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Index-*p* Abelianization Data of *p*-Class Tower Groups

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Abstract

Given a fixed prime number *p*, the multiplet of abelian type invariants of the *p*-class groups of all unramified cyclic degree *p* extensions of a number field *K* is called its IPAD (index-*p* abelianization data). These invariants have proved to be a valuable information for determining the Galois group G_p^2 of the second Hilbert *p*-class field and the *p*-capitulation type $\varkappa$ of *K*. For $p = 3$ and a number field *K* with elementary *p*-class group of rank two, all possible IPADs are given in the complete form of several infinite sequences. Iterated IPADs of second order are used to identify the group G_p^∞ of the maximal unramified pro-*p* extension of *K*.

Keywords

***p*-Class Groups, *p*-Principalization Types, *p*-Class Field Towers, Quadratic Fields, Second *p*-Class Groups, *p*-Class Tower Groups, Coclass Graphs**

1. Introduction

After a thorough discussion of the terminology used in this article, such as the logarithmic and power form of *abelian type invariants* in Section 2, and *multilayered transfer target types* (TTTs), ordered and accumulated *index-* p *abelianization data* (IPADs) up to the third order in Section 3, we state the main results on IPADs of exceptional form in Section 3.1, and on IPADs in parametrized infinite sequences in Section 3.2. These main theorems give all possible IPADs of number fields K with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$.

Before we turn to applications in extreme computing, that is, squeezing the computational algebra systems PARI/GP [1] and MAGMA [2]-[4] to their limits in Section 5, where we show how to detect *malformed* IPADs in Section 5.1, and how to complete partial p-capitulation types in Section 5.2, we have to establish a componentwise correspondence between *transfer kernel types* (TKTs) and IPADs in Section 4 by exploiting details

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of proofs which were given in [5].

Iterated IPADs *of second order* are used in Section 6 for the indirect calculation of TKTs in Section 6.1, and for determining the *exact length* $\ell_p(K)$ *of the* p*-class tower* of a number field K in Section 6.2. These sophisticated techniques prove $\ell_3(K)=3$ for quadratic number fields $K=\mathbb{Q}(\sqrt{d})$ with $d\in\{342664, 957013\}$ (the first *real quadratic* fields) and $d=-3896$ (the first tough *complex quadratic* field after the easy $d=-9748$ ([6], Cor.4.1.1), which resisted all attempts up to now.

Finally, we emphasize that IPADs of *infinite* p*-class towers* reveal an unknown wealth of possible *fine structure* in Section 7 on complex quadratic fields K having a 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3,3)$.

2. Abelian Type Invariants

Let p be a prime number and A be a finite abelian p-group. According to the main theorem on finitely generated abelian groups, there exists a non-negative integer $r\ge 0$, the *rank* of A, and a sequence $n_1,\cdots,n_r$ of positive integers such that $n_1\le n_2\le\cdots\le n_r$ and

$$A\simeq(\mathbb{Z}/p^{n_1}\mathbb{Z})\oplus\cdots\oplus(\mathbb{Z}/p^{n_r}\mathbb{Z}) \tag{1}$$

The powers $d_i:=p^{n_i}$, $1\le i\le r$, are known as the *elementary divisors* of A, since $d_i|d_{i+1}$ for each $1\le i\le r-1$. It is convenient to collect equal elementary divisors in formal powers with positive exponents $r_1,\cdots,r_s$ such that $r_1+\cdots+r_s=r$, $0\le s\le r$, and

$$n_1=\cdots=n_{r_1}<n_{r_1+1}=\cdots=n_{r_1+r_2}<\cdots<n_{r_1+\cdots+r_{s-1}+1}=\cdots=n_{r_1+\cdots+r_s}$$

The cumbersome subscripts can be avoided by defining $m_j:=n_{r_1+\cdots+r_j}$ for each $1\le j\le s$. Then

$$A\simeq(\mathbb{Z}/p^{m_1}\mathbb{Z})^{r_1}\oplus\cdots\oplus(\mathbb{Z}/p^{m_s}\mathbb{Z})^{r_s} \tag{2}$$

and we can define:

Definition 2.1. *The abelian type invariants* (*ATI*) *of* A *are given by the sequence*

$$\mathrm{ATI}(A):=(m_1^{r_1},\cdots,m_s^{r_s}) \tag{3}$$

of strictly increasing positive integers $m_1<\cdots<m_s$ with multiplicities $r_1,\cdots,r_s$ written as formal exponents indicating iteration.

Remark 2.1. *The integers* m_j *are the* p*-logarithms of the elementary divisors* d_i.

1) For abelian type invariants of high complexity, the *logarithmic form* in Definition 2.1 requires considerably less space (e.g. in Section 7, Table 2) than the usual *power form*

$$\left(\overbrace{p^{m_1},\cdots,p^{m_1}}^{r_1},\cdots,\overbrace{p^{m_s},\cdots,p^{m_s}}^{r_s}\right) \tag{4}$$

2) For brevity, we can even omit the commas separating the entries of the logarithmic form of abelian type invariants, provided all the m_j remain smaller than 10.

3) A further advantage of the brief logarithmic notation is the independence of the prime p, in particular when p-groups with distinct p are being compared.

4) Finally, since our preference is to select generators of finite p-groups with decreasing orders, we agree to write abelian type invariants from the right to the left, in both forms.

Example 2.1. *For instance, if* $p=3$, *then the abelian type invariants* (21^4) *in logarithmic form correspond to the power form* $(9,3,3,3,3)$ *and* (2^21^2) *corresponds to* $(9,9,3,3)$.

Now let G be an arbitrary finite p-group or infinite topological pro-p group with derived subgroup G' and finite abelianization $G^{ab}:=G/G'$.

Definition 2.2. The abelian type invariants of the commutator quotient group G/G' are called the abelian quotient invariants (AQI) *of* G, that is, $\mathrm{AQI}(G):=\mathrm{ATI}(G^{ab})$.

3. Index-*p* Abelianization Data

Let p be a fixed prime number and K be a number field with p-class group $\mathrm{Cl}_p(K)$ of order p^v, where

$v \ge 0$ denotes a non-negative integer.

According to the Artin reciprocity law of class field theory [7], $\mathrm{Cl}_p(K)$ is isomorphic to the commutator quotient group G/G' of the Galois group $G = G_p^\infty(K) := \mathrm{Gal}\left(F_p^\infty(K)\middle|K\right)$ of the maximal unramified pro- p extension $F_n^\infty(K)$ of K. G is called the p*-class tower group* (briefly p*-tower group*) of K. The fixed field of the commutator subgroup G' in $F_p^\infty(K)$ is the maximal abelian unramified p-extension of K, that is the (first) Hilbert p-class field $F_p^1(K)$ of K with Galois group $\mathrm{Gal}\left(F_p^1(K)\middle|K\right) \simeq \mathrm{Gal}\left(F_p^\infty(K)\middle|K\right)/\mathrm{Gal}\left(F_p^\infty(K)\middle|F_p^1(K)\right) = G/G'$. The derived subgroup G' is a closed (and open) subgroup of finite index $(G:G') = p^v$ in the topological pro- p group G.

Definition 3.1. *For each integer* $0 \le n \le v$, *the system*

$$\mathrm{Lyr}_n(K) = \left\{K \le L \le F_p^1(K) \middle| [L:K] = p^n\right\} \tag{5}$$

of intermediate fields $K \le L \le F_p^1(K)$ with relative degree $[L:K] = p^n$ is called the n*-th layer* of abelian un-ramified p-extensions of K. In particular, for $n = 0$, K forms the *bottom layer* $\mathrm{Lyr}_0(K) = \{K\}$, and for $n = v$, $F_p^1(K)$ forms the *top layer* $\mathrm{Lyr}_v(K) = \left\{F_p^1(K)\right\}$.

Now let $0 \le n \le v$ be a fixed integer and suppose that $K \le L \le F_p^1(K)$ belongs to the n-th layer. Then the Galois group $H = \mathrm{Gal}\left(F_p^\infty(K)\middle|L\right)$ is of finite index $(G:H) = [L:K] = p^n$ in the p-tower group G of K and the quotient $G/H \simeq \mathrm{Gal}(L|K)$ is abelian, since H contains the commutator subgroup $G' = \mathrm{Gal}\left(F_p^\infty(K)\middle|F_p^1(K)\right)$ of G.

Definition 3.2. *For each integer* $0 \le n \le v$, *the system*

$$\mathrm{Lyr}_n(G) := \left\{G' \le H \le G \middle| (G:H) = p^n\right\} \tag{6}$$

of intermediate groups $G' \le H \le G$ with index $(G:H) = p^n$ is called the n*-th layer* of normal subgroups of G with abelian quotients G/H. In particular, for $n = 0$, G forms the *top layer* $\mathrm{Lyr}_0(G) = \{G\}$, and for n = v, G' forms the *bottom layer* $\mathrm{Lyr}_v(K) = \{G'\}$.

A further application of Artin's reciprocity law [7] shows that

$$H/H' = \mathrm{Gal}\left(F_p^\infty(K)\middle|L\right)/\mathrm{Gal}\left(F_p^\infty(K)\middle|F_p^1(L)\right) \simeq \mathrm{Gal}\left(\left(F_p^1(L)\right)\middle|L\right) \simeq \mathrm{Cl}_p(L) \tag{7}$$

for every subgroup $H \in \mathrm{Lyr}_n(G)$ and its corresponding extension field $L \in \mathrm{Lyr}_n(K)$, where $0 \le n \le v$ is fixed (but arbitrary).

Since the abelianization $H^{ab} = H/H'$ forms the target of the Artin transfer homomorphism $T_{G,H}: G \to H/H'$ from G to H [8], we introduced a preliminary instance of the following terminology in ([9], Dfn. 1.1, p. 403).

Definition 3.3. For each integer $0 \le n \le v$, the multiplet $\tau_n(G) = (H/H')_{H \in \mathrm{Lyr}_n(G)}$, where each member H/H' is interpreted rather as its abelian type invariants, is called the n-th layer of the transfer target type (TTT) of the pro- p group G,

$$\tau(G) := [\tau_0(G); \cdots; \tau_v(G)], \quad \text{where} \quad \tau_n(G) = (H/H')_{H \in \mathrm{Lyr}_n(G)} \quad \text{for each } 0 \le n \le v \tag{8}$$

Similarly, the multiplet $\tau_n(K) = \left(\mathrm{Cl}_p(L)\right)_{L \in \mathrm{Lyr}_n(K)}$, where each member $\mathrm{Cl}_p(L)$ is interpreted rather as its abelian type invariants, is called the n-th layer of the *transfer target type* (TTT) of the number field K,

$$\tau(K) := [\tau_0(K); \cdots; \tau_v(K)], \quad \text{where} \quad \tau_n(K) = \left(\mathrm{Cl}_p(L)\right)_{L \in \mathrm{Lyr}_n(K)} \quad \text{for each } 0 \le n \le v \tag{9}$$

Remark 3.1.

1) If it is necessary to specify the underlying prime number p, then the symbol $\tau(p,G)$, resp. $\tau(p,K)$, can be used for the TTT.

2) Suppose that $0 < n < v$. If an ordering is defined for the elements of $\mathrm{Lyr}_n(G)$, resp. $\mathrm{Lyr}_n(K)$, then the same ordering is applied to the members of the layer $\tau_n(G)$, resp. $\tau_n(K)$, and the TTT layer is called *ordered*. Otherwise, the TTT layer is called *unordered* or *accumulated*, since equal components are collected in powers with formal exponents denoting iteration.

3) In view of the considerations in Equation (7), it is clear that we have the equality

$$\tau(G)=\tau(K) \tag{10}$$

in the sense of componentwise isomorphisms.

Since it is increasingly difficult to compute the structure of the p-class groups $\mathrm{Cl}_p(L)$ of extension fields $L\in\mathrm{Lyr}_n(K)$ in higher layers with $n\ge 2$, it is frequently sufficient to make use of information in the first layer only, that is the layer of subgroups with index p. Therefore, Boston, Bush and Hajir [10] invented the following *first order approximation* of the TTT, a concept which had been used in earlier work already [11]-[15], without explicit terminology.

Definition 3.4. *The restriction*

$$\begin{aligned}\tau^{(1)}(G)&:=\left[\tau_0(G);\tau_1(G)\right], \text{ resp.}\\ \tau^{(1)}(K)&:=\left[\tau_0(K);\tau_1(K)\right],\end{aligned} \tag{11}$$

of the TTT $\tau(G)$, resp. $\tau(K)$, to the zeroth and first layer is called the *index-* p *abelianization data* (IPAD) of G, resp. K.

So, the complete TTT is an extension of the IPAD. However, there also exists another extension of the IPAD which is not covered by the TTT. It has also been used already in previous investigations by Boston, Bush and Nover [12] [14] [15] and is constructed from the usual IPAD $\left[\tau_0(K);\tau_1(K)\right]$ of K, firstly, by observing that $\tau_1(K)=\left(\mathrm{Cl}_p(L)\right)_{L\in\mathrm{Lyr}_1(K)}$ can be viewed as $\tau_1(K)=\left(\tau_0(L)\right)_{L\in\mathrm{Lyr}_1(K)}$ and, secondly, by extending each $\tau_0(L)$ to the IPAD $\left[\tau_0(L);\tau_1(L)\right]$ of L.

Definition 3.5. *The family*

$$\begin{aligned}\tau^{(2)}(G)&:=\left[\tau_0(G);\left(\tau_0(H);\tau_1(H)\right)_{H\in\mathrm{Lyr}_1(G)}\right], \quad \text{resp.}\\ \tau^{(2)}(K)&:=\left[\tau_0(K);\left(\tau_0(L);\tau_1(L)\right)_{L\in\mathrm{Lyr}_1(K)}\right],\end{aligned} \tag{12}$$

is called the *iterated* IPAD *of second order* of G, resp. K.

The concept of iterated IPADs as given in Dfn. 3.5 is restricted to the second order and first layers, and thus is open for further generalization (higher orders and higher layers). Since it could be useful for 2-power extensions, whose absolute degrees increase moderately and remain manageable by MAGMA [4] or PARI/GP [1], we briefly indicate how the *iterated* IPAD *of third order* could be defined:

$$\begin{aligned}\tau^{(3)}(G)&:=\left[\tau_0(G);\left(\tau_0(H);\left(\tau_0(I);\tau_1(I)\right)_{I\in\mathrm{Lyr}_1(H)}\right)_{H\in\mathrm{Lyr}_1(G)}\right], \quad \text{resp.}\\ \tau^{(3)}(K)&:=\left[\tau_0(K);\left(\tau_0(L);\left(\tau_0(M);\tau_1(M)\right)_{M\in\mathrm{Lyr}_1(L)}\right)_{L\in\mathrm{Lyr}_1(K)}\right].\end{aligned} \tag{13}$$

3.1. Sporadic IPADs

In the next two central theorems, we present complete specifications of all possible IPADs of pro- p groups G for $p=3$ in the simplest case of an abelianization G/G' of type $(3,3)$. We start with pro-3 groups G whose metabelianizations G/G'' are vertices on sporadic parts of coclass graphs outside of coclass trees ([9], Section 2, p. 410), ([16] Section 10).

Since the abelian type invariants of the members of TTT layers will depend on the parity of the nilpotency class c or coclass r, a more economic notation, avoiding the tedious distinction of the cases odd or even, is provided by the following definition ([5], Section 3).

Definition 3.6. *For an integer* $n\ge 2$, *the nearly homocyclic abelian* 3*-group* $A(3,n)$ *of order* 3^n *is defined by its type invariants* $(q+r,q)\hat{=}\left(3^{q+r},3^q\right)$, *where the quotient* $q\ge 1$ *and the remainder* $0\le r<2$ *are determined uniquely by the Euclidean division* $n=2q+r$. *Two degenerate cases are included by putting* $A(3,1):=(1)\hat{=}(3)$ *the cyclic group* C_3 *of order* 3 *and* $A(3,0):=(0)\hat{=}1$ *the trivial group of order* 1.

In the following theorems, we use the *identifiers* in the SmallGroups Library [17] [18].

Theorem 3.1. (*First Main Theorem on* $p=3$, $G/G'\simeq(3,3)$, *and* G/G'' *of small class*)

Let G be a pro-3 group having a transfer target type $\tau(G)=\left[\tau_0(G);\tau_1(G);\tau_2(G)\right]$ with top layer component $\tau_0(G)=1^2$. Let $0\le k\le 1$ denote the defect of commutativity ([9], 3.1.1, p.412, and 3.3.2, p. 429) of the metabelianization G/G'' of G. Then the ordered first layer $\tau_1(G)$ and the bottom layer $\tau_2(G)$ are given in the following way (exceptional cases in **boldface** font).

1) If G/G'' is of coclass $\mathrm{cc}(G/G'')=1$ and nilpotency class $c:=\mathrm{cl}(G/G'')\le 3$, then

$$\begin{aligned}
&\tau_1(G)=\left((1)^4\right); && \tau_2(G)=(0), && \text{if } c=1, G\simeq\langle 9,2\rangle,\\
&\tau_1(G)=\left((1^2)^4\right); && \tau_2(G)=(1), && \text{if } c=2, G\simeq\langle 27,3\rangle,\\
&\tau_1(G)=\left(1^2,(\mathbf{2})^{\mathbf{3}}\right); && \tau_2(G)=(1), && \text{if } c=2, G\simeq\langle \mathbf{27,4}\rangle,\\
&\tau_1(G)=\left(\mathbf{1^3},(1^2)^3\right); && \tau_2(G)=(1^2), && \text{if } c=3, G\simeq\langle \mathbf{81,7}\rangle,\\
&\tau_1(G)=\left(21,(1^2)^3\right); && \tau_2(G)=(1^2), && \text{if } c=3, G\simeq\langle 81,8|9|10\rangle.
\end{aligned}\tag{14}$$

where generally $G''=1$.

2) If G/G'' is of coclass $\mathrm{cc}(G/G'')=2$ and nilpotency class $c:=\mathrm{cl}(G/G'')=3$, then

$$\begin{aligned}
&\tau_1(G)=\left((21)^2,1^3,21\right); && \tau_2(G)=(1^3), && \text{if } G\simeq\langle 243,5\rangle \text{ or } G/G''\simeq\langle 243,6\rangle,\\
&\tau_1(G)=\left((21)^2,(1^3)^2\right); && \tau_2(G)=(1^3), && \text{if } G/G''\simeq\langle 243,3\rangle,\\
&\tau_1(G)=\left(1^3,21,1^3,21\right); && \tau_2(G)=(1^3), && \text{if } G\simeq\langle 243,7\rangle,\\
&\tau_1(G)=\left((\mathbf{1^3})^{\mathbf{2}},21,\mathbf{1^3}\right); && \tau_2(G)=(1^3), && \text{if } G/G''\simeq\langle \mathbf{243,4}\rangle,\\
&\tau_1(G)=\left((21)^4\right); && \tau_2(G)=(1^3), && \text{if } G/G''\simeq\langle 243,8|9\rangle.
\end{aligned}\tag{15}$$

where $G''=1$ can be warranted for $G/G''\simeq\langle 243,5|7\rangle$ only.

However, if $c:=\mathrm{cl}(G/G'')=4$ with $k=1$, then

$$\begin{aligned}
&\tau_1(G)=\left((21)^2,(1^3)^2\right); && \tau_2(G)=(21^2), && \text{if } G/G''\simeq\langle 729,37|38|39\rangle,\\
&\tau_1(G)=\left((21)^2,(1^3)^2\right); && \tau_2(G)=(\mathbf{1^4}), && \text{if } G/G''\simeq\langle \mathbf{729,34|35|36}\rangle,\\
&\tau_1(G)=\left((\mathbf{1^3})^{\mathbf{2}},21,\mathbf{1^3}\right); && \tau_2(G)=(21^2), && \text{if } G/G''\simeq\langle \mathbf{729,44|45|46|47}\rangle,\\
&\tau_1(G)=\left((21)^4\right); && \tau_2(G)=(\mathbf{1^4}), && \text{if } G/G''\simeq\langle \mathbf{729,56|57}\rangle.
\end{aligned}\tag{16}$$

3) If G/G'' is of coclass $r:=\mathrm{cc}(G/G'')\ge 3$ and nilpotency class $c:=\mathrm{cl}(G/G'')=r+1$, then

$$\tau_1(G)=\left(A(3,r+1)^2,(1^3)^2\right);\quad \tau_2(G)=A(3,r)\times A(3,r-1)\ \text{ and } k=0 \tag{17}$$

However, if $c=r+2$, then

$$\begin{aligned}
&\tau_1(G)=\left(A(3,r+2),A(3,r+1),(1^3)^2\right); && \tau_2(G)=A(3,r+1)\times A(3,r-1), && \text{if } k=0\\
&\tau_1(G)=\left(A(3,r+1)^2,(1^3)^2\right); && \tau_2(G)=A(3,r+1)\times A(3,r-1), && \text{if } k=1, \text{regular case},\\
&\tau_1(G)=\left(A(3,r+1)^2,(1^3)^2\right); && \tau_2(G)=\mathbf{A(3,r)}\times\mathbf{A(3,r)}, && \text{if } k=1, \text{irregular case},
\end{aligned}\tag{18}$$

where the **irregular** case can only occur for even class and coclass $c=r+2\equiv 0 \pmod 2$, positive defect of commutativity $k=1$, and relational parameter $\rho=-1$ in ([5], Equation (3.6), p. 424) or ([9], Equation (3.3), p. 430).

Proof. Since this proof heavily relies on our earlier paper [5], it should be pointed out that, for a p-group G, the index of nilpotency $m=c+1$ is used generally instead of the nilpotency class $\mathrm{cl}(G)=c=m-1$ and the invariant $e=r+1$ frequently (but not always) replaces the coclass $\mathrm{cc}(G)=r=e-1$ in that paper.

1) Using the association between the identifier of G in the SmallGroups Library [17] [18] and the transfer kernel type (TKT) [19], which is visualized in ([5], Figure 3.1, p. 423) and ([16], Figure 3), this claim follows from ([5], Thm. 4.1, p. 427, and Tbl. 4.1, p. 429).

2) For $c=3$, resp. $c=4$ with $k=1$, the statement is a consequence of ([5], Thm. 4.2 and Tbl. 4.3, p. 434), resp. ([5], Thm. 4.3 and Tbl. 4.5, p. 438), when the association between the identifier of G in the SmallGroups database and the TKT is taken into consideration, as visualized in ([5], Figure 4.1, p. 433) and ([16], Figure 4).

3) All the regular cases behave completely similar as the general case in Theorem 3.2, item (3), Equation (22). In the irregular case, only the bottom layer $\tau_2(G)$, consisting of the abelian quotient invariants G'/G'' of the derived subgroup G', is exceptional and must be taken from ([5], Appendix 8, Thm. 8.8, p.461). □

3.2. Infinite IPAD Sequences

Now we come to the IPADs of pro-p groups G whose metabelianizations G/G'' are members of infinite periodic sequences, inclusively mainlines, of coclass trees.

Theorem 3.2. (*Second Main Theorem on* $p=3$, $G/G'\simeq(3,3)$, *and* G/G'' *of large class*)

Let G be a pro-3 group having a transfer target type $\tau(G)=[\tau_0(G);\tau_1(G);\tau_2(G)]$ with top layer component $\tau_0(G)=1^2$. Let $0\le k\le 1$ denote the defect of commutativity ([9], Section 3.1.1, p. 412, and Section 3.3.2, p. 429) of the metabelianization G/G'' of G. Then the ordered first layer $\tau_1(G)$ and the bottom layer $\tau_2(G)$ are given in the following way.

1) If G/G'' is of coclass $\mathrm{cc}(G/G'')=1$ and nilpotency class $c:=\mathrm{cl}(G/G'')\ge 4$, then

$$\begin{aligned}\tau_1(G)&=\left(A(3,c-k),\left(1^2\right)^3\right);\\ \tau_2(G)&=A(3,c-1).\end{aligned}\tag{19}$$

2) If G/G'' is of coclass $\mathrm{cc}(G/G'')=2$ and nilpotency class $c:=\mathrm{cl}(G/G'')\ge 5$, or $c=4$ with $k=0$, then

$$\begin{aligned}\tau_1(G)&=\left(A(3,c-k),21,\left(1^3\right)^2\right)\quad\text{or}\\ \tau_1(G)&=\left(A(3,c-k),21,1^3,21\right)\quad\text{or}\\ \tau_1(G)&=\left(A(3,c-k),(21)^3\right),\end{aligned}\tag{20}$$

in dependence on the coclass tree $G/G''\in\mathcal{T}^2(\langle 729,i\rangle)$, $i\in\{40,49,54\}$, but uniformly

$$\tau_2(G)=A(3,c-1)\times A(3,1)\tag{21}$$

3) If G/G'' is of coclass $r:=\mathrm{cc}(G/G'')\ge 3$ and nilpotency class $c:=\mathrm{cl}(G/G'')\ge r+3$, or $c=r+2$ with $k=0$, then

$$\begin{aligned}\tau_1(G)&=\left(A(3,c-k),A(3,r+1),\left(1^3\right)^2\right);\\ \tau_2(G)&=A(3,c-1)\times A(3,r-1).\end{aligned}\tag{22}$$

The first member H_1/H_1' of the ordered first layer $\tau_1(G)$ reveals a **uni-polarization** (dependence on the nilpotency class c), whereas the other three members H_i/H_i', $2\le i\le 4$, show a **stabilization** (independence of c) for fixed coclass r.

Proof. Again, we make use of [5], and we point out that, for a p-group G, the index of nilpotency $m=c+1$ is used generally instead of the nilpotency class $\mathrm{cl}(G)=c=m-1$ and the invariant $e=r+1$ frequently (but not always) replaces the coclass $\mathrm{cc}(G)=r=e-1$ in that paper.

1) All components of $\tau_1(G)$ are given in ([5], Section 3.1, Thm. 3.1, Equation (3.4)-(3.5), p. 421) when their ordering is defined by the special selection of generators ([5], Section 3.1, Equation (3.1)-(3.2), p. 420).

There is only a unique coclass tree with 3-groups of coclass 1.

2) The first component of $\tau_1(G)$ is given in ([5], Section 3.2, Thm.3.2, Equation (3.7), p. 424), and the last three components of $\tau_1(G)$ are given in ([5], Section 4.5, Thm. 4.4, p. 440) and ([5], Section 4.5, Tbl. 4.7, p. 441), when their ordering is defined by the special selection of generators ([5], Section 3.2, Equation (3.6), p. 424). The invariant $\varepsilon \in \{0,1,2\}$ ([5], Section 4, p. 426), which counts IPAD components of rank 3, decides to which of the mentioned three coclass trees the group G belongs ([9], Figure 3.6-3.7, pp. 442-443).

3) The first two components of $\tau_1(G)$ are given in ([5], Section 3.2, Thm.3.2, Equation (3.7)-(3.8), p. 424), and the last two components of $\tau_1(G)$ are given in ([5], Section 4.6, Thm. 4.5, p. 444), when their ordering is defined by the special selection of generators ([5], Section 3.2, Equation (3.6), p. 424). For coclass bigger than 2, it is irrelevant to which of the four (in the case of odd coclass r) or six (in the case of even coclass r) coclass trees the group G belongs. The IPAD is independent of this detailed information, provided that $c \geq r+3$.

Finally, the bottom layer $\tau_2(G)$, consisting of the abelian quotient invariants G'/G'' of the derived subgroup G', is generally taken from ([5], Appendix 8, Thm. 8.8, p. 461). □

4. Componentwise Correspondence of IPAD and TKT

Within this section, where generally $p=3$, we employ some special terminology.

Definition 4.1. *We say a class of a base field* K *remains resistant if it does not capitulate in any unramified cyclic cubic extension* $L|K$.

When the 3-class group of K is of type $(3,3)$ the next layer of unramified abelian extensions is already the top layer consisting of the Hilbert 3-class field $F_3^1(K)$, where a resistant class must capitulate, according to the Hilbert/Artin/Furtwängler principal ideal theorem [8].

Our desire is to show that the components of the ordered IPAD and TKT [9] [19] are in a strict correspondence to each other. For this purpose, we exploit details of the proofs given in [5], where generators of metabelian 3-groups G with $G/G' \simeq (3,3)$ were selected in a canonical way, particularly adequate for theoretical aspects ([5], Section 3.2, p. 424), ([9], Section 3.3.2, p. 429).

Since we now prefer a more computational aspect, we translate the results into a form which is given by the computational algebra system MAGMA [4].

To be specific, we choose the vertices of two important coclass trees for illustrating these peculiar techniques. The vertices of depth (distance from the mainline) at most 1 of both coclass trees, with roots $\langle 243,6\rangle$ and $\langle 243,8\rangle$ ([9], Figure 3.6-3.7, pp. 442-443), are metabelian 3-groups G with order $|G| \geq 3^5$, nilpotency class $c = \mathrm{cl}(G) \geq 3$, and fixed coclass $\mathrm{cc}(G) = 2$.

4.1. The Coclass Tree $\mathcal{T}^2(\langle 243,6\rangle)$

Remark 4.1. *The first layers of the TTT and TKT of vertices of depth at most* 1 *of the coclass tree* $\mathcal{T}^2(\langle 243,6\rangle)$ ([9], Figure 3.6, p. 442) *consist of four components each, and share the following common properties with respect to Magma's selection of generators*:

1) *polarization* (dependence on the class c) at the first component of TTT and TKT,

2) *Stabilization* (independence of the class c) at the last three components,

3) Rank 3 at the second TTT component ($\varepsilon = 1$ in [5]).

Using the class c, resp. an asterisk, as wildcard characters, these common properties can be summarized as follows, now including the details of the stabilization:

$$\tau_1(G) = \left[A(3,c), 1^3, (21)^2\right], \quad \text{and} \quad \varkappa_1(G) = (*,1,2,2) \tag{23}$$

However, to assure the general applicability of the theorems and corollaries in this section, we aim at independency of the selection of generators (and thus invariance under permutations).

Theorem 4.1. (*in field theoretic terminology*)

1) The class associated with the polarization becomes principal in the extension with rank 3.

2) The class associated with rank 3 becomes principal in both extensions of type (21), in particular, $\varkappa_1(G)$ cannot be a permutation and can have at most one fixed point.

In the sequel, we use designations of special TKTs which were developed in [19]-[21].

Remark 4.2. *Aside from the common properties, there also arise variations due to the polarization, which we* *first express with respect to Magma's selection of generators*:

1) The TKT is E.6, $\varkappa_1(G)=(1,1,2,2)$, if and only if the polarized extension reveals a *fixed point* principalization.

2) The TKT is E.14, $\varkappa_1(G)\in\{(3,1,2,2),(4,1,2,2)\}$, if and only if one of the classes associated with type (21) becomes principal in the polarized extension.

3) The TKT is H.4, $\varkappa_1(G)=(2,1,2,2)$, if and only if the class associated with rank 3 becomes principal in the polarized extension.

4) The TKT is c.18, $\varkappa_1(G)=(0,1,2,2)$, if and only if the polarized extension reveals a *total* principalization (indicated by 0).

Corollary 4.1. (*in field theoretic terminology*)

1) For the TKTs E.6 and H.4, both classes associated with type (21) remain resistant, for TKT E.14 only one of them.

2) All extensions satisfy Taussky's condition (B) [22], with the single exception of the polarized extension in the case of TKT E.6 or c.18, which satisfies condition (A).

3) TKT E.6 has a single fixed point, E.14 contains a 3-cycle, and H.4 contains a 2-cycle.

Proof. (of Theorem 4.1 and Corollary 4.1)

Observe that in [5], the index of nilpotency $m=c+1$ and the invariant $e=r+1$ are used rather than the nilpotency class $c=m-1$ and the coclass $r=e-1$. The claims are a consequence of ([5], 4.5, Tbl. 4.7, p. 441), when we perform a transformation from the first layer TKT and TTT

$$\varkappa_1(G)=(*,3,1,3),\quad \tau_1(G)=\left[A(3,c),21,1^3,21\right]$$

with respect to the canonical generators, to the corresponding invariants

$$\varkappa_1(G)=(*,1,2,2),\quad \tau_1(G)=\left[A(3,c),1^3,(21)^2\right]$$

with respect to Magma's generators. □

4.2. The Coclass Tree $\mathcal{T}^2(\langle 243,8\rangle)$

Remark 4.3. *The first layer TTT and TKT of vertices of depth at most* 1 *of the coclass tree* $\mathcal{T}^2(\langle 243,8\rangle)$ ([9], Figure 3.7, p. 443) *consist of four components each, and share the following common properties with respect to Magma's choice of generators*:

1) *polarization* (dependence on the class c) at the second component of TTT and TKT,

2) *stabilization* (independence of the class c) at the other three components,

3) rank 3 does not occur at any TTT component ($\varepsilon=0$ in [5]).

Using the class c, resp. an asterisk, as wildcard characters, the common properties can be summarized as follows, now including details of the stabilization:

$$\tau_1(G)=\left[21,A(3,c),(21)^2\right],\quad\text{and}\quad \varkappa_1(G)=(2,*,3,4) \tag{24}$$

Again, we have to ensure the general applicability of the following theorem and corollary, which must be independent of the choice of generators (and thus invariant under permutations).

Theorem 4.2. (*in field theoretic terminology*)

1) Two extensions of type (21) reveal fixed point principalization satisfying condition (A) [22].

2) The remaining extension of type (21) satisfies condition (B), since the class associated with the polarization becomes principal there.

Remark 4.4. *Next, we come to variations caused by the polarization, which we now express with respect to Magma's choice of generators*:

1) The TKT is E.8, $\varkappa_1(G)=(2,2,3,4)$, if and only if the polarized extension reveals a *fixed point* principalization.

2) The TKT is E.9, $\varkappa_1(G)=\{(2,3,3,4),(2,4,3,4)\}$, if and only if one of the classes associated with fixed points becomes principal in the polarized extension.

3) The TKT is G.16, $\varkappa_1(G)=(2,1,3,4)$, if and only if the class associated with type (21), satisfying condition (B), becomes principal in the polarized extension.

4) The TKT is c.21, $\varkappa_1(G)=(2,0,3,4)$, if and only if the polarized extension reveals a *total* principalization (indicated by 0).

Corollary 4.2. (*in field theoretic terminology*)

1) For the TKTs E.8 and E.9, the class associated with type (21), satisfying condition (B), remains resistant.

2) The polarized extension satisfies condition (B) [22] in the case of TKT E.9 or G.16, and it satisfies condition (A) in the case of TKT E.8 or c.21.

3) TKT G.16 is a permutation containing a 2-cycle, and TKT E.8 is the unique TKT possessing three fixed points.

Proof. (of Theorem 4.2 and Corollary 4.2)

In our paper [5], the index of nilpotency $m=c+1$ and the invariant $e=r+1$ are used rather than the nilpotency class $c=m-1$ and the coclass $r=e-1$. All claims are a consequence of ([5], 4.5, Tbl. 4.7, p. 441), provided we perform a transformation from the first layer TKT and TTT

$$\varkappa_1(G)=(*,2,3,1),\quad \tau_1(G)=\left[A(3,c),(21)^3\right]$$

with respect to the canonical generators, to the corresponding invariants

$$\varkappa_1(G)=(2,*,3,4),\quad \tau_1(G)=\left[21,A(3,c),(21)^2\right]$$

with respect to Magma's generators. □

5. Applications in Extreme Computing

5.1. Application 1: Sifting Malformed IPADs

Definition 5.1. *An IPAD with bottom layer component* $\tau_0(K)=(3,3)$ *is called malformed if it is not covered by Theorems* 3.1 *and* 3.2.

To verify predicted asymptotic densities of maximal unramified pro-3 extensions in the article [10] numerically, the IPADs of all complex quadratic fields $K=\mathbb{Q}\left(\sqrt{d}\right)$ with discriminants $-10^8<d<0$ and 3-class rank $r_3(K)=2$ were computed by Boston, Bush and Hajir with the aid of PARI/GP [1]. In particular, there occurred 276375, resp. 122444, such fields with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$, resp. $(9,3)$.

Example 5.1.

A check of all 276375 IPADs for complex quadratic fields with type $(3,3)$ in the range $-10^8<d<0$ of discriminants, for which Theorem 3.2 states that the 3-class groups of the 4 unramified cyclic cubic extensions can only have 3-rank 2, except for the unique type $(3,3,3)$, revealed that the following 5 IPADs were computed erroneously by the used version of PARI/GP [1] in [10]. The successful recomputation was done with MAGMA [4].

1) For $d=-96174803$, the erroneous IPAD $\tau^{(1)}(K)=\left[(3,3);(3,3,3),(9,3,3,3),(27,9)^2\right]$ contained the malformed component $(9,3,3,3)$ instead of the correct $(3,3,3)$. The transfer kernel type (TKT) [9] [19] [21] turned out to be F.12.

2) For $d=-77254244$, the erroneous IPAD $\tau^{(1)}(K)=\left[(3,3);(3,3,3)^2,(3,3,3,3),(9,3)\right]$ contained the malformed component $(3,3,3,3)$ instead of the correct $(3,3,3)$. Its TKT is H.4.

3) For $d=-73847683$, the erroneous IPAD $\tau^{(1)}(K)=\left[(3,3);(3,3,3),(9,3,3),(9,3)^2\right]$ contained the malformed component $(9,3,3)$ instead of the correct $(9,3)$. The TKT is D.10.

4) For $d=-81412223$, the erroneous IPAD $\tau^{(1)}(K)=\left[(3,3);(9,3,3),(9,3)^2,(27,9)\right]$ contained the malformed component $(9,3,3)$ instead of the correct $(9,3)$. This could be a TKT E.8 or E.9 or G.16.

5) For $d=-82300871$, the erroneous IPAD $\tau^{(1)}(K)=\left[(3,3);(3,3,3),(9,3),(9,9,3),(27,9)\right]$ contained the malformed component $(9,9,3)$ instead of the correct $(9,3)$. This could be a TKT E.6 or E.14 or H.4.

For the last two cases, Magma failed to determine the TKT. Nevertheless, none of the discriminants

$$d\in\{-73847683,-77254244,-81412223,-82300871,-96174803\}$$

is particularly spectacular.

Example 5.2.

We also checked all 122444 IPADs for complex quadratic fields with type $(9,3)$ in the range $-10^8 < d < 0$ of discriminants, Again, we found exactly 5 errors among these IPADs which had been computed by PARI/GP [1] in [10]. For the recomputation we used MAGMA [4]. The study of this extensive material was very helpful for the deeper understanding of 3-groups having abelianization of type $(9,3)$. Systematic results in the style of Theorems 3.1 and 3.2 will be given in a forthcoming paper. The abbreviation pTKT means the *punctured* TKT.

1) For $d = -94304231$, the erroneous IPAD $\tau^{(1)}(K) = [(9,3);(9,3,3),(27,3),(9,9,9),(27,9)]$ contained the malformed component $(27,9)$ instead of the correct $(27,3)$. This could be a homocyclic pTKT B.2 or C.4 or D.5.

2) For $d = -79749087$, the erroneous IPAD $\tau^{(1)}(K) = \left[(9,3);(9,3,3)^2,(27,3,3),(27,3)\right]$ contained the malformed component $(27,3,3)$ instead of the correct $(27,3)$. It is a pTKT D.11, $\varkappa_1(K) = (4,3,2;3)$.

3) For $d = -74771240$, the erroneous IPAD $\tau^{(1)}(K) = [(9,3);(9,3,3),(27,3,3),(27,3),(9,9,9)]$ contained the malformed component $(27,3,3)$ instead of the correct $(27,3)$. It could be a homocyclic pTKT B.2 or C.4 or D.5.

4) For $d = -70204919$, the erroneous IPAD $\tau^{(1)}(K) = \left[(9,3);(9,3,3),(27,3)^2,(81,27,27)\right]$ contained the malformed component $(81,27,27)$ instead of the correct $(81,27,3)$. This could be a pTKT B.2 or C.4 or D.5 in the first excited state.

5) For $d = -86139199$, the erroneous IPAD $\tau^{(1)}(K) = \left[(9,3);(81,3,3,3),(9,3,3),(27,3)^2\right]$ contained the malformed component $(81,3,3,3)$ instead of the correct $(9,3,3)$. This is clearly a pTKT D.11, $\varkappa_1(K) = (4,3,2;3)$.

Again, none of the corresponding discriminants

$$d \in \{-70204919, -74771240, -79749087, -86139199, -94304231\}$$

is particularly spectacular.

We emphasize that, in both Examples 5.1 and 5.2, the errors of PARI/GP [1] occured in the upper limit range of absolute discriminants above 70 millions. This seems to be a critical region of extreme computing where current computational algebra systems become unstable. MAGMA [4] also often fails to compute the TKT in that range.

Fortunately, there appeared a single discriminant only for each of the 5 erroneous IPADs, in both examples. This indicates that the errors are not systematic but rather stochastic.

5.2. Application 2: Completing Partial Capitulation Types

Example 5.3. *For the discriminant $d = -3849267$ of the complex quadratic field $K = \mathbb{Q}(\sqrt{d})$ with 3-class group of type $(3,3)$, we constructed the four unramified cyclic cubic extensions $L_i|K$, $1 \le i \le 4$, and computed the IPAD $\tau^{(1)}(K) = \left[1^2;(54,21,1^3,21)\right]$ with the aid of MAGMA* [4].

According to Equation (20) in Theorem 3.2, the second 3-class group G of K must be of coclass $\mathrm{cc}(G) = 2$, and the polarized component 54 of the IPAD shows that $c - k = 5 + 4 = 9$ and thus the nilpotency class $c = \mathrm{cl}(G)$ and the defect of commutativity k are given by either $c = 9$, $k = 0$, or $c = 10$, $k = 1$. Further, in view of the rank-3 component 1^3 of the IPAD, G must be a vertex of the coclass tree $\mathcal{T}^2(\langle 729, 49\rangle)$.

When we tried to determine the 3-principalization type $\varkappa := \varkappa_1(3,K)$, Magma succeeded in calculating $\varkappa(1) = 3$ and $\varkappa(2) = 3$ but unfortunately failed to give $\varkappa(3)$ and $\varkappa(4)$. With respect to the complete IPAD, Theorem 4.1 enforces $\varkappa(3) = 1$ (item (1)) and $\varkappa(4) = 3$ (item (2)), and therefore the partial result $\varkappa = (3,3,*,*)$ is completed to $\varkappa = (3,3,1,3)$. According to item (3) of Remark 4.2 or item (3) of Corollary 4.1, K is of TKT H.4. Our experience suggests that this TKT compels the arrangement $c = 10$, $k = 1$, expressed by the *weak leaf conjecture* ([9], Cnj. 3.1, p. 423).

Example 5.4. *For the discriminant $d = -4928155$ of the complex quadratic field $K = \mathbb{Q}(\sqrt{d})$ with 3-class* *group of type* $(3,3)$, *we constructed the four unramified cyclic cubic extensions* $L_i|K$, $1 \le i \le 4$, *and computed the IPAD* $\tau^{(1)}(K) = \left[1^2; \left(21, 54, (21)^2\right)\right]$ *with the aid of MAGMA* [4].

According to Equation (20) in Theorem 3.2, the second 3-class group G of K must be of coclass $\mathrm{cc}(G) = 2$, and the polarized component 54 of the IPAD shows that $c - k = 5 + 4 = 9$ and thus the nilpotency class $c = \mathrm{cl}(G)$ and the defect of commutativity k are given by either $c = 9$, $k = 0$, or $c = 10$, $k = 1$. Further, due to the lack of a rank-3 component 1^3 in the IPAD, G must be a vertex of the coclass tree $\mathcal{T}^2(\langle 729, 54\rangle)$.

Next, we tried to determine the 3-principalization type $\varkappa := \varkappa_1(3, K)$. Magma succeeded in calculating two fixed points $\varkappa(1) = 1$ and $\varkappa(2) = 2$ but unfortunately failed to give $\varkappa(3)$ and $\varkappa(4)$. With respect to the complete IPAD, Theorem 4.2 enforces $\varkappa(3) = 3$ or $\varkappa(4) = 4$ (item (1)), and $\varkappa(4) = 2$ or $\varkappa(3) = 2$ (item (2)), and therefore the partial result $\varkappa = (1, 2, *, *)$ is completed to $\varkappa = (1, 2, 3, 2)$ or $\varkappa = (1, 2, 2, 4)$. According to item (1) of Remark 4.4 or item (3) of Corollary 4.2, K is of TKT E.8, and this TKT enforces the arrangement $c = 9$, $k = 0$, since $k = 1$ is impossible.

Example 5.5. *For the discriminant* $d = -65433643$ *of the complex quadratic field* $K = \mathbb{Q}\left(\sqrt{d}\right)$ *with 3-class group of type* $(3,3)$, *we constructed the four unramified cyclic cubic extensions* $L_i|K$, $1 \le i \le 4$, *and computed the IPAD* $\tau^{(1)}(K) = \left[1^2; \left(65, 1^3, (21)^2\right)\right]$ *with the aid of MAGMA* [4].

According to Equation (20) in Theorem 3.2, the second 3-class group G of K must be of coclass $\mathrm{cc}(G) = 2$, and the polarized component 65 of the IPAD shows that $c - k = 6 + 5 = 11$ and thus the nilpotency class $c = \mathrm{cl}(G)$ and the defect of commutativity k are given by either $c = 11$, $k = 0$, or $c = 12$, $k = 1$. Further, in view of the rank-3 component 1^3 of the IPAD, G must be a vertex of the coclass tree $\mathcal{T}^2(\langle 729, 49\rangle)$.

Then we tried to determine the 3-principalization type $\varkappa := \varkappa_1(3, K)$. Magma succeeded in calculating $\varkappa(1) = 4$ and $\varkappa(2) = 1$ but unfortunately failed to give $\varkappa(3)$ and $\varkappa(4)$. With respect to the complete IPAD, Theorem 4.1 enforces $\varkappa(3) = 2$ and $\varkappa(4) = 2$ (item (2)), whereas the claim in item (1) is confirmed, and therefore the partial result $\varkappa = (4, 1, *, *)$ is completed to $\varkappa = (4, 1, 2, 2)$. According to item (2) of Remark 4.2 or item (3) of Corollary 4.1, K is of TKT E.14, and this TKT enforces the arrangement $c = 11$, $k = 0$, since $k = 1$ is impossible.

6. Iterated IPADs of Second Order

In this section, where generally $p = 3$, we apply *iterated* IPADs *of second order* for indirectly computing p-capitulation types in Section 6.1 and finding the exact length of p-class towers in Section 6.2.

6.1. *p*-Capitulation Type

By means of the following theorem, the exact 3-principalization type $\varkappa$ [9] [19] [21] of *real* quadratic fields $K = \mathbb{Q}\left(\sqrt{d}\right)$, $d > 0$, can be determined indirectly with the aid of information on the structure of 3-class groups of number fields of absolute degree $6 \times 3 = 18$.

Theorem 6.1. (*Indirect computation of the* p*-capitulation type*)

Suppose that $p = 3$ and let K be a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and 3-tower group G.

1) If the IPAD of K is given by

$$\tau^{(1)}(K) = \left[1^2; \left(21, \left(1^2\right)^3\right)\right]$$

then by Equation (14) in Theorem 3.1,

$$G'' = 1, \quad G \simeq G/G'', \quad \mathrm{cc}(G) = 1, \quad \text{and} \quad G \in \{\langle 81,8\rangle, \langle 81,9\rangle, \langle 81,10\rangle\}$$

in particular, the length of the 3-class tower of K is given by $\ell_3(K) = 2$.

2) If the first layer $\mathrm{Lyr}_1(K)$ of abelian unramified extensions of K consists of $L_1, \cdots, L_4$, then the iterated IPAD of second order

$$\tau^{(2)}(K) = \left[\tau_0(K); \left(\tau_0(L_i); \tau_1(L_i)\right)_{1\le i\le 4}\right], \quad \text{with} \quad \tau_0(K) = 1^2$$

admits a sharp decision about the group G and the first layer of the transfer kernel type

$$\varkappa(K) = \left[\varkappa_0(K); \varkappa_1(K); \varkappa_2(K)\right] \quad \text{where trivially} \quad \varkappa_0(K) = 1, \varkappa_2(K) = 0$$

$$\begin{aligned} \left[\tau_0(L_1); \tau_1(L_1)\right] &= \left[21; \left(1^2, (2)^3\right)\right], \\ \left[\tau_0(L_i); \tau_1(L_i)\right] &= \left[1^2; \left(1^2, (\mathbf{2})^{\mathbf{3}}\right)\right], \quad \text{for} \quad 2 \le i \le 4, \end{aligned} \tag{25}$$

if and only if $G \simeq \langle \mathbf{81,10} \rangle$, and thus $\varkappa_1(K) = (1,0,0,0)$.

$$\begin{aligned} \left[\tau_0(L_1); \tau_1(L_1)\right] &= 2\left[1; \left(1^2, (2)^3\right)\right], \\ \left[\tau_0(L_2); \tau_1(L_2)\right] &= \left[1^2; \left(1^2\right)^4\right], \\ \left[\tau_0(L_i); \tau_1(L_i)\right] &= \left[1^2; \left(1^2, (\mathbf{2})^{\mathbf{3}}\right)\right], \quad \text{for} \quad 3 \le i \le 4, \end{aligned} \tag{26}$$

if and only if $G \simeq \langle \mathbf{81,8} \rangle$, and thus $\varkappa_1(K) = (2,0,0,0)$.

$$\begin{aligned} \left\langle \tau_0(L_1); \tau_1(L_1) \right\rangle &= \left[21; \left(1^2, (2)^3\right)\right], \\ \left\langle \tau_0(L_i); \tau_1(L_i) \right\rangle &= \left[1^2; \left(1^2\right)^4\right], \quad \text{for} \quad 2 \le i \le 4, \end{aligned} \tag{27}$$

if and only if $G \simeq \langle \mathbf{81,9} \rangle$, and thus $\varkappa_1(K) = (0,0,0,0)$.

Proof. It suffices to compute the iterated IPADs of the groups $G = \langle 81,8|9|10 \rangle$ with the aid of MAGMA [4], simply by investigating their four maximal normal subgroups $H_i \lhd G$, $1 \le i \le 4$. □

Example 6.1. *A possible future application of Theorem* 6.1 *could be, for instance, the separation of the capitulation types a*.2, $\varkappa_1(K) = (1,0,0,0)$, *and a*.3, $\varkappa_1(K) = (2,0,0,0)$, *among the* 1386 *real quadratic fields* $K = \mathbb{Q}\left(\sqrt{d}\right)$, $0 < d < 10^7$, *with* 3-*class group* $\mathrm{Cl}_3(K)$ *of type* $(3,3)$ *and IPAD* $\tau^{(1)}(K) = \left[1^2; \left(21, \left(1^2\right)^3\right)\right]$, *which was outside of our reach in all investigations of* ([23], *Tbl*. 2, *p*. 496), ([5], *Tbl*. 6.1, *p*. 451) *and* ([9], Figure 3.2, *p*. 422). *The reason why we expect this enterprise to be promising is that our experience with MAGMA* [4] *shows that computing class groups can become slow but remains sound and stable for huge discriminants* d, *whereas the calculation of capitulation kernels frequently fails*.

6.2. Length of the *p*-Class Tower

In this section, we use the iterated IPAD of second order $\tau^{(2)}(K) = \left[\tau_0(K); \left(\tau_0(L); \tau_1(L)\right)_{L \in \mathrm{Lyr}_1(K)}\right]$ for the indirect computation of the length $\ell_p(K)$ of the p-class tower of a number field K with p-tower group G, where p denotes a fixed prime.

We begin with theorems which permit a decision between finitely many possibilities for the length $\ell_p(K)$ in Section 6.2.1. These results systematically extend the investigations of complex quadratic fields in [6] to arbitrary base fields K and yield the first examples of *real quadratic* fields having a p-class tower of exact length three. Then, in Section 6.2.3, we launch the first successful attack against *quadratic* fields $K = \mathbb{Q}\left(\sqrt{d}\right)$ whose second p-class group G/G'' possesses an infinite cover ([16], Dfn.21.2), and for which the problem of finding the length $\ell_p(K)$ was completely unsolved up to now [13]. In Section 6.2.2, some prerequisites concerning periodic bifurcations are provided.

6.2.1. Second *p*-Class Groups with Finite Cover

In the following theorems, we must use *extended identifiers* [4] [24] [25] of finite 3-groups with order bigger than 3^7, which lie outside of the SmallGroups database [17] [18]. This was explained in ([16], Section 9).

Theorem 6.2. (*Length* $\ell_3(K)$ *of the* 3-*class tower for* $G/G'' \in \mathcal{T}^2(\langle 243,6 \rangle)$)

Suppose that $p=3$ and let K be a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and 3-tower group G.

1) If the IPAD of K is given by

$$\tau^{(1)}(K)=\left[1^2;\left(32,1^3,(21)^2\right)\right]$$

and the first layer TKT $\varkappa_1(K)$ neither contains a total principalization nor a 2-cycle, then there are two possibilities $\ell_3(K)\in\{2,3\}$ for the length of the 3-class tower of K.

2) If the first layer $\mathrm{Lyr}_1(K)$ of abelian unramified extensions of K consists of $L_1,\cdots,L_4$, then the iterated IPAD of second order

$$\tau^{(2)}(K)=\left[\tau_0(K);\left(\tau_0(L_i);\tau_1(L_i)\right)_{1\le i\le 4}\right],\quad \text{with}\quad \tau_0(K)=1^2$$

admits a sharp decision about the length $\ell_3(K)$.

$$\begin{aligned}
\left[\tau_0(L_1);\tau_1(L_1)\right]&=\left[32;\left(2^21,\left(31^2\right)^3\right)\right],\\
\left[\tau_0(L_2);\tau_1(L_2)\right]&=\left[1^3;\left(2^21,\left(\mathbf{1^3}\right)^{\mathbf{3}},\left(1^2\right)^9\right)\right],\\
\left[\tau_0(L_i);\tau_1(L_i)\right]&=\left[21;\left(2^21,\left(\mathbf{21}\right)^{\mathbf{3}}\right)\right],\quad \text{for}\quad 3\le i\le 4,
\end{aligned}\tag{28}$$

if and only if $G\simeq\langle\mathbf{2187,288}|\mathbf{289}|\mathbf{290}\rangle$, and thus $\ell_3(K)=2$.

$$\begin{aligned}
\left[\tau_0(L_1);\tau_1(L_1)\right]&=\left[32;\left(2^21,\left(31^2\right)^3\right)\right],\\
\left[\tau_0(L_2);\tau_1(L_2)\right]&=\left[1^3;\left(2^21,\left(\mathbf{21^2}\right)^{\mathbf{3}},\left(1^2\right)^9\right)\right],\\
\left[\tau_0(L_i);\tau_1(L_i)\right]&=\left[21;\left(2^21,\left(\mathbf{31}\right)^{\mathbf{3}}\right)\right],\quad \text{for}\quad 3\le i\le 4,
\end{aligned}\tag{29}$$

if and only if $G\simeq\langle\mathbf{729,49}\rangle-\mathbf{\#2;4}|\mathbf{5}|\mathbf{6}$, and thus $\ell_3(K)=3$.

Proof. According to Theorem 3.2, an IPAD of the form $\tau^{(1)}(K)=\left[1^2;\left(32,1^3,(21)^2\right)\right]$ indicates that the metabelianization of the group G belongs to the coclass tree $\mathcal{T}^2(\langle 243,6\rangle)$ ([9], Figure 3.6, p. 442) and has nilpotency class $3+2=5$, due to the polarization.

According to Section 4.1, the lack of a total principalization excludes the TKT c.18 and the absence of a 2-cycle discourages the TKT H.4, whence the group G must be of TKT E.6 or E.14.

By means of the techniques described in ([6], Proof of Thm. 4.1), a search in the complete descendant tree $\mathcal{T}(\langle 243,6\rangle)$, not restricted to groups of coclass 2, yields exactly six candidates for the group *G*: three metabelian groups $\langle 2187,i\rangle$ with $i\in\{288,289,290\}$, and three groups of derived length 3 and order 3^8 with generalized identifiers $\langle 729,49\rangle-\#2;i$ $(i\in\{4,5,6\})$. There cannot exist adequate groups of bigger orders ([6], Cor. 3.0.1). The former three groups are charcterized by Equations (28) the latter three groups (see [16], 21.2, Figure 8) by Equations (29).

Finally, we have $\ell_3(K)=\mathrm{dl}(G)$. □

Theorem 6.3. (*Length* $\ell_p(K)$ *of the* 3*-class tower for* $G/G''\in\mathcal{T}^2(\langle 243,8\rangle)$)

Suppose that $p=3$ and let K be a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and 3-tower group G.

1) If the IPAD of K is given by

$$\tau^{(1)}(K)=\left[1^2;\left(32,(21)^3\right)\right]$$

and the first layer TKT $\varkappa_1(K)$ neither contains a total principalization nor a 2-cycle, then there are two possibilities $\ell_3(K)\in\{2,3\}$ for the length of the 3-class tower of K.

2) If the first layer $\mathrm{Lyr}_1(K)$ of abelian unramified extensions of K consists of $L_1,\cdots,L_4$, then the iterated IPAD of second order

$$\tau^{(2)}(K) = \left[\tau_0(K); \left(\tau_0(L_i); \tau_1(L_i)\right)_{1\le i\le 4}\right], \quad \text{with} \quad \tau_0(K) = 1^2$$

admits a sharp decision about the length $\ell_3(K)$.

$$\begin{aligned} \left[\tau_0(L_1); \tau_1(L_1)\right] &= \left[32; \left(2^2 1, \left(31^2\right)^3\right)\right], \\ \left[\tau_0(L_i); \tau_1(L_i)\right] &= \left[21; \left(2^2 1, (\mathbf{21})^{\mathbf{3}}\right)\right], \quad \text{for} \quad 2 \le i \le 4, \end{aligned} \tag{30}$$

if and only if $G \simeq \langle \mathbf{2187, 302} | \mathbf{304} | \mathbf{306} \rangle$, and thus $\ell_3(K) = 2$.

$$\begin{aligned} \left[\tau_0(L_1); \tau_1(L_1)\right] &= \left[32; \left(2^2 1, \left(31^2\right)^3\right)\right], \\ \left[\tau_0(L_i); \tau_1(L_i)\right] &= \left[21; \left(2^2 1, (\mathbf{31})^{\mathbf{3}}\right)\right], \quad \text{for} \quad 2 \le i \le 4, \end{aligned} \tag{31}$$

if and only if $G \simeq \langle \mathbf{729, 54} \rangle - \mathbf{\#2; 2} | \mathbf{4} | \mathbf{6}$, and thus $\ell_3(K) = 3$.

Proof. According to Theorem 3.2, an IPAD of the form $\tau^{(1)}(K) = \left[1^2; \left(32, (21)^3\right)\right]$ indicates that the metabelianization of the group G belongs to the coclass tree $\mathcal{T}^2(\langle 243, 8\rangle)$ ([9], Figure 3.7, p. 443) and has nilpotency class $3 + 2 = 5$, due to the polarization.

According to Section 4.2, the lack of a total principalization excludes the TKT c.21 and the absence of a 2-cycle discourages the TKT G.16, whence the group G must be of TKT E.8 or E.9.

As we have shown in detail in [6], Proof of Thm. 4.1], a search in the complete descendant tree $\mathcal{T}^2(\langle 243, 8\rangle)$, not restricted to groups of coclass 2, yields exactly six candidates for the group G: three metabelian groups $\langle 2187, i\rangle$ with $i \in \{302, 304, 306\}$, and three groups of derived length 3 and order 3^8 with generalized identifiers $\langle 729, 54\rangle - \#2; i$ $\left(i \in \{2, 4, 6\}\right)$. There cannot exist adequate groups of bigger orders ([6], Cor. 3.0.1). The former three groups are characterized by Equations (30) the latter three groups (see [16], Section 21.2, Figure 9) by Equations (31).

Eventually, the 3-tower length of K, $\ell_3(K) = \mathrm{dl}(G)$, coincides with the derived length of G. □

Example 6.2. (*The **first real quadratic** field* K *with* $\ell_3(K) = 3$.) *In June* 2006, *we discovered the smallest discriminant* $D = 342664$ *of a real quadratic field* $K = \mathbb{Q}\left(\sqrt{D}\right)$ *with* 3-*class group of type* $(3,3)$ *whose* 3-*tower group* G *possesses the transfer kernel type* E.9, $\varkappa = (2,3,3,4)$ ([19], *Tbl.* 4, *p.* 498).

The complex quadratic analogue $k = \mathbb{Q}\left(\sqrt{-9748}\right)$ was known since 1934 by the famous paper of Scholz and Taussky [20]. However, it required almost 80 years until M.R. Bush and ourselves ([6], Cor. 4.1.1) succeeded in providing the first faultless proof that k has a 3-class tower of exact length $\ell_3(k) = 3$ with 3-tower group G isomorphic to one of the two Schur σ-groups $\langle 729, 54\rangle - \#2; i$ $\left(i \in \{2, 6\}\right)$ of order 3^8.

For $K = \mathbb{Q}\left(\sqrt{342664}\right)$, the methods in [6] do not admit a final decision about the length $\ell_3(K)$. They only yield four possible 3-tower groups of *K*, namely either the two unbalanced groups $\langle 2187, i\rangle$ with $i \in \{302, 306\}$ and relation rank $r = 3$ bigger than the generator rank $d = 2$ or the two Schur σ-groups $\langle 729, 54\rangle - \#2; i$ with $i \in \{2, 6\}$ and $r = 2$ equal to $d = 2$.

In October 2014, we succeeded in proving that three of the unramified cyclic cubic extensions $L_i | K$ reveal the critical IPAD component $\tau_1(L_i) = \left(2^2 1, (\mathbf{31})^{\mathbf{3}}\right)$ in Equation (31) of Theorem 6.3, item (2), whence also $G \simeq \langle 729, 54\rangle - \#2; 2 | 6$ and $\ell_3(K) = 3$. This was done by computing 3-class groups of number fields of absolute degree 6 × 3 = 18 with the aid of MAGMA [4].

6.2.2. Another Descendant Tree with Periodic Bifurcations

In ([16], Section 21.2), we provided computational evidence of periodic bifurcations in pruned descendant trees with roots $\langle 243, 6 | 8\rangle$. Now we want to show that the immediate descendant $\langle 729, 45\rangle$ of another root $\langle 243, 4\rangle$ possesses an infinite balanced cover ([16], Dfn. 21.2). For this purpose we construct an extensive finite part of the descendant tree $\mathcal{T}(\langle 243, 4\rangle)$ with the aid of MAGMA [4] and we combine the gained insight with results of L. Bartholdi and M.R. Bush [13]. The group $\langle 729, 45\rangle$ has been called the *non-CF group* N

by J. Ascione, G. Havas and C.R. Leedham-Green [26] [27].

For brevity, we give 3-logarithms of abelian type invariants in the following theorem and we denote iteration by formal exponents, for instance, $1^3 := (1,1,1) \hat{=} (3,3,3)$, $(2,1) \hat{=} (9,3)$, $(1^3)^3 := (1^3,1^3,1^3)$, and we eliminate initial anomalies of generalized identifiers by putting $\langle 243,4\rangle(-\#1;1-\#2;1)^j-\#1;1 := \langle 243,4\rangle(-\#1;1-\#2;1)^j-\#1;2$, for $0 \le j \le 2$, formally. (Observe that actually $\langle 729,45\rangle = \langle 243,4\rangle - \#1;2$, whereas $\langle 243,4\rangle - \#1;1 = \langle 729,44\rangle$.)

Theorem 6.4. *Let* ℓ *be a positive integer bounded from above by* 8.

1) In the descendant tree $\mathcal{T}(G)$ of $G = \langle 243,4\rangle$, there exists a unique path with length 2ℓ of σ-groups ([16], Dfn. 10.1),

$$G = \delta^0(G) \leftarrow \delta^1(G) \leftarrow \cdots \leftarrow \delta^{2\ell}(G)$$

and (reverse) directed edges with alternating step sizes 1 and 2 such that $\delta^i(G) = \pi(\delta^{i+1}(G))$, for all $0 \le i \le 2\ell-1$ (δ is a section of the surjection π on this path), and all the vertices with even superscript $i = 2j$, $j \ge 0$,

$$\delta^{2j}(G) = G(-\#1;1-\#2;1)^j \tag{32}$$

resp. all the vertices with odd superscript $i = 2j+1$ $j \ge 0$,

$$\delta^{2j+1}(G) = G(-\#1;1-\#2;1)^j - \#1;1 \tag{33}$$

of this path share the following common invariants, respectively:

-the uniform IPAD

$$\tau^{(1)}(\delta^i(G)) = \left[1^2;\left((1^3)^3,21\right)\right] \tag{34}$$

-the uniform transfer kernel type

$$\varkappa(\delta^i(G)) = \left[1;(4,1,1,1);0\right] \tag{35}$$

-the 3-multiplicator rank and the nuclear rank [14] [16],

$$\mu(\delta^{2j}(G)) = 3, \quad \nu(\delta^{2j}(G)) = 1 \tag{36}$$

resp., giving rise to the bifurcations for odd $i = 2j+1$,

$$\mu(\delta^{2j+1}(G)) = 4, \quad \nu(\delta^{2j+1}(G)) = 2 \tag{37}$$

-and the counters of immediate descendants [16] [18],

$$N_1(\delta^{2j}(G)) = 4, \quad C_1(\delta^{2j}(G)) = 4 \tag{38}$$

resp.

$$N_1(\delta^{2j+1}(G)) = 4, \quad C_1(\delta^{2j+1}(G)) = 0, \quad N_2(\delta^{2j+1}(G)) = 2, \quad C_2(\delta^{2j+1}(G)) = 1 \tag{39}$$

determining the local structure of the descendant tree.

2) A few other invariants of the vertices $\delta^i(G)$ depend on the superscript i,

-the 3-logarithm of the order, the nilpotency class and the coclass,

$$\log_3(\mathrm{ord}(\delta^{2j}(G))) = 3j+5, \quad \mathrm{cl}(\delta^{2j}(G)) = 2j+3, \quad \mathrm{cc}(\delta^{2j}(G)) = j+2 \tag{40}$$

resp.

$$\log_3(\mathrm{ord}(\delta^{2j+1}(G))) = 3j+6, \quad \mathrm{cl}(\delta^{2j+1}(G)) = 2j+4, \quad \mathrm{cc}(\delta^{2j+1}(G)) = j+2 \tag{41}$$

Theorem 6.4 provides the scaffold of the pruned descendant tree $\mathcal{T}_*(G)$ of $G = \langle 243,4\rangle$ with the distinguished path and periodic bifurcations.

With respect to our intended applications, however, the following Corollaries 6.1 and 6.2 are of the greatest importance.

Corollary 6.1. *Let* $0 \le i \le 2\ell$ *be a non-negative integer, where* $\ell \le 8$.

Whereas the vertices with even superscript $i = 2j$, $j \ge 0$, that is, $\delta^{2j}(G) = G(-\#1;1-\#2;1)^j$, are merely links in the distinguished path, the vertices with odd superscript $i = 2j+1$ $j \ge 0$, that is, $\delta^{2j+1}(G) = G(-\#1;1-\#2;1)^j - \#1;1$, reveal the essential periodic bifurcations with the following properties.

1) The regular component $\mathcal{T}^{j+2}\left(\delta^{2j+1}(G)\right)$ of the descendant tree $\mathcal{T}\left(\delta^{2j+1}(G)\right)$ is a finite tree which contains the root and four terminal σ-groups,

$$T_{j,k} := \delta^{2j+1}(G) - \#1;k \quad \text{with } 1 \le k \le 4$$

2) The irregular component $\mathcal{T}^{j+3}\left(\delta^{2j+1}(G)\right)$ of the descendant tree $\mathcal{T}\left(\delta^{2j+1}(G)\right)$ is a forest which contains a single isolated Schur σ-group

$$S_j := \delta^{2j+1}(G) - \#2;2$$

and additionally contains the next vertex of the distinguished path $\delta^{2j+1}(G) - \#2;1 = \delta^{2(j+1)}(G)$.

Remark 6.1. *We apply a sifting strategy for reducing the entire descendant tree* $\mathcal{T}(G)$ *to a pruned descendant tree* $\mathcal{T}_*(G)$ *by filtering all vertices which are* σ*-groups. The process consists of*

1) keeping the unique terminal step size-2 descendant, which is exactly the Schur σ-group S_j, the unique capable step size-2 descendant, and the 4 terminal step size-1 descendants $T_{j,k}$, $1 \le k \le 4$, which are σ-groups, of $\delta^{2j+1}(G)$, $j \ge 0$, and

2) eliminating ([16], Section 20, (F1)) the 3 non-σ groups, together with their complete descendant trees, among the 4 capable step size-1 descendants of $\delta^{2j}(G)$, $j \ge 0$.

A finite part at the top of the resulting tree $\mathcal{T}_*(G)$ is drawn in **Figure 1**.

Proof. (of Theorem 6.4, and Corollary 6.1)

The p-group generation algorithm [28] [29], which is implemented in the computational algebra system MAGMA [2]-[4], was used for constructing the pruned descendant tree $\mathcal{T}_*(G)$ with root $G = \langle 243, 4 \rangle$ which was defined as the disjoint union of all regular components, *i.e.* finite trees

$$\mathcal{T}^{j+2}\left(\delta^{2j+1}(G)\right) = \left\{\delta^{2j+1}(G), T_{j,1}, \cdots, T_{j,4}\right\}$$

with the descendants $\delta^{2j+1}(G) = G(-\#1;1-\#2;1)^j - \#1;1$, $0 \le j \le \ell$, of G as roots, together with 2 siblings in the irregular component

$$\mathcal{T}^{j+3}\left(\delta^{2j+1}(G)\right) = \left\{S_j, \delta^{2j+1}(G) - \#2;1\right\}$$

one of them the Schur σ-group S_j with 3-multiplicator rank $\mu = 2$ and nuclear rank $\nu = 0$, the other $\delta^{2j+1}(G) - \#2;1 = \delta^{2(j+1)}(G)$ the next vertex of the distinguished path.

The vertical construction was terminated for $j = \ell = 8$ at nilpotency class 2 × 8 + 3 = 19. The horizontal construction was extended up to coclass $8 + 2 = 10$, where the consumption of CPU time became annoying. □

The extent to which we constructed the pruned descendant tree $\mathcal{T}_*(\langle 243, 4 \rangle)$ suggests the following conjecture.

Conjecture 6.1. *Theorem* 6.4 *and Corollary* 6.1 *remain true for an arbitrarily large positive integer* ℓ, *not necessarily bounded by* 8.

For the convenience of the reader, we now recall ([16], Dfn. 21.2):

Definition 6.1. *Let* P *be a finite metabelian* p*-group, then the set of all finite non-metabelian* p*-groups* D *whose second derived quotient* D/D'' *is isomorphic to* P *is called the cover* $\operatorname{cov}(P)$ *of* P. *The subset* $\operatorname{cov}_*(P)$ *consisting of Schur* σ*-groups in* $\operatorname{cov}(P)$ *is called the balanced cover of* P.

Corollary 6.2. *The group* $\langle 729, 45 \rangle$, *which does not have a balanced presentation, has an infinite cover and even an infinite balanced cover. More precisely, the covers are given explicitly by*

$$\begin{aligned} &\operatorname{cov}(\langle 729, 45 \rangle) = \mathcal{T}_*(\langle 729, 45 \rangle) \setminus \{\langle 729, 45 \rangle\}, \\ &\operatorname{cov}_*(\langle 729, 45 \rangle) = \left\{ S_j \,\middle|\, j \ge 0 \right\}. \end{aligned} \tag{42}$$

For the proof, we have to recall that L. Bartholdi and M.R. Bush ([13], Thm. 2.1, p. 160, and Prop. 3.6, p. 165) have constructed an *infinite family* $(G_n)_{n \ge 1}$ of Schur σ-groups with strictly increasing 3-power order, nilpotency class, coclass, and increasing *unbounded derived length*

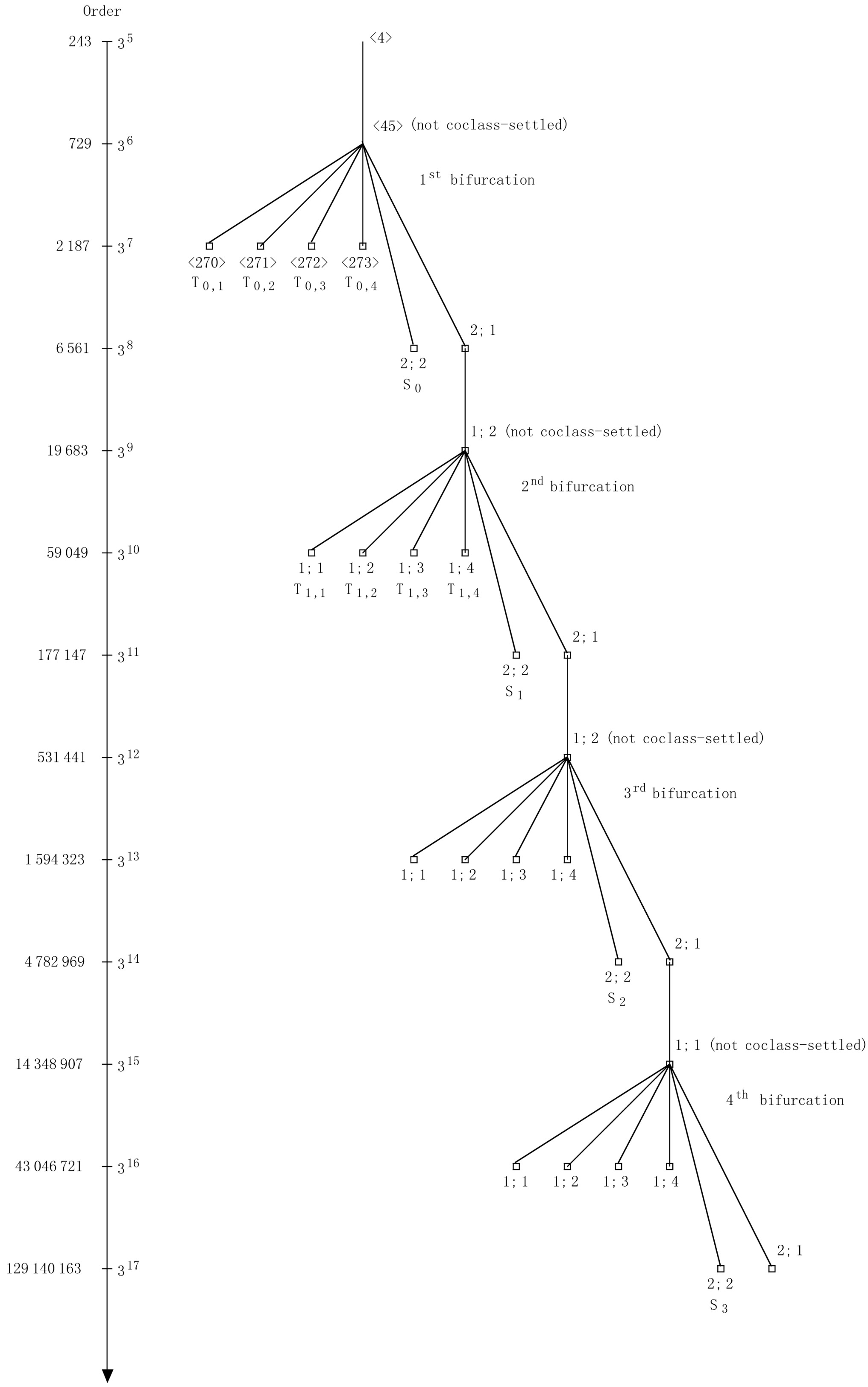

Figure 1. Periodic bifurcations in the pruned descendant tree $\mathcal{T}_*(\langle 243,4\rangle)$.

$$\log_3(\mathrm{ord}(G_n)) = 3n+2,\quad \mathrm{cl}(G_n) = 2n+1,\quad \mathrm{cc}(G_n) = n+1,\quad \mathrm{dl}(G_n) = \lfloor \log_2(3n+3) \rfloor \tag{43}$$

such that $G_1 \simeq \langle 243,5 \rangle$ is the well-known metabelian two-stage tower group $G_3^2(K)$ of the complex quadratic field $K = \mathbb{Q}(\sqrt{-4027})$ ([9], Thm. 1.5, p. 407), ([20], pp. 22-25), ([30], p. 20) with IPAD $\tau^{(1)}(G_1) = [1^2;1^3,(21)^3]$. All the other members G_n of the family with $n \geq 2$ are non-metabelian and share the common IPAD $\tau^{(1)}(G_n) = [1^2;(1^3)^3,21]$.

Proof. Several issues must be clarified for the groups G_n with $n \geq 2$.

Firstly, according to the Equations (15) and (16) in Theorem 3.1, there are only five possibilities for the metabelianizations G_n/G_n'', namely the groups $\langle 243,4 \rangle$ and $\langle 729,44|45|46|47 \rangle$. However, we can eliminate $\langle 243,4 \rangle$, because it has an empty cover by ([9], Thm. 1.4, p. 406). We can also cancel $\langle 729,44|46|47 \rangle$ by ([6], Lem. 3.3), since they fail to be σ-groups. In fact, the same arguments as for the G_n $(n \geq 2)$ apply to all vertices of the pruned tree $\mathcal{T}_*(\langle 729,45 \rangle) \setminus \{\langle 729,45 \rangle\}$, since they are σ-groups, by definition. This shows that

$$\mathrm{cov}_*(\langle 729,45 \rangle) \supseteq \{G_n \mid n \geq 2\} \quad \text{and} \quad \mathrm{cov}(\langle 729,45 \rangle) \supseteq \mathcal{T}_*(\langle 729,45 \rangle) \setminus \{\langle 729,45 \rangle\} \tag{44}$$

Secondly, we have to make sure that the G_n $(n \geq 2)$, and more generally all finite 3-groups G with IPAD $[1^2;(1^3)^3,21]$, are descendants of $\langle 243,4 \rangle$. Since the abelian 3-group with type invariants 21 is not a quotient of 1^3, ([6], Cor. 3.0.1) shows that such a group G cannot be descendant of any other sibling $\langle 243,i \rangle$ with $i \in \{3,5,6,7,8,9\}$ of $\langle 243,4 \rangle$, which contains at least two maximal subgroups of type 21. According to Equation (14) in Theorem 3.1, the unique remaining possibility for an ancestor having the required IPAD $[1^2;(1^2)^4]$ is the extra special group $\langle 27,3 \rangle$. However, this group has been exhausted completely already, since it gives rise to exactly the seven immediate step size-2 descendants $\langle 243,i \rangle$ with $3 \leq i \leq 9$, mentioned above.

Finally, the Schur σ-groups G_n $(n \geq 2)$ must necessarily coincide with the periodic sequence S_j $(j \geq 0)$ of Schur σ-groups, whose initial sections were constructed in Corollary 6.1, and which must be continued inductively in the sense of Conjecture 6.1:

$$G_n \simeq S_{n-2} \quad \text{for all } n \geq 2 \tag{45}$$

Indeed, since each S_j is a sibling of $\delta^{2(j+1)}$, for $j \geq 0$, having identical invariants, Equation (40) in Theorem 6.4 shows that all invariants of S_j coincide with those of G_{j+2}

$$\begin{aligned}
\log_3(\mathrm{ord}(S_j)) &= 3(j+1)+5 = 3(j+2)+2 = \log_3(\mathrm{ord}(G_{j+2})),\\
\mathrm{cl}(S_j) &= 2(j+1)+3 = 2(j+2)+1 = \mathrm{cl}(G_{j+2}),\\
\mathrm{cc}(S_j) &= (j+1)+2 = (j+2)+1 = \mathrm{cc}(G_{j+2}).
\end{aligned} \tag{46}$$

Consequently, the inclusions in Equation (44) can be replaced by equalities, and the claims of Corollary 6.2 are proved. □

Remark 6.2.

1) The claims of Conjecture 6.1 are strongly supported by the proven infinitude of the family $(G_n)_{n\geq 2}$ of Schur σ-groups in [13] and their obvious coincidence with the periodic sequence $(S_j)_{j\geq 0}$ in Corollary 6.1.

2) The IPAD $\tau^{(1)} = (1^2;(1^3)^3,21)$ is a considerably more restrictive condition for a pro-3 group than the TKT H.4, $\varkappa = [1;(4,1,1,1);0]$, which is necessarily the TKT of all descendants of $\langle 243,4 \rangle$, by ([6], Cor.3.0.2). However, the TKT H.4 is unable to characterize the vertices of the tree of $\langle 243,4 \rangle$ uniquely, since there are lots of vertices with TKT H.4 in the tree of $\langle 243,3 \rangle$, and, as we have seen in Remark 4.2, the same is true for the tree of $\langle 243,6 \rangle$, since $(2,1,2,2)$ and $(4,1,1,1)$ are equivalent ([21], Section 3, p. 79).

3) The exact specification of the infinite cover, resp. the infinite and entirely non-metabelian balanced cover, of $\langle 729,45 \rangle$ in Corollary 6.2 implies that the length $\ell_3(K)$ of the 3-class tower of a real, resp. complex,

quadratic field K with IPAD $\tau^{(1)}(K)=\left[1^2;\left((1^3)^3,21\right)\right]$ can take any value bigger than 1, resp. 2, or even ∞. In the complex case, the 3-tower group must be a Schur σ-group, according to [6] [10] [31] [32].

6.2.3. Second *p*-Class Groups with Infinite Cover

As a final coronation of this section, we show that our new IPAD strategies are powerful enough to enable the determination of the length $\ell_3(K)$ for some quadratic fields $K=\mathbb{Q}\left(\sqrt{d}\right)$ with IPAD $\tau^{(1)}(K)=\left[1^2;\left((1^3)^3,21\right)\right]$ with the aid of information on the structure of 3-class groups of number fields of absolute degree $6\cdot 9=54$. This is close to the limits of what can currently be done with MAGMA [4] on powerful machines.

For this purpose, we extend the concept of iterated IPADs of second order

$$\tau^{(2)}(K)=\left[\tau_0(K);\left(\tau^{(1)}(L)\right)_{L\in\mathrm{Lyr}_1(K)}\right]=\left[\tau_0(K);\left(\tau_0(L);\tau_1(L)\right)_{L\in\mathrm{Lyr}_1(K)}\right]$$

once more by adding the second layers $\tau_2(L)$ to all IPADs $\tau^{(1)}(L)$ of unramified degree-p extensions $L|K$. The resulting *iterated multi-layered IPAD of second order* is indicated by an asterisk

$$\tau_*^{(2)}(K)=\left[\tau_0(K);\left(\tau_0(L);\tau_1(L);\tau_2(L)\right)_{L\in\mathrm{Lyr}_1(K)}\right]$$

Theorem 6.5. (*Length* $\ell_p(K)$ *of the* 3*-class tower for* $G/G''\in\mathcal{T}\left(\langle 243,4\rangle\right)$)
Suppose that $p=3$ and let $K=\mathbb{Q}\left(\sqrt{d}\right)$ be a quadratic number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and 3-tower group G.

1) If the IPAD of K is given by

$$\tau^{(1)}(K)=\left[1^2;\left((1^3)^3,21\right)\right]$$

then the first layer TKT is $\varkappa_1(K)=(4,1,1,1)$ and there exist infinitely many possibilities $\ell_3(K)\geq 2$ for the length of the 3-class tower of K.

2) If the first layer $\mathrm{Lyr}_1(K)$ of abelian unramified extensions of K consists of $L_1,\cdots,L_4$, then the iterated multi-layered IPAD of second order

$$\tau_*^{(2)}(K)=\left[\tau_0(K);\left(\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right)_{1\leq i\leq 4}\right],\quad\text{with}\quad \tau_0(K)=1^2$$

admits certain partial decisions about the length $\ell_3(K)$ as follows.

$$\begin{aligned}
&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left((1^3)^4,(1^2)^9\right);(1^2)^{13}\right],\\
&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(1^3,(21)^3,(1^2)^9\right);\left((1^2)^4,(2)^9\right)\right],\quad\text{for}\quad 2\leq i\leq 3,\\
&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(1^3,(21)^3\right)\left(1^2\right)^4\right].
\end{aligned}\tag{47}$$

if and only if $G\simeq\langle\mathbf{243,4}\rangle$, and thus $\ell_3(K)=2$.

$$\begin{aligned}
&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(\mathbf{21^2},(1^3)^3,(1^2)^9\right);\left(1^3,(21)^3,(1^2)^9\right)\right],\\
&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(\mathbf{21^2},\mathbf{(21)^{12}}\right);\left(21^2,(21)^{12}\right)\right],\quad\text{for}\quad 2\leq i\leq 3,\\
&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(\mathbf{21^2},(21)^3\right);\left(21^2,(2^2)^3\right)\right].
\end{aligned}\tag{48}$$

if and only if $G\simeq\langle\mathbf{729,45}\rangle$, and thus $\ell_3(K)=2$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(\left(\mathbf{21^2}\right)^{\mathbf{4}},\left(1^2\right)^9\right);\left(21^2,\left(1^3\right)^3,(21)^9\right)\right],\\&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(21^2,(21)^{12}\right);\left(21^2,(21)^{12}\right)\right],\quad\text{for}\quad 2\le i\le 3,\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(21)^3\right);\left(21^2,\left(2^2\right)^3\right)\right].\end{aligned}\tag{49}$$

if and only if $G\simeq T_{0,1}=\langle\mathbf{2187,270}\rangle$, and thus $\ell_3(K)=3$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(21^2,\left(1^3\right)^3,\left(1^2\right)^9\right);\left(21^2,\left(2^2\right)^3,\left(1^2\right)^9\right)\right],\\&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(21^2,(21)^{12}\right);\left(21^2,(21)^{12}\right)\right],\quad\text{for}\quad 2\le i\le 3,\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(\mathbf{31})^{\mathbf{3}}\right);\left(21^2,(21)^3\right)\right].\end{aligned}\tag{50}$$

if and only if either $G\simeq T_{0,2}=\langle\mathbf{2187,271}\rangle$ or $G\simeq T_{0,3}=\langle\mathbf{2187,272}\rangle$, and thus $\ell_3(K)=3$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(21^2,\left(1^3\right)^3,\left(1^2\right)^9\right);\left(21^2,(21)^3,\left(1^2\right)^9\right)\right],\\&\left[\tau_0(L_2);\tau_1(L_2);\tau_2(L_2)\right]=\left[1^3;\left(21^2,(21)^{12}\right);\left(21^2,(21)^{12}\right)\right],\\&\left[\tau_0(L_3);\tau_1(L_3);\tau_2(L_3)\right]=\left[1^3;\left(\left(\mathbf{21^2}\right)^{\mathbf{4}},\left(\mathbf{2^2}\right)^{\mathbf{9}}\right);\left(21^2\right)^{13}\right],\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(21)^3\right);\left(21^2,(21)^3\right)\right].\end{aligned}\tag{51}$$

if and only if $G\simeq T_{0,4}=\langle\mathbf{2187,273}\rangle$, and thus $\ell_3(K)=3$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(\left(\mathbf{21^2}\right)^{\mathbf{4}},\left(1^2\right)^9\right);\left(2^21,\left(1^3\right)^3,\left(\mathbf{2^2}\right)^{\mathbf{3}},(21)^6\right)\right],\\&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(\left(\mathbf{21^2}\right)^{\mathbf{4}},\left(\mathbf{2^2}\right)^{\mathbf{9}}\right);\left(2^21,\left(\mathbf{21^2}\right)^{\mathbf{12}}\right)\right],\quad\text{for}\quad 2\le i\le 3,\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(\mathbf{31})^{\mathbf{3}}\right);\left(2^21,\left(\mathbf{2^2}\right)^{\mathbf{3}}\right)\right].\end{aligned}\tag{52}$$

if and only if either $G\simeq S_0=\langle\mathbf{729,45}\rangle-\mathbf{\#2;2}$ or $G\simeq\langle\mathbf{729,45}\rangle-\mathbf{\#2;1}$, both of order 3^8, and thus $\ell_3(K)=3$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(\left(21^2\right)^4,\left(1^2\right)^9\right);\left(2^21,\left(1^3\right)^3,(\mathbf{32})^{\mathbf{3}},(21)^6\right)\right],\\&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(\left(21^2\right)^4,\left(2^2\right)^9\right);\left(\mathbf{2^21},\left(\mathbf{31^2}\right)^{\mathbf{3}},\left(21^2\right)^9\right)\right],\quad\text{for}\quad 2\le i\le 3,\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(31)^3\right);\left(2^21,(\mathbf{32})^{\mathbf{3}}\right)\right].\end{aligned}\tag{53}$$

occurs for $G\simeq\langle\mathbf{729,45}\rangle-\mathbf{\#2;1-\#1;2}$ of order 3^9, where $\ell_3(K)=3$.

$$\begin{aligned}&\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right]=\left[1^3;\left(\left(21^2\right)^4,\left(1^2\right)^9\right);\left(2^21,\left(1^3\right)^3,(\mathbf{32})^{\mathbf{3}},(21)^6\right)\right],\\&\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right]=\left[1^3;\left(\left(21^2\right)^4,\left(2^2\right)^9\right);\mathbf{2^21},\left(\mathbf{31^2}\right)^{\mathbf{3}},\left(21^2\right)^9\right],\quad\text{for}\quad 2\le i\le 3,\\&\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right]=\left[21;\left(21^2,(31)^3\right);\left(2^21,(\mathbf{32})^{\mathbf{3}}\right)\right].\end{aligned}\tag{54}$$

occurs for either $G\simeq T_{1,1}=\langle\mathbf{729,45}\rangle-\mathbf{\#2;1-\#1;2-\#1;1}$
or $G\simeq T_{1,2}=\langle\mathbf{729,45}\rangle-\mathbf{\#2;1-\#1;2-\#1;2}$
or $G\simeq T_{1,3}=\langle\mathbf{729,45}\rangle-\mathbf{\#2;1-\#1;2-\#1;3}$, all of order 3^{10}, where $\ell_3(K)=3$.

$$\begin{aligned}
\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right] &= \left[1^3;\left((21^2)^4,(1^2)^9\right);\left(2^21,(1^3)^3,(\mathbf{32})^3,(21)^6\right)\right],\\
\left[\tau_0(L_2);\tau_1(L_2);\tau_2(L_2)\right] &= \left[1^3;\left((21^2)^4,(2^2)^9\right);\mathbf{2^21},(\mathbf{31^2})^{\mathbf{3}},(21^2)^9\right],\\
\left[\tau_0(L_3);\tau_1(L_3);\tau_2(L_3)\right] &= \left[1^3;\left((21^2)^4,(2^2)^9\right);(\mathbf{2^21})^{\mathbf{4}},(\mathbf{31^2})^{\mathbf{9}}\right],\\
\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right] &= \left[21;\left(21^2,(31)^3\right);\left(2^21,(\mathbf{32})^3\right)\right].
\end{aligned} \tag{55}$$

occurs for $G \simeq T_{1,4} = \langle \mathbf{729,45}\rangle - \mathbf{\#2;1-\#1;2-\#1;4}$ of order 3^{10}, where $\ell_3(K)=3$.

$$\begin{aligned}
\left[\tau_0(L_1);\tau_1(L_1);\tau_2(L_1)\right] &= \left[1^3;\left((21^2)^4,(1^2)^9\right);\left(2^21,(1^3)^3,(\mathbf{32})^3,(21)^6\right)\right],\\
\left[\tau_0(L_i);\tau_1(L_i);\tau_2(L_i)\right] &= \left[1^3;\left((21^2)^4,(2^2)^9\right);\left((\mathbf{2^21})^{\mathbf{4}},(\mathbf{31^2})^{\mathbf{9}}\right)\right], \quad \text{for} \quad 2\le i\le 3,\\
\left[\tau_0(L_4);\tau_1(L_4);\tau_2(L_4)\right] &= \left[21;\left(21^2,(31)^3\right);\left(2^21,(\mathbf{32})^3\right)\right].
\end{aligned} \tag{56}$$

occurs for either

$G \simeq \langle \mathbf{729,45}\rangle(\mathbf{-\#2;1-\#1;2})^{\mathbf{j}} \mathbf{-\#2;1|2}$, $1\le j\le 2$, of order 3^{8+3j}, where $\ell_3(K)=3$,

or $G \simeq \langle \mathbf{729,45}\rangle(\mathbf{-\#2;1-\#1;2})^{\mathbf{3}} \mathbf{-\#2;1|2}$ of order 3^{17}, where $\ell_3(K)=4$.

The various groups which occur in Theorem 6.5 are drawn in the tree diagram of **Figure 1**. This diagram impressively visualizes periodically repeating bifurcations in the descendant tree $\mathcal{T}_*(N)$ of Ascione's non-CF group $N \simeq \langle 729,45\rangle$. An essential difference to the trees $\mathcal{T}_*(Q)$ and $\mathcal{T}_*(U)$ in ([16], Section 21.2), where $Q \simeq \langle 729,49\rangle$ and $U \simeq \langle 729,54\rangle$, is that the regular component $\mathcal{T}_*^r(V)$ of each vertex V with coclass $\mathrm{cc}(V)=r$ and nuclear rank $\nu(V)=2$, giving rise to a bifurcation, is not a coclass tree but rather a finite tree with root and four terminal immediate descendants. The irregular component $\mathcal{T}_*^{r+1}(V)$ is a forest containing an isolated Schur σ-group and the parent of the next vertex with bifurcation (together with its regular component).

Proof. If we restricted the statements to complex quadratic fields it would be sufficient to consider the Schur σ-groups S_j, $j\ge 0$, in the descendant tree $\mathcal{T}(\langle 243,4\rangle)$. But since we are also interested in the behaviour of real quadratic fields, we must determine the iterated multi-layered IPADs $\tau_*^{(2)}(V)$ of second order of all vertices $V\in\mathcal{T}_*(\langle 243,4\rangle)$ in the pruned descendant tree consisting of σ-groups. We do this for the groups up to the order 3^{17}, which were constructed in Section 6.2.2, by investigating their four maximal normal subgroups $H_i \lhd V$, $1\le i\le 4$, with the aid of MAGMA [4]. Finally, we make use of the relation $\ell_3(K)=\mathrm{dl}(G)$. □

We point out that Equation (56) cannot be used for a conclusive identification.

Example 6.3. (*The **first quadratic fields with TKT H.4** and* $\ell_3(K)=3$) *In December* 2009, *we discovered the smallest discriminant* $d=957013$ *of a real quadratic field* $K=\mathbb{Q}(\sqrt{d})$ *with* 3-*class group of type* $(3,3)$ *whose* 3-*tower group* G *possesses the transfer kernel type H*.4, $\varkappa=(4,1,1,1)$ ([19], *Tbl*. 4, *p*. 498). *In January* 2010, *we found two further examples* $d\in\{1571953,1734184\}$.

The complex quadratic analogue $d=-3896$ was known since 1982 by the paper of Heider and Schmithals ([30], p. 20), whereas Scholz and Taussky gave an erroneous TKT F.7, $\varkappa=(2,1,1,2)$ ([20], Footnote, p. 36). There is another example $d=-6583$ in ([30], Tbl. 3, p. 19) and in November 1989, we found $d\in\{-23428,-25447\}$ ([21], p. 84). All these fields share the same IPAD $\tau^{(1)}(K)=\left[1^2;\left((1^3)^3,21\right)\right]$.

In February 2015, we succeeded in proving that the unramified cyclic cubic extensions $L_i|K$, for $d\in\{-3896,-25447\}$, resp. $d=-23428$, resp. $d\in\{957013,1571953,1734184\}$, reveal the critical first and second layer IPAD components in Equation (52), resp. (56), resp. (51), of Theorem 6.5, item (2), whence certainly $\ell_3(K)=3$, for all these fields, with the single exception of $K=\mathbb{Q}(\sqrt{-23428})$, where also $\ell_3(K)\ge 4$ could be possible.

However, the 3-class tower groups G of K are different: for $d\in\{-3896,-25447\}$ we have the Schur σ-group $G\simeq\langle 729,45\rangle-\#2;2$ of order 3^8, and for $d\in\{957013,1571953,1734184\}$ we have the unbalanced group $G \simeq \langle 2187, 273 \rangle$, which is distinguished by possessing three pairwise non-isomorphic maximal subgroups of type 1^3. For $d = -23428$, the group remains unknown.

The justifications were conducted by computing 3-class groups of number fields of absolute degree 6 × 9 = 54 with the aid of MAGMA [4] on a machine with two XEON 8-core CPUs and 256 GB RAM.

7. Complex Quadratic Fields of 3-Rank Three

In this concluding section we present another intriguing application of IPADs.

Due to Koch and Venkov [32], it is known that a complex quadratic field K with 3-class rank $r_3(K) \ge 3$ has an infinite 3-class field tower $K < F_3^1(K) < F_3^2(K) < \cdots < F_3^\infty(K)$ of length $\ell_3(K) = \infty$. In the time between 1973 and 1978, Diaz y Diaz [33] [34] and Buell [35] have determined the smallest absolute discriminants $|d|$ of such fields. Recently, we have launched a computational project which aims at verifying these classical results and adding sophisticated arithmetical details. Below the bound 10^7 there exist 25 discriminants d of this kind, and 14 of the corresponding fields K have a 3-class group $\mathrm{Cl}_3(K)$ of elementary abelian type $(3,3,3)$. For each of these 14 fields, we determine the type of 3-principalization $\varkappa := \varkappa_1(3,K)$ in the thirteen unramified cyclic cubic extensions $L_1, \cdots, L_{13}$ of K, and the structure of the 3-class groups $\mathrm{Cl}_3(L_i)$ of these extensions, *i.e.*, the IPAD of *K*. We characterize the metabelian Galois group $G = G_3^2(K) = \mathrm{Gal}\left(F_3^2(K) \middle| K\right)$ of the second Hilbert 3-class field $F_3^2(K)$ by means of kernels and targets of its Artin transfer homomorphisms [8] to maximal subgroups. We provide evidence of a wealth of structure in the set of infinite topological 3-class field tower groups $G_3^\infty(K) = \mathrm{Gal}\left(F_3^\infty(K) \middle| K\right)$ by showing that the 14 groups G are pairwise non-isomorphic.

We summarize our results and their obvious conclusion in the following theorem.

Theorem 7.1.

There exist exactly 14 complex quadratic number fields $K = \mathbb{Q}\left(\sqrt{d}\right)$ with 3-class groups $\mathrm{Cl}_3(K)$ of type $(3,3,3)$ and discriminants in the range $-10^7 < d < 0$. They have pairwise non-isomorphic

1) second and higher 3-class groups $\mathrm{Gal}\left(F_3^n(K) \middle| K\right)$, $n \ge 2$,

2) infinite topological 3-class field tower groups $\mathrm{Gal}\left(F_3^\infty(K) \middle| K\right)$.

Before we come to the proof of Theorem 7.1 in Section 7.3, we collect basic numerical data concerning fields with $r_3(K) = 3$ in Section 7.1, and we completely determine sophisticated arithmetical invariants in Section 7.2 for all fields with $\mathrm{Cl}_3(K)$ of type $(3,3,3)$. The first attempt to do so for the smallest absolute discriminant $|d| = 3321607$ with $r_3(K) = 3$ is due to Heider and Schmithals in ([30], Section 4, Tbl. 2, p. 18), but it resulted in partial success only.

7.1. Discriminants $-10^7 < d < 0$ of Fields $K = \mathbb{Q}\left(\sqrt{d}\right)$ with Rank $r_3(K) = 3$

Since one of our aims is to investigate tendencies for the coclass of second and higher p-class groups $G_p^n(K) = \mathrm{Gal}\left(F_p^n(K) \middle| K\right)$, $n \ge 2$, [9] [23] of a series of algebraic number fields K with infinite p-class field tower, for an odd prime $p \ge 3$, the most obvious choice which suggests itself is to take the smallest possible prime $p = 3$ and to select complex quadratic fields $K = \mathbb{Q}\left(\sqrt{d}\right)$, $d < 0$, having the simplest possible 3-class group $\mathrm{Cl}_3(K)$ of rank 3, that is, of elementary abelian type $(3,3,3)$.

The reason is that Koch and Venkov [32] have improved the lower bound of Golod, Shafarevich [31] [36] and Vinberg [37] for the p-class rank $r_p(K)$, which ensures an infinite p-class tower of a complex quadratic field K, from 4 to 3.

However, quadratic fields with 3-rank $r_3(K) = 3$ are sparse. Diaz y Diaz and Buell [33]-[35] [38] have determined the minimal absolute discriminant of such fields to be $|d| = 3321607$.

To provide an independent verification, we used the computational algebra system MAGMA [2]-[4] for compiling a list of all quadratic fundamental discriminants $-10^7 < d < 0$ of fields $K = \mathbb{Q}\left(\sqrt{d}\right)$ with 3-class rank $r_3(K) = 3$. In 16 hours of CPU time we obtained the 25 desired discriminants and the abelian type invariants (here written in 3-power form) of the corresponding 3-class groups $\mathrm{Cl}_3(K)$, and also of the complete class groups $\mathrm{Cl}(K)$, as given in **Table 1**. There appeared only one discriminant $d = -7503391$ (No. 16) which is not contained in ([34], Appendix 1, p.68) already.

There are 14 discriminants, starting with $d = -4447704$, such that $\mathrm{Cl}_3(K)$ is elementary abelian of type

Table 1. Data collection for $r_3(K)=(3)$ and $-10^7<d<0$.

No.	Discriminant d	$\mathrm{Cl}_3(K)$	$\mathrm{Cl}(K)$
1	−3321607	$(9,3,3)$	$(63,3,3)$
2	−3640387	$(9,3,3)$	$(18,3,3)$
3	−4019207	$(9,3,3)$	$(207,3,3)$
4	−4447704	$(3,3,3)$	$(24,6,6)$
5	−4472360	$(3,3,3)$	$(30,6,6)$
6	−4818916	$(3,3,3)$	$(48,3,3)$
7	−4897363	$(3,3,3)$	$(33,3,3)$
8	−5048347	$(9,3,3)$	$(18,6,3)$
9	−5067967	$(3,3,3)$	$(69,3,3)$
10	−5153431	$(27,3,3)$	$(216,3,3)$
11	−5288968	$(9,3,3)$	$(72,3,3)$
12	−5769988	$(3,3,3)$	$(12,6,6)$
13	−6562327	$(9,3,3)$	$(126,3,3)$
14	−7016747	$(9,3,3)$	$(99,3,3)$
15	−7060148	$(3,3,3)$	$(60,6,3)$
16	−7503391	$(9,3,3)$	$(90,6,3)$
17	−7546164	$(9,3,3)$	$(18,6,6,2)$
18	−8124503	$(9,3,3)$	$(261,3,3)$
19	−8180671	$(3,3,3)$	$(159,3,3)$
20	−8721735	$(3,3,3)$	$(60,6,6)$
21	−8819519	$(3,3,3)$	$(276,3,3)$
22	−8992363	$(3,3,3)$	$(48,3,3)$
23	−9379703	$(3,3,3)$	$(210,3,3)$
24	−9487991	$(3,3,3)$	$(381,3,3)$
25	−9778603	$(3,3,3)$	$(48,3,3)$

$(3,3,3)$, and 10 discriminants, starting with -3321607, such that $\mathrm{Cl}_3(K)$ is of non-elementary type $(9,3,3)$. For the single discriminant $d=-5153431$, we have a 3-class group of type $(27,3,3)$. We have published this information in the Online Encyclopedia of Integer Sequences (OEIS) [39], sequences A244574 and A244575.

7.2. Arithmetic Invariants of Fields $K=\mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K)\simeq(3,3,3)$

After the preliminary data collection in Section 7.1, we restrict ourselves to the 14 cases with elementary abelian 3-class group of type $(3,3,3)$. The complex quadratic field $K=\mathbb{Q}(\sqrt{d})$ possesses 13 unramified cyclic cubic extensions $L_1,\cdots,L_{13}$ with dihedral absolute Galois group $\mathrm{Gal}(L_i|\mathbb{Q})$ of order six [23]. Based on Fieker's technique [40], we use the computational algebra system MAGMA [3] [4] to construct these extensions and to calculate their arithmetical invariants. In **Table 2**, which is continued in **Table 3** on the following page, we present the kernel $\varkappa_i$ of the 3-principalization of K in L_i [21] [23], the occupation numbers $o(\varkappa)_i$ of the principalization kernels [19], and the abelian type invariants τ_i, resp. τ_i^0, of the 3-class group $\mathrm{Cl}_3(L_i)$, resp. $\mathrm{Cl}_3(K_i)$, for each $1\le i\le 13$ [5] [9]. Here, we denote by K_i the unique real non-Galois absolutely

Table 2. Pattern recognition via ordered IPADs.

No.	Discriminant												
i	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$d=-4447704$												
$\varkappa$	8	1	8	8	10	8	6	13	8	2	10	8	9
$o(\varkappa)$	1	1	0	0	0	1	0	6	1	2	0	0	1
τ	2^21^2	21^4	2^21^2	32^21	2^21^2	21^4	2^21^2	2^21^2	21^4	21^4	21^4	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
2	$d=-4472360$												
$\varkappa$	1	12	6	13	6	10	4	1	10	10	1	8	4
$o(\varkappa)$	2	0	0	2	0	2	0	1	0	3	0	1	2
τ	2^21^2	2^21^2	2^21^2	21^4	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	32^21	2^21^2	21^4
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21	1^2	1^2
3	$d=-4818916$												
$\varkappa$	6	9	13	1	5	6	9	4	11	7	1	3	4
$o(\varkappa)$	2	0	1	2	1	2	1	0	2	0	1	0	1
τ	2^21^2	2^21^2	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	431^3	2^21^2	21^4	32^21	2^21^2
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	31	1^2	1^2	21	1^2
4	$d=-4897363$												
$\varkappa$	3	8	11	2	6	6	12	7	2	2	9	13	6
$o(\varkappa)$	0	3	1	0	0	3	1	1	1	0	1	1	1
τ	2^21^2	321^3	431^3	2^21^2	21^4	2^21^2	32^21	21^4	2^21^2	2^21^2	2^21^2	2^21^2	2^21^2
τ^0	1^2	21	31	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2
5	$d=-5067967$												
$\varkappa$	8	6	9	2	3	7	12	7	1	4	3	9	4
$o(\varkappa)$	1	1	2	2	0	1	2	1	2	0	0	1	0
τ	21^4	21^4	2^21^2	21^4	2^21^2	2^21^2	2^21^2	32^21	2^21^2	21^4	21^4	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2
6	$d=-5769988$												
$\varkappa$	12	11	7	6	1	1	10	10	9	6	4	3	13
$o(\varkappa)$	2	0	1	1	0	2	1	0	1	2	1	1	1
τ	321^3	2^21^2	21^4	321^3	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	2^21^2	21^4	32^21
τ^0	21	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21

Table 3. Pattern recognition (continued).

No.	Discriminant												
i	1	2	3	4	5	6	7	8	9	10	11	12	13
7							$d = -7060148$						
$\varkappa$	2	4	4	9	4	10	8	10	10	1	6	8	3
$o(\varkappa)$	1	1	1	3	0	1	0	2	1	3	0	0	0
τ	21^4	21^4	32^21	2^21^2	32^21	21^4	2^21^2	321^3	21^4	21^4	2^21^2	2^21^2	321^3
τ^0	1^2	1^2	21	1^2	21	1^2	1^2	21	1^2	1^2	1^2	1^2	21
8							$d = -8180671$						
$\varkappa$	12	9	2	6	10	6	8	2	10	10	9	11	4
$o(\varkappa)$	0	2	0	1	0	2	0	1	2	3	1	1	0
τ	321^3	2^21^2	21^4	2^21^2	2^21^2	2^21^2	2^21^2	21^4	2^21^2	2^21^2	2^21^2	21^4	2^21^2
τ^0	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
9							$d = -8721735$						
$\varkappa$	5	2	5	1	10	13	4	7	11	3	9	8	8
$o(\varkappa)$	1	1	1	1	2	0	1	2	1	1	1	0	1
τ	2^21^2	21^4	21^4	21^4	321^3	2^21^2	21^4	2^21^2	321^3	321^3	21^4	321^3	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	21	21	1^2	21	1^2
10							$d = -8819519$						
$\varkappa$	2	7	8	12	4	12	9	5	5	3	10	6	10
$o(\varkappa)$	0	1	1	1	2	1	1	1	1	2	0	2	0
τ	2^21^2	21^4	32^21	2^21^2	2^21^2	2^21^2	2^21^2	2^21^2	21^4	2^21^2	2^21^2	2^21^2	1^6
τ^0	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
11							$d = -8992363$						
$\varkappa$	12	10	2	12	9	5	10	10	2	12	6	9	7
$o(\varkappa)$	0	2	0	0	1	1	1	0	2	3	0	3	0
τ	2^21^2	21^4	2^21^2	2^21^2	32^21	2^21^2	21^4	21^4	21^4	2^21^2	2^21^2	21^4	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
12							$d = -9379703$						
$\varkappa$	8	11	8	13	9	5	6	1	2	13	4	12	3
$o(\varkappa)$	1	1	1	1	1	1	0	2	1	0	1	1	2
τ	21^4	2^21^2	321^3	2^21^2	21^4	21^4	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	2^21^2
τ^0	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
13							$d = -9487991$						
$\varkappa$	4	2	2	11	13	9	12	9	8	1	1	12	3
$o(\varkappa)$	2	2	1	1	0	0	0	1	2	0	1	2	1
τ	2^21^2	2^21^2	2^21^2	2^21^2	321^3	2^21^2	2^21^2	21^4	2^21^2	21^4	2^21^2	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
14							$d = -9778603$						
$\varkappa$	10	6	6	9	9	10	8	10	13	5	12	6	10
$o(\varkappa)$	0	0	0	0	1	3	0	1	2	4	0	1	1
τ	2^21^2	321^3	21^4	32^21	32^21	2^21^2	2^21^2	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2
τ^0	1^2	21	1^2	21	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2

cubic subfield of L_i. For brevity, we give 3-logarithms of abelian type invariants and we denote iteration by formal exponents. Note that the multiplets $\varkappa$ and τ are *ordered* and in componentwise mutual correspondence, in the sense of §4.

In **Table 4**, we classify each of the 14 complex quadratic fields $K = \mathbb{Q}\left(\sqrt{d}\right)$ of type $(3,3,3)$ according to the counters of the abelian type invariants of the 3-class groups $\mathrm{Cl}_3(L_i)$ of the 13 unramified cyclic cubic extensions L_i, that is the *accumulated* (unordered) form of the IPAD of K. Whereas the dominant part of these groups is of order $3^6 = 729$, there always exist(s) at least one and at most four distinguished groups of bigger order, usually $3^8 = 6561$ and occasionally even $3^{10} = 59049$. According to the number of distinguished groups, we speak about *uni*-, *bi*-, *tri*- or *tetra*-polarization. If the maximal value of the order is 3^8, then we have a *ground* state, otherwise an *excited* state.

7.3. Proof of Theorem 7.1

Proof. According to ([9], Thm.1.1 and Dfn. 1.1, pp. 402-403), the information given in **Table 4** consists of isomorphism invariants of the metabelian Galois group $G = \mathrm{Gal}\left(F_3^2(K) \middle| K\right)$ of the second Hilbert 3-class field of K [23]. Consequently, with respect to the 13 abelian type invariants of the 3-class groups $\mathrm{Cl}_3(L_i)$ alone, only the groups G for $d \in \{-4447704, -5067967, -8992363\}$ could be isomorphic. However, **Table 2** and **Table 3** show that these three groups differ with respect to another isomorphism invariant, the 3-principalization type $\varkappa$ [19] [21], since the corresponding maximal occupation numbers of the multiplet $o(\varkappa)$ are $6, 2, 3$, respectively. □

7.4. Final Remark

We would like to emphasize that Theorem 7.1 provides evidence of *a wealth of structure* in the set of infinite 3-class field towers, which was unknown up to now, since the common practice is to consider a 3-class field tower as done when some criterion in the style of Golod-Shafarevich-Vinberg [31] [36] [37] or Koch-Venkov [32] ensures just its infinity. However, this perspective is very coarse and our result proves that it can be refined considerably.

It would be interesting to extend the range of discriminants $-10^7 < d < 0$ and to find the first examples of isomorphic infinite 3-class field towers.

Table 4. Accumulative (unordered) form of IPADs.

No.	Discriminant d	$2^2 1^2$	21^4	1^6	$32^2 1$	321^3	431^3	Polarization	State
1	−4447704	7	5	0	1	0	0	uni	Ground
2	−4472360	8	4	0	1	0	0	uni	Ground
3	−4818916	8	3	0	1	0	1	bi	Excited
4	−4897363	8	2	0	1	1	1	tri	Excited
5	−5067967	7	5	0	1	0	0	uni	Ground
6	−5769988	6	4	0	1	2	0	tri	Ground
7	−7060148	4	5	0	2	2	0	tetra	Ground
8	−8180671	9	3	0	0	1	0	uni	Ground
9	−8721735	4	5	0	3	1	0	tetra	Ground
10	−8819519	9	2	1	1	0	0	uni	Ground
11	−8992363	7	5	0	1	0	0	uni	Ground
12	−9379703	7	5	0	0	1	0	uni	Ground
13	−9487991	10	2	0	0	1	0	uni	Ground
14	−9778603	7	3	0	2	1	0	tri	Ground

Another very difficult remaining open problem is the actual identification of the metabelianizations of the 3-tower groups G of the 14 fields. The complexity of this task is due to unmanageable descendant numbers of certain vertices, e.g. $\langle 243,37\rangle$ and $\langle 729,122\rangle$, in the tree with root $\langle 27,5\rangle$.

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