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On Co-Primarily Packed Modules

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Abstract

Let R be a commutative ring with 1, and *M* is a (left) R-module. We introduce the concept of coprimarily packed submodules as a proper submodule *N* of an R-module *M* which is said to be Coprimarily Packed Submodule. If $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule of *M* for each $\alpha\in\Lambda$, then $N + \bigcap_{i=1}^{n} N_{\alpha_i} \neq M$ for some $\alpha_1,\alpha_2,\cdots,\alpha_n \in \Lambda$. When there exists $\beta\in\Lambda$ such that $N + N_\beta \neq M$; *N* is called Strongly Coprimarily Packed submodule. In this paper, we list some basic properties of this concept. We end this paper by explaining the relations between p-compactly packed and coprimarily packed submodules, and also the relations between strongly p-compactly packed and strongly coprimarily packed submodules.

Keywords

Coprimarily Packed Submodule, Strongly Coprimarily Packed Submodule, Bezout Module

1. Introduction

Coprimely packed rings were introduced by Erdoˇgdu for the first time in [1]. Al-Ani gave an analogous concept in modules [2], that is, a proper submodule N of an R-module M which is called Coprimely Packed. If $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$ where N_α is a prime submodule of M for each $\alpha\in\Lambda$, then $N + \bigcap_{i=1}^{n} N_{\alpha_i} \neq M$ for some $\alpha_1,\alpha_2,\cdots,\alpha_n \in \Lambda$. If there exists $\beta\in\Lambda$ such that $N + N_\beta \neq M$, then N is called Strongly Coprimely Packed submodule.

In this paper, we discuss the situation where the union of a family of primary submodules of M is considered.

In [2], the concept of compactly packed modules was introduced. We generalized this concept to the concept of p-compactly packed modules in [3], that is, a proper submodule N of an R-module M which is called P-Compactly Packed. If for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$, there exist $\alpha_1,\alpha_2,\cdots,\alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta\in\Lambda$, then N is called Strongly

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P-Compactly Packed. A module M is said to be P-Compactly Packed (Strongly P-Compactly Packed), if every proper submodule of M is p-compactly packed (strongly p-compactly packed).

In this paper, we introduce the definitions of coprimarily packed and strongly coprimarily packed module and discuss some of their properties. We end this paper by explaining the relations between p-compactly packed and coprimarily packed submodules, and also the relations between strongly p-compactly packed and strongly coprimarily packed submodules.

2. Coprimarily Packed and Strongly Coprimarily Packed Submodules

In this section we introduce the definition of coprimarily packed and strongly coprimarily packed module and discuss some of their properties.

2.1. Definition

Let N be a proper submodule of an R-module M. N is said to be Coprimarily Packed Submodule if whenever $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule of M for each $\alpha \in \Lambda$, then $N + \bigcap_{i=1}^{n} N_{\alpha_i} \neq M$ for some $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Lambda$. When there exists $\beta \in \Lambda$ such that $N + N_\beta \neq M$, N is called Strongly Coprimarily Packed submodule.

A module M is called Coprimarily Packed (Strongly Coprimarily Packed) module if every proper submodule of M is coprimarily packed (strongly coprimarily packed) submodule. It is clear that every strongly coprimarily packed submodule is a coprimarily packed submodule.

In the following proposition, we discuss the behavior of strongly coprimarily packed module under homomorphism.

2.2. Proposition

Let $f: M \to M'$ be an epimorphism. If M is an R-module such that $\ker f \subseteq N$ for every primary submodule N of M, then M is a strongly coprimarily packed module if and only if M' is a strongly coprimarily packed module.

Proof. Suppose that M is a strongly coprimarily packed module and let $N' \subseteq \bigcup_{\alpha\in\Lambda} W'_\alpha$ where N' is a proper submodule of M' and W'_α is a primary submodule of M' for each $\alpha \in \Lambda$, so

$$f^{-1}(N') \subseteq f^{-1}\left(\bigcup_{\alpha\in\Lambda} W'_\alpha\right) = \bigcup_{\alpha\in\Lambda} f^{-1}(W_\alpha).$$

$f^{-1}(W_\alpha)$ is a primary submodule of M for each $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ such that

$$f^{-1}(N') + f^{-1}(W'_\beta) \neq M.$$

We must show $N' + W'_\beta \neq M'$.

Suppose $N' + W'_\beta = M'$, let $m' \in M'$ so there exists $a' \in N'$ and $b' \in W'_\beta$ such that $a' + b' = m'$. Since f is an epimorphism there exists $m, a, b \in M$ such that

$$f(m) = m', \quad f(a) = a', \text{ and } \quad f(b) = b'.$$

Thus $f(m) = f(a+b)$, so $f(m-(a+b)) = 0$, this implies $m-(a+b) \in \ker f$. Since $a \in f^{-1}(N')$, $b \in f^{-1}(W'_\beta)$, and $\ker f \subseteq f^{-1}(W'_\beta)$, so $m - a \in f^{-1}(W'_\beta)$, hence

$$m \in f^{-1}(W'_\beta) + f^{-1}(N'),$$

this implies

$$M = f^{-1}(W'_\beta) + f^{-1}(N')$$

which is a contradiction. So M' is a strongly coprimarily packed module.

Conversely, suppose M' is a strongly coprimarily packed module and let $N \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$, where N is a proper submodule of M and W_α is a primary submodule of M for each $\alpha \in \Lambda$. Hence

$$f(N) \subseteq f\left(\bigcup_{\alpha\in\Lambda} W_\alpha\right) = \bigcup_{\alpha\in\Lambda} f(W_\alpha)$$

and since $\ker f \subseteq W_\alpha$ for each $\alpha \in \Lambda$, $f(W_\alpha)$ is a primary submodule of M', there exists $\beta \in \Lambda$ such that $f(N) + f(W_\beta) \neq M'$.

Suppose $N + W_\beta = M$ and let $x \in M'$, since f is an epimorphism, there exists $y \in M$ such that $f(y) = x$ and there exists $n \in N$ and $u \in W_\beta$ such that $y = n + u$. Then $f(y) = f(n+u)$, hence $x = f(n) + f(u)$, so $x \in f(N) + f(W_\beta)$). It follows $M' \subseteq f(N) + f(W_\beta)$ which is a contradiction, thus $N + W_\beta \neq M$, so M is a strongly coprimarily packed modul.

The following proposition gives a characterization of strongly coprimarily packed submodules in a multiplication or finitely generated module.

2.3. Proposition

Let M be a finitely generated or multiplication R-module. A proper submodule N is strongly coprimarily packed if and only if whenever $N \subseteq \bigcup_{\alpha\in\Lambda} L_\alpha$ where L_α is a maximal submodule of M for each $\alpha \in \Lambda$ then there exists $\beta \in \Lambda$ such that $N \subseteq L_\beta$.

Proof. Suppose N is a strongly coprimarily packed submodule and let $N \subseteq \bigcup_{\alpha\in\Lambda} L_\alpha$ where L_α is a maximal submodule of M for each $\alpha \in \Lambda$, hence L_α is a primary submodule, so there exists $\beta \in \Lambda$ such that $N + L_\beta \neq M$. But $L_\beta \subseteq N + L_\beta$ and L_β is a maximal submodule, hence $N + L_\beta = L_\beta$ thus $N \subseteq L_\beta$.

Conversely, let $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule for each $\alpha \in \Lambda$. There exists a maximal submodule L_α that contains N_α for each $\alpha \in \Lambda$, hence $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha \subseteq \bigcup_{\alpha\in\Lambda} L_\alpha$. By hypothesis, there exists $\beta \in \Lambda$ such that $N \subseteq L_\beta$, but $N_\beta \subseteq L_\beta$ so $N + N_\beta \subseteq L_\beta \neq M$, thus N is a strongly coprimarily packed submodule.

Recall that an R-module M is called Bezout Module if every finitely generated submodule of M is cyclic.

In the following proposition we will give a characterization for strongly coprimarily packed multiplication module.

2.4. Proposition

Let M be a multiplication R-module. If one of the following holds:

1) M is a cyclic module.

2) R is a Bezout ring.

3) M is a Bezout module.

Then M is strongly coprimarily packed module if and only if every primary submodule is strongly coprimarily packed.

Proof. Let N be a proper submodule of a module M such that $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where W_α is a maximal submodule of M for each $\alpha \in \Lambda$, then by proposition (2.3), it is enough to show that there exists $\beta \in \Lambda$ such that $N \subseteq W_\beta$.

First, if $\bigcup_{\alpha\in\Lambda} W_\alpha = M$, since N is a submodule of a multiplication module, there exists a primary submodule L that contains N. By hypothesis, L is strongly coprimarily packed submodule and $N \subseteq L \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha = M$, so there exists $\beta \in \Lambda$ such that $L \subseteq W_\beta$ hence $N \subseteq L \subseteq W_\beta$.

Now, if $\bigcup_{\alpha\in\Lambda} W_\alpha \neq M$, let $S^* = M - \bigcup_{\alpha\in\Lambda} W_\alpha$ and $S = R - \bigcup_{\alpha\in\Lambda} \sqrt{[W_\alpha : M]}$, so S^* is an S-closed subset of M. Since $N \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$, so $N \subseteq M - S^*$ thus there exists a submodule K that contains N and K is a maximal in $M - S^*$ [1], K is prime [1], so it is primary submodule. Thus by hypothesis K is strongly coprimarily packed and since $K \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$ and by proposition (2.3) there exists $\beta \in \Lambda$ such that $N \subseteq K \subseteq W_\beta$.

We end this Paper by looking at the relations between the strongly p-compactly packed modules and strongly coprimarily packed modules.

Recall that a proper submodule N of an R-module M is called P-Compactly Packed if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$, there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called Strongly P-Compactly Packed. A module M is said to be P-Compactly Packed (Strongly P-Compactly Packed) if every proper submodule of M is p-compactly packed (strongly p-compactly packed).

It is easy to show that every strongly p-compactly packed submodule is a strongly coprimarily packed submodule.

2.5. Proposition

If M is a p-compactly packed module, which cannot be written as a finite union of primary submodules, then M is a coprimarily packed module.

Proof. Let $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N is a proper submodule and N_α is a primary submodule of M for each $\alpha\in\Lambda$. Since M is a p-compactly packed module then there exists $\alpha_1,\alpha_2,\cdots,\alpha_n\in\Lambda$ such that $N\subseteq\bigcup_{i=1}^{n} N_{\alpha_i}$.

We claim that $N+\bigcap_{i=1}^{n} N_{\alpha_i}\neq M$, let $x\in N+\bigcap_{i=1}^{n} N_{\alpha_i}$ so there exists $n\in N$ and $b\in\bigcap_{i=1}^{n} N_{\alpha_i}$ such that $x=n+b$. Then there exists $j\in\{1,\cdots,n\}$ such that $n\in N_{\alpha_j}$, hence $x\in N_{\alpha_j}$, so $x\in\bigcup_{i=1}^{n} N_{\alpha_i}$, thus

$$N+\bigcap_{i=1}^{n} N_{\alpha_i}\subseteq\bigcup_{i=1}^{n} N_{\alpha_i}.$$

By hypothesis $\bigcup_{i=1}^{n} N_{\alpha_i}\neq M$ therefore $N+\bigcap_{i=1}^{n} N_{\alpha_i}\neq M$.

2.6. Definition

Let M be a non-zero module, M is called Primary Module if the zero-submodule of M is a primary submodule.

2.7. Proposition

If M is a multiplication or finitely generated strongly p-compactly packed module, then M is a strongly coprimarily packed module. The converse holds if M is a primary module such that every primary submodule of M contains no non-trivial primary submodule.

Proof. Suppose M is a primary module such that every primary submodule of M contains no non-trivial primary submodule. Let N be a proper submodule of M such that $N\subseteq\bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule of M for each $\alpha\in\Lambda$. Without loss of generality we can suppose that $N_\alpha\neq 0$, for each $\alpha\in\Lambda$. Then N_α is a maximal submodule of M, for each $\alpha\in\Lambda$. Since M is strongly coprimarily packed module, there exists $\beta\in\Lambda$, such that $N+N_\beta\neq M$; but N_β is a maximal submodule and $N_\beta\subseteq N+N_\beta$. This implies $N_\beta=N+N_\beta$, and hence $N\subseteq N_\beta$. Therefore M is a strongly p-compactly packed module.

The other direction is trivial.

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