ for some $j \in \{1, \cdots, p\}$, but this follows from Lemma 7 if w is even, and from Lemma 6 if w is odd. This proves that $\mathfrak{B}$ is a Lie superalgebra. □

Funding

Supported by FONDECYT process 2010/11100092.

References

[1] Humphreys, J.E. (1972) Introduction to Lie Algebras and Representation Theory. Springer Verlag, New York. http://dx.doi.org/10.1007/978-1-4612-6398-2

[2] Jacobson, N. (1979) Lie Algebras. Dover, New York.

[3] Kac, V.G. (1977) Lie Superalgebras. *Advances in Mathematics*, **26**, 8-96. http://dx.doi.org/10.1016/0001-8708(77)90017-2

[4] Malcev, A.I. (1955) Analytic Loops. *Mat. Sb. N.S*, **36**, 569-576.

[5] Sagle, A.A. (1961) Malcev Algebras. *Transactions of the American Mathematical Society*, **101**, 426-458. http://dx.doi.org/10.1090/S0002-9947-1961-0143791-X

[6] Perez-Izquierdo, J.M. and Shestakov, I.P. (2004) An Envelope for Malcev Algebras. *Journal of Algebra*, **272**, 379-393. http://dx.doi.org/10.1016/S0021-8693(03)00389-2

[7] Shestakov, I.P. (1998) Speciality Problem for Malcev Algebras and Poisson Malcev Algebras. Nonassociative Algebra and Its Applications. São Paulo, 365-371.

[8] Shestakov, I.P. and Zhukavets, N. (2007) The Free Malcev Superalgebras on One Odd Generator and Related Superalgebras. *Journal of Mathematical Sciences*, **140**, 243-249. http://dx.doi.org/10.1007/s10958-007-0421-x

[9] Shestakov, I.P. and Zhukavets, N. (2006) The Universal Multiplicative Envelope of the Free Malcev Superalgebra on One Odd Generator. *Communications in Algebra*, **34**, 1319-1344. http://dx.doi.org/10.1080/00927870500454570

[10] Shestakov, I.P. and Zhukavets, N. (2006) The Malcev Poisson Supelalgebra of the Free Malcev Superalgebra on One Odd Generator. *Journal of Algebra and Its Applications*, **5**, 521-535. http://dx.doi.org/10.1142/S0219498806001867

[11] Gainov, A.T. (1957) Identical Relations for Binary-Lie Rings. *Uspehi Mat. Nauk* (*N.S.*), **12**, 141-146.

[12] Arenas, M. and Shestakov, I.P. (2011) On Speciality of Binary-Lie Algebras. *Journal of Algebra and Its Applications*, **10**, 257-268. http://dx.doi.org/10.1142/S0219498811004550

[13] Filipov, V.T. (2008) Binary-Lie Algebras Satisfying the Third Engel Condition. *Sibirskii Matematicheskii Zhurnal*, **49**, 928-933. http://dx.doi.org/10.1007/s11202-008-0071-3

[14] Gainov, A.T. (1963) Binary Lie Algebras of Lower Ranks. *Algebra i Logika*, **2**, 21-40.

[15] Grishkov, A.N. (1977) On the Theory of Finite Dimensional Binary Lie Algebras. *Algebra i Logika*, **16**, 549-556. http://dx.doi.org/10.1007/BF01669476

[16] Grishkov, A.N. (1980) Structure and Representaion of Binary Lie Algebras. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, **44**, 999-1030.

[17] Grishkov, A.N. (1988) Finite-Dimensional Solvable Binary Lie Algebras. *Sibirskii Matematicheskii Zhurnal*, **29**, 48-52.

[18] Kuzmin, E.N. (1998) Binary Lie Algebras of Small Dimentions. *Algebra and Logic*, **37**, 181-186. http://dx.doi.org/10.1007/BF02671589

[19] Shestakov, I.P. (1993) Prime Malcev Superalgebras. *Mathematics*, **182**, 1357-1366. http://dx.doi.org/10.1070/SM1993v074n01ABEH003337