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Crystallography in the Spaces E^2, E^3, E^4, E^5, $\cdots$ N°II Isomorphism Classes and Study of Five Crystal Families of Space E^5

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Abstract

In the paper N°II, we describe some isomorphism classes and we apply their properties to the study of five crystal families of space E^5. The names of these families are the following ones (monoclinic di iso squares)-al, decadic-al, (monoclinic di iso hexagons)-al, (rhombotopic $\cos\alpha = -1/4$)-al and rhombotopic $\cos\alpha = -1/5$. The meaning of these names will be given in Paragraphs 5 and 6 with some geometric properties of their cell.

Keywords

Crystal Families of Space E^5, Names, Point Groups of the Families, Rhombotopic Crystal Families

1. Introduction

Besides the study of the isomorphism classes and in order to complete the study of the crystal families of space E^5, five families are studied in this paper:

1) The crystal families N°XXIII (monoclinic di iso squares)-al, N°XXIV decadic-al and N°XXV (monoclinic di iso hexagons)-al. The suffix "al" means that the family cell in space E^5 is a right hyper prism based on a cell of space E^4. The WPV holohedry symbols are ([8] 2 2) $\perp$ m, group of order 32 for the family N°XXIII, ([10] 2 2) $\perp$ m, group of order 40 for the family N°XXIV and ([12] 2 2) $\perp$ m, and group of order 48 for the family N°XXV. These families have 7, 12, and 7 crystallographic (cr) point groups respectively. We remark that the WPV symbol of the holohedry of these families contains double rotations of order 8, 10 and 12 (Para-

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graphs 2, 3 and 4).

2) The crystal families N°XXX (rhombotopic $\cos\alpha = -1/4$)-al and N°XXXII rhombotopic $\cos\alpha = -1/5$. The meaning of these names will be given in Paragraphs 5 and 6. The family N°XXX splits in two sub-families: the centred family N°XXXa with group $\left[\bar{\bar{5}}\right]\times\bar{4}3\ \mathrm{m}$ of order 240 for holohedry and 8 cr point groups; the primitive family N°XXX with group $\left(\left[10\right]\times\bar{4}3\ \mathrm{m}\right)\perp \mathrm{m}$ of order 480 for holohedry and 15 cr point groups. The WPV holohedry symbol of family N°XXXII is $\left(\left[5\right]\times\bar{4}3\ \mathrm{m}\times 36\right)\times\bar{1}_5$ of order 1440 and this family has 10 cr point groups. Some cr point groups of the family N°XXX are obtained from those of the rhombotopic $\cos\alpha = -1/4$ family of space E^4 while all the 10 cr point groups of the family N°XXXII belong to space E^5. Moreover, this family is one of the irreducible families of space E^5 [1].

The mark "×" means direct product.

To end the study of the crystal families of space E^5, the crystal families (hypercube 4 dim.)-al (N°XXVIII), (di iso hexagons)-al (N°XXIX) and hypercube 5 dim. N°XXXI will be described in the next paper.

The results about the cr point groups are obtained from our Scientific Software SS E5 (explanations are given paper N°1).

2. (Monoclinic di iso Squares)-al Crystal Family. Point Groups. Isomorphism Classes

2.1. Recall

Let be denoted e_i $(i = 1,\cdots,5)$, the 5 vectors of a basis of space E^5. The metric tensor of the quadratic form defining a cell is a symmetric tensor with the scalar products $e_i \,.\, e_j$ as elements $\forall i,\ \forall j$.

2.2. Cell of the (Monoclinic di iso Squares)-al Crystal Family (N° XXIII)

The cell of the "(monoclinic di iso squares)-al" family is a right hyper prism, generalization of the right prism of space E^3. The word "al" is the abbreviation of the adjective orthogonal".

The metric tensor of the quadratic form defining the cell of this family is as follows (matrix N°1):

Matrix N°1 associated with the cell of the (monoclinic di iso squares)-al family

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & b & 0 \\ b & 0 & a & 0 & 0 \\ 0 & b & 0 & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Caption $\|e_i\|^2 = a,\ i = 1,\cdots,4;\ \|e_5\|^2 = c;\ b = e_1\cdot e_3 = e_2\cdot e_4 = e_3\cdot e_1 = e_4\cdot e_2$

As the name suggests, this cell is a right hyper prism which basis is built from two equal squares in space E^4. These two squares are built in the planes defined by the axes (e_1, e_2) for the first one and by the axes (e_3, e_4) for the second one; these two planes are not orthogonal but the angles between them depend on one angular parameter *b*. It is the reason why the word "monoclinic" appears in the family name. One of the two length parameters is the length $\sqrt{a}$ of the two square sides, the other one is the length $\sqrt{c}$ of the edge of the hyper prism (see the caption of the matrix N°1).

The angles between the axes e_1 and e_3 on one hand, between the axes e_2 and e_4 on the other hand have equal values while the axis e_5 is orthogonal to the space defined by the axes e_1, e_2, e_3, e_4.

2.3. Crystallographic Point Groups of the (Monoclinic di iso Squares)-al Family

The WPV symbols of all cr (cr for crystallographic) point groups of the family (monoclinic di iso squares)-al contain groups $[8]$ or $\left[\bar{\bar{8}}\right]$ either alone or as product with one or two binary groups such as m, 2, $\bar{1}$. It is a characteristic property of the cr point groups of this family. We recall that [8] is the WPV symbol of the cyclic group generated by the double rotation through angles $2\pi/8$ and $3\times 2\pi/8$ denoted 8^18^3, these two rotations 8^1 and 8^3 take place into the planes (e_1,e_3) and (e_2,e_4). The elements of the cyclic group [8], isomorphic to group C_8, are 8^18^3, 4^14^3, 8^38^1, $\bar{1}_4$, 8^58^7, 4^34^1, 8^78^5, 1.

In the same way, $\left[\bar{\bar{8}}\right]$ is the WPV symbol of a cyclic group of order 8 generated by the symmetry operation $8^18^3\times\bar{1}_5$, $\bar{1}_5$ being the total homothetie of space E^5. The group $\left[\bar{\bar{8}}\right]$ has for elements $8^18^3m_u$, 4^14^3, $8^38^1m_u$, $\bar{1}_4$, $8^58^7m_u$, 4^34^1, $8^78^5m_u$, 1, where u is the unit vector of the axis e_5 orthogonal to space E^4.

Group $\left[\bar{\bar{8}}\right]$ cannot be confused with the following two groups of order 16, $[8]\times\bar{1}_5$ and $[8]\perp\bar{1}_5$, the last one acts into a space of dimension 9 = 4 + 5 and belongs to the crystal family "(monoclinic di iso squares) decaclinic".

Table 1 lists the seven cr point groups of family XXIII which belong to four isomorphism classes, C_8, D_8, $C_8\times C_2$, $D_8\times C_2$, D_8 is the dihedral group of order 16 (see Annex, paper N˚I).

Remark

WPV symbols $([8]\ 2\ 2)$, $\left(\left[\bar{\bar{8}}\right]\ 2\ \bar{1}\right)$, and $([8]\ \bar{1}\ \bar{1})$ of the groups of the isomorphism class D_8 are very similar to Hermann-Mauguin symbols 4 mm, 422, $\bar{4}2$ m (groups of the isomorphism class D_4).

3. (Monoclinic di iso Hexagons)-al Crystal Family. Point Groups. Isomorphism Classes

3.1. Cell of the (Monoclinic di iso Hexagons)-al Crystal Family (N˚XXV)

Family N˚XXV, (monoclinic di iso hexagons)-al, presents great similarities with family N˚XXIII (monoclinic di iso squares)-al. The metric tensor of the quadratic form defining the cell of this family is as follows (matrix N˚2):

Matrix N˚2 associated with the cell of the (monoclinic di iso hexagons)-al family in space E^5

$$\begin{pmatrix} a & -a/2 & 0 & b & 0 \\ -a/2 & a & -b & 0 & 0 \\ 0 & -b & a & -a/2 & 0 \\ b & 0 & -a/2 & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Caption: $\|e_i\|^2 = a,\ i=1,\cdots,4;\ \|e_5\|^2 = c;\ b = e_1\cdot e_4 = -e_2\cdot e_3 = -e_3\cdot e_2 = e_4\cdot e_1$

The cell of this family is a righthyper prism which basis in space E^4 is built from two equal hexagons into the planes defined by the axes (e_1,e_2) for the first one and by the axes (e_3,e_4) for the second one; these planes are not orthogonal and the angles between these two planes depend on one angular parameter *b*. It is the reason

Table 1. Crystallographic point groups of (monoclinic di iso squares)-al family of space E^5.

Classes	Orders	Point groups and their WPV symbols		Arrangements
		Space E^4	Space E^5	
C_8	8	[8]	$\left[\bar{\bar{8}}\right]$	4(8) 2(4) 1(2)
D_8	16	([8] 2 2)	$\left(\left[\bar{\bar{8}}\right]\ 2\ \bar{1}\right)$, $([8]\ \bar{1}\ \bar{1})$	4(8) 2(4) 9(2)
$C_8\times C_2$	16		$[8]\perp m$	8(8) 4(4) 3(2)
$D_8\times C_2$	32		$([8]\ 2\ 2)\perp m$ (hol.)	8(8) 4(4) 19(2)

Table caption: First column: Symbols of the isomorphism classes. Second column: Order classes. Third column: WPV symbol of the point groups of the (monoclinic di iso squares)-al family in space E^4. Fourth column: WPV symbol of the point groups of the (monoclinic di iso squares)-al family in space E^5. Fifth column: Lists of the elements of the isomorphism classes denoted arrangement. SS E5 gives this information.

why the word "monoclinic" appears in the family name. One of the two length parameters is the length $\sqrt{a}$ of the two hexagon sides, the other one is the length $\sqrt{c}$ of the edge of the hyper prism, b is the angular parameter (see caption of matrix N˚2).

3.2. Crystallographic Point Groups of the (Monoclinic di iso Hexagons)-al Crystal Family

All WPV point group symbols of this family contain the group [12] or $\left[\overline{\overline{12}}\right]$ either alone or as product with one or two binary groups such as m, 2, $\bar{1}_4$. It is a characteristic property of the cr point groups of this family.

We recall that [12] is the WPV symbol of the cyclic group of order 12, isomorphic to group C_{12}. It is generated by the double rotation $12^1 12^5$ through angles 2π/12 and 5 × 2π/12, each of these rotations 12^1 and 12^5 takes place into the planes (e_1, e_3) and (e_2, e_4). The group [12] elements are the following ones $12^1 12^5$, $6^1 6^5$, $4^1 4^1$, $3^1 3^2$, $12^5 12^1$, $\bar{1}_4$, $12^7 12^{11}$, $3^2 3^1$, $4^3 4^3$, $6^5 6^1$, $12^{11} 12^7$, 1. In the same way, the group denoted $\left[\overline{\overline{12}}\right]$ is the WPV symbol of the cyclic group of order 12 generated by the symmetry operation $12^1 12^5 \bar{1}_5$ or $12^7 12^{11} m_u$. The elements of the group $\left[\overline{\overline{12}}\right]$ are $12^7 12^{11} m_u$, $6^1 6^5$, $4^3 4^3 m_u$, $3^1 3^2$, $12^{11} 12^7 m_u$, $\bar{1}_4$, $12^1 12^5 m_u$, $3^2 3^1$, $4^1 4^1 m_u$, $6^5 6^1$, $12^5 12^1 m_u$, 1.

Group $\left[\overline{\overline{12}}\right]$ cannot be confused with the following two groups of order 24, $[12] \times \bar{1}_5$ and $[12] \perp \bar{1}_5$, the last group acting into a space of dimension 9 = 4 + 5.

Table 2 lists the seven cr point groups of family N˚XXV, these groups belong to four isomorphism classes C_{12}, D_{12}, $C_{12} \times C_2$, $D_{12} \times C_2$ (D_{12} is the dihedral group of order 24).

Remark

WPV symbols $([12]\ 2\ 2)$, $\left(\left[\overline{\overline{12}}\right]\ 2\ \bar{1}\right)$ and $([12]\ \bar{1}\ \bar{1})$ of the groups of the isomorphism class D_{12} are very similar to Hermann-Mauguin symbols 4 mm, 422, $\bar{4}2$ m (groups of the isomorphism class D_4).

4. Decadic-al Crystal Family. Point Groups. Isomorphism Classes 6

4.1. Cell of the Decadic-al Crystal Family (N˚ XXIV)

The word "decadic" means that double rotations of order 10 belong to several cr point groups of this family.

The cell of the decadic-al family is a right hyper prism, the basis of which is a particular parallelotope of space E^4. The faces of this cell are equal lozenges in planes (e_1, e_2) and (e_3, e_4), so are the faces in planes (e_1, e_2) and (e_3, e_4) but these lozenges are different from the previous ones. The metric tensor of the quadratic form of the decadic-al family is as follows:

Table 2. Crystallographic point groups of (monoclinic di iso hexagons)-al family.

Classes	Orders	Cr. Point groups and their WPV symbols		Arrangements
		Space E^4	Space E^5	
C_{12}	12	[12]	$\left[\overline{\overline{12}}\right]$	4(12) 2(6) 2(4) 2(3) 1(2)
D_{12}	24	([12] 2 2)	$([12]\ \bar{1}\ \bar{1})$, $\left(\left[\overline{\overline{12}}\right]\ 2\ \bar{1}\right)$	4(12) 2(6) 2(4) 2(3) 13(2)
$C_{12} \times C_2$	24		$[12] \perp m$	8(12) 6(6) 4(4) 2(3) 3(2)
$D_{12} \times C_2$	48		$([12]\ 2\ 2) \perp m$ (hol.)	8(12) 6(6) 4(4) 2(3) 27(2)

Table caption: First column: Symbols of the isomorphism classes. Second column: Order of these classes. Third column: WPV symbol of the point groups of the (monoclinic di iso hexagons)-al family in space E^4. Fourth column: WPV symbol of the point groups of the (monoclinic di iso hexagons)-al family in space E^5. Fifth column: Lists of the elements of the isomorphism classes. SS E5 gives this information.

Matrix N˚3 associated with the cell of the (monoclinic di iso hexagons)-al family

$$\begin{pmatrix} a & b & -1/2(a+2b) & -1/2(a+2b) & 0 \\ b & a & b & -1/2(a+2b) & 0 \\ -1/2(a+2b) & b & a & b & 0 \\ -1/2(a+2b) & -1/2(a+2b) & b & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Caption: $\|e_i\|^2 = a,\ i = 1,\cdots,4;\ \|e_5\|^2 = c;\ -1/2(a+2b) = e_1 \cdot e_3 = e_2 \cdot e_4 = e_3 \cdot e_1 = e_4 \cdot e_2$

This metric tensor depends on three parameters of length *a*, *b* and *c*; the parameter $\sqrt{a}$ is the length of the side of the first lozenge, $\sqrt{b}$ the one of the second lozenge and $\sqrt{c}$ the hyperprism side. The angles between the axes depend on the two parameters *a* and *b*.

Family N˚XXIV splits in two sub-families, the centred sub-family N˚XXIVa and the primitive sub-family N˚XXIV.

4.2. Crystallographic Point Groups of the Two Decadic-al Crystal Sub-Families

All the WPV point group symbols of these two families contain groups $[5]$ or $\left[\bar{\bar{5}}\right]$ (family N˚XXIVa) and group [10] (family N˚XXIV) either alone or as product with a binary group such as 2, $\bar{1}$, m, ⋯ It is a characteristic property of the cr point groups of these families. Symbols $[5]$, $\left[\bar{\bar{5}}\right]$, $[10]$ have the same meaning as [8] and $\left[\bar{\bar{8}}\right]$ or $[12]$ and $\left[\bar{\bar{12}}\right]$ (see Paragraphs 2-3 and 3-2).

The five cr point groups of the sub-family N˚XXIVa belong to four isomorphism classes C_5, D_5, $C_{10} = C_5 \times C_2$, $D_{10} = D_5 \times C_2$ and the seven cr point groups of the sub-family N˚XXIV belong to four isomorphism classes C_{10}, D_{10}, $C_{10} \times C_2$, $D_{10} \times C_2$. Groups D_5 and D_{10} are dihedral groups of orders 10 and 20. Table 3 gives the list of the point groups of the decadic-al sub-families. Only groups of centred sub-family XXIVa are pointed out.

Remarks

- One group of class C_{10} belongs to the centred sub-family and two to the primitive sub-family.
- $\left[\bar{\bar{5}}\right] = [5] \times \bar{1}_5$. This group is a cyclic group of order 10, generated by the symmetry $10^1 10^3 m_u$ operation. Its elements are $10^1 10^3 m_u$, $5^1 5^3$, $10^3 10^9 m_u$, $5^2 5^1$, $\bar{1}_5$, $5^3 5^4$, $10^7 10^1 m_u$, $5^4 5^2$, $10^9 10^7 m_u$, 1. We explain how the planes of the double rotation $10^1 10^3$ can be defined. Indeed,thanks to the symmetry operation list, it is easy to find the generators of the different cr point groups. With respect to the general basis e_1, e_2, e_3, e_4, e_5 or *x*, *y*, *z*, *t*, *u*, the double rotation [10] is generated by the symmetry operation described by the following matrix A:

Table 3. Crystallographic point groups of the two decadic-al sub-families.

Classes	Orders	Point groups and their WPV symbols		Arrangements
		Space E^4	Space E^5	
C_5	5	[5] (XXIVa)		4(5)
$C_{10} = C_5 \times C_2$	10	[10]	$\left[\bar{\bar{5}}\right]$ (XXIVa), [5] ⊥ m	4(10) 4(5) 1(2)
D_5	10	([5] 2) (XXIVa)	$([5]\ \bar{1})$ (XXIVa)	4(5) 5(2)
$D_{10} = D_5 \times C_2$	20	([10] 2 2)	([5] 2) ⊥ m $\left(\left[\bar{\bar{5}}\right]\ 2\ \bar{1}\right)$ (hol. XXIVa)	4(10) 4(5) 11(2)
$C_{10} \times C_2$	20		[10] ⊥ m, $([10]\ \bar{1}\ \bar{1})$	12(10) 4(5) 3(2)
$D_{10} \times C_2$	40		([10] 2 2) ⊥ m (hol.)	12(10) 4(5) 23(2)

Table caption: First column: Symbol of the isomorphism classes. Second column: Order of these classes. Third column: WPV symbols of the point groups of the decadic-al family in space E^4. Fourth column: WPV symbols of the point groups of the decadic-al family in space E^5. Fifth column: Lists of the symmetry elements with their number of every isomorphism class.

Matrix N°4 associated with the double rotation $10^1 10^3$

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is possible to calculate the roots of the characteristic polynomial of the matrix A and the eigen subspaces associated to the different eigen values. As the double rotation [10] takes place in a four-dimensional space, the eigenvalue 1 appears. The other eigen values of the matrix A are $\exp(2i\pi/10)$ and $\exp(3\times 2i\pi/10)$. This double rotation $10^1 10^3$ takes place in the two planes defined by the axes:

$-x - 0.80902y + 0.30902z + 0.30902t$ and $-0.58779y + 0.95106z - 0.95106t$ for the rotation 10^1

$-x + 0.30902y - 0.80902z - 0.80902t$, $-0.95106y - 0.58779z + 0.58779$ for the rotation 10^3.

The coefficients which define the planes are

$$\cos(2i\pi/10) = -0.80902,\ \sin(2i\pi/10) = 0.58779$$

$$\cos(3\times 2i\pi/10) = 0.30902,\ \sin(3\times 2i\pi/10) = 0.95106$$

For all the symmetry operations of the group, we repeat the same process. For instance, the double rotation [5] is generated by the element $5^1 5^3$ which is described by the matrix A^2:

Matrix N°5 associated with the double rotation $5^1 5^3$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are numbers $\exp(2\times 2i\pi/10)$ and $\exp(2\times 3\times 2i\pi/10)$ and 1. The same process leads to the same eigen subspaces for double rotation [5] as for double rotation [10].

5. (Rhombotopic $\cos\alpha = -1/4$)-al Crystal Family. Point Groups. Isomorphism Classes

The crystal family N°XXX splits into two sub-families, one primitive sub-family N°XXX, one centred sub-family N°XXXa. This family is a particular case of the (rhombotopic $\cos\alpha = -1/n$)-al family. The cell of this family is a right hyper prism which basis is a regular rhombotope. We remind some properties of these types of cells.

5.1. Generalities about the (Rhombotopic $\cos\alpha = -1/n$)-al Family

The cell of the (rhombotopic $\cos\alpha = -1/n$)-al family is a right hyperprism of the $(n+1)$-dimensional space. The base of this cell is a regular rhombotope of the n-dimensional space. The metric tensor associated to the basis in space E^{n+1} is the matrix N°6. The n vectors of the lattice basis of the space E^n have the same norm $\sqrt{a}$ and the cosine of the angle between two of them is $-1/n$. The norm of the $(n+1)^{\text{th}}$ vector is $\sqrt{c}$ and this vector is orthogonal to space E^n.

Matrix N°6 associated with the cell of the (rhombotopic $\cos\alpha = -1/n$)-al crystal family

$$\begin{pmatrix} a & -a/n & -a/n & \cdots & 0 \\ -a/n & a & -a/n & \cdots & 0 \\ -a/n & -a/n & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}$$

Caption:
$\|e_i\|^2 = a,\ \forall i,\ i = 1,\cdots,n;\ \|e_{n+1}\|^2 = c;\ \forall i, \forall j \neq i, i = 1,\cdots,4,\ j = 1,\cdots,4;\ e_i \cdot e_j = -a/n$ and $\cos(e_i e_j) = -1/n$

5.1.1. Description of Some Simplexes

A simplex is the generalization of a triangle. Any set of $(n+1)$ points which do not lie in one $(n-1)$-dimensional space are the vertices of an n-dimensional simplex. *A* regular simplex has all edges equal, all faces equal and so on. For instance, regular simplexes are the equilateral triangle (space E^2), the regular tetrahedron (space E^3), the regular pentatope (space E^4) in [2] [3].

A regular simplex is not a crystal cell in (E^3). To obtain a crystal cell, we must add a second regular simplex symmetrical to the previous one at the centre of the initial cell and so the lattice of the crystal (rhombotopic $\cos\alpha = -1/n$) family is obtained.

Some additional properties are given in the Annex.

5.1.2. Point Groups of the Simplexes and of the (Rhombotopic $\cos\alpha = -1/n$) Family

Now, it is easy to find the WPV point groups of some regular simplexes and the holohedries of the associated crystallographic family together with its order. Table 4 gives point groups of some simplexes and of some rhombotopic crystal families.

[7] is the abridged WPV symbol of a cyclic group of order 7 generated by the triple rotation of space E^6, $7^1_{xy}7^3_{zt}7^5_{uv}$.

5.2. Primitive (Rhombotopic $\cos\alpha = -1/4$)-al Sub-Family

The cell of the primitive (rhombotopic $\cos\alpha = -1/4$)-al sub-family is a right hyper prism which basis is the regular rhombotope of space E^4. Hence, the previous results are applicable. This family can be considered as a particular case of the decadic-al family if the parameter b is equal to $-a/4$.

The metric tensor of the quadratic form of the primitive family N˚XXX is as follows (matrix N˚7):

Matrix N˚7 associated with the cell of the (rhombotopic $\cos\alpha = -1/4$)-al family

$$\begin{pmatrix} a & -a/4 & -a/4 & -a/4 & 0 \\ -a/4 & a & -a/4 & -a/4 & 0 \\ -a/4 & -a/4 & a & -a/4 & 0 \\ -a/4 & -a/4 & -a/4 & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Caption:
$\|e_i\|^2 = a;\ \forall i, i = 1,\cdots,4;\ \|e_5\|^2 = c;\ \forall i, \forall j \neq i,\ i = 1,\cdots,4,\ j = 1,\cdots,4;\ e_i \cdot e_j = -a/4$ and $\cos(e_i e_j) = -1/4$

This metric tensor depends on two parameters of length *a* and *c*.

Table 4. Crystallographic point groups of some regular simplexes and of the (rhombotopic $\cos\alpha = -1/n$) family holohedries.

WPV point group symbols of simplexes	Orders	Spaces	WPV symbol holohedries of the (rhombotopic $\cos\alpha = -1/n$) families	Orders
3 m	3 × 2 = 6	E^2	3 m × 2 = 6 mm	12
$\bar{4}3$ m	4 × 3 × 2 = 24	E^3	$\bar{4}3\ \text{m} \times \bar{1}$	48
$\bar{4}3\ \text{m} \times [5]$	24 × 5 = 120	E^4	$(\bar{4}3\ \text{m} \times [5]) \times \bar{1}_4$	240
$(\bar{4}3\ \text{m} \times [5]) \times \overline{\overline{63}}$	120 × 6 = 720	E^5	$((\bar{4}3\ \text{m} \times [5]) \times \overline{\overline{63}}) \times \bar{1}_5$	1440
$(\bar{4}3\ \text{m} \times [5]) \times \overline{\overline{63}} \times [7]$	720 × 7 = 5040	E^6	$((\bar{4}3\ \text{m} \times [5]) \times \overline{\overline{63}} \times [7]) \times \bar{1}_6$	10080

Table caption: First column: WPV point group symbols of simplexes. Second column: Order of these point groups. Third column: Space of these simplexes. Fourth column: WPV symbol holohedries of the (rhombotopic $\cos\alpha = -1/n$) families. Fifth column: Order of these point groups.

5.3. Isomorphism Classes of the Two Sub-Families (Rhombotopic $\cos\alpha = -1/4$)-al of Space E^5

The isomorphism classes have point groups belonging to the two sub-families N°XXXa and N°XXX so that it is useful to gather them, only groups of sub-family N°XXXa are pointed out. To sum up, the eight cr point groupsof sub-family N°XXXa and the five teen cr point groups of sub-family N°XXX belong to nine isomorphism classes. Table 5 gives all this results.

Remarks

- In the isomorphism class $S_5 \times C_2$, only one group, [10] × (42 3 2), is a positive point group, two groups are of the type $g_4 \perp m$ with group g_4 acting in space E^4.
- The list of symmetry operations of all cr point groups of space E^5 established by Veysseyre [4] gives ten mirrors to the group $[5]\times\bar{4}3\ \mathrm{m}$. The symmetry group of the regular tetrahedron $\bar{4}3\ \mathrm{m}$ has six mirrors. The(rhombotopic $\cos\alpha = -1/4$)-al cell is bounded by five regular tetrahedrons, therefore, we can expect to find 6 × 5 = 30 mirrors, but each mirror of this cell belongs to three adjacent tetrahedrons, then, it remains 30/3 = 10 different mirrors. In the same way, the list of the symmetry operations gives ten symmetry operations $\bar{1}_4$ to the group $[5]\times\left(42\ 3\ \bar{1}_4\right)$. To each mirror m, we can associate a total homothetie $\bar{1}_4$ of space E^4 orthogonal to the mirror. It is the reason why we find the same number of mirrors and of total homotheties.
- Groups (42 3 2) and $\left(42\ 3\ \bar{1}_4\right)$ are two iso cubic groups isomorphic to cr cubic point group $\bar{4}3\ \mathrm{m}$. Iso cubic point groups have been defined and studied in [5].

Table 5. Crystallographic point groups of the two (rhombotopic $\cos\alpha = -1/4$)-al sub-families.

Classes & orders	Arrangements	Point groups and their WPV symbols	
		Space E^4	Space E^5
$C_5 \times C_4$ (20)	4(5) 10(4) 5(2)	$[5]\times\bar{4}$ (XXXa)	$[5]\times 42$ (XXXa)
$C_{10} \times C_4$ (40)	4(10) 4(5) 20(4) 11(2)	$[10]\times\bar{4}$	$[\bar{\bar{5}}]\times 42$ (XXXa), [10].42
$(C_5 \times C_4) \times C2$ (40)	4(10) 4(5) 20(4) 11(2)		$([5]\times\bar{4})\perp \mathrm{m}$
$(C_{10} \times C_4) \times C_2$ (80)	12(10) 4(5) 40(4) 23(2)		$([10]\times\bar{4})\perp \mathrm{m}$
A_5 (60)	24(5) 20(3) 15(2)		$[5]\times 23$ (XXXa)
S_5 (120)	24(5) 20(6) 20(3)30(4) 25(2)	$[5]\times\bar{4}3\ \mathrm{m}$ (XXXa)	[5] × (42 3 $\bar{1}_4$) (XXXa), $[5]\times(\bar{4}\ 3\ \bar{1})$, $[5]\times(42\ 3\ 2)$
$A_5 \times C_2$ (120)	24(10) 24(5) 20(6)20(3) 31(2)	[10] × 23	$([5]\times 23)\perp \mathrm{m}$, $[\bar{\bar{5}}]\times 23$ (XXXa)
$S_5 \times C_2$ (240)	24(10) 24(5) 60(6) 20(3) 60(4) 51(2)	[10] × $\bar{4}3m$	$[10]\times(42\ 3\ 2)$, $([5]\times\bar{4}3\ \mathrm{m})\perp \mathrm{m}$, $[\bar{\bar{5}}]\times(42\ 3\ 2)$, $([5]\times(\bar{4}\ 3\ \bar{1}))\perp \mathrm{m}$, $[\bar{\bar{5}}]\times\bar{4}3\ \mathrm{m}$ (hol. XXXa)
$A_5 \times C_2 \times C_2$ (240)	72(10) 24(5) 60(6) 20(3)63(2)		$([10]\times 23)\perp \mathrm{m}$
$S_5 \times C_2 \times C_2$ (480)	72(10) 24(5) 140(6)120(4) 20(3) 103(2)		$([10]\times\bar{4}3\ \mathrm{m})\perp \mathrm{m}$ (hol.XXX)

Table caption: First column: Symbols of the isomorphism classes and these orders. Second column: Lists of the symmetry elements with their number of every isomorphism class. Third column WPV symbols of the point groups of the (rhombotopic $\cos\alpha = -1/4$)-al family in space E^4. Fourth column: WPV symbols of the point groups of the (rhombotopic $\cos\alpha = -1/4$)-al family in space E^5.

6. Rhombotopic $\cos\alpha = -1/5$ Crystal Family. Point Groups. Isomorphism Classes

This family (N˚XXXII) is a particular case of the families studied Paragraph 5-1. The metric tensor of the quadratic form of family XXXII is as follows (Matrix N˚8):

Matrix N˚8 associated with the cell of the rhombotopic $\cos\alpha = -1/5$ family

$$\begin{pmatrix} a & -a/5 & -a/5 & -a/5 & -a/5 \\ -a/5 & a & -a/5 & -a/5 & -a/5 \\ -a/5 & -a/5 & a & -a/5 & -a/5 \\ -a/5 & -a/5 & -a/5 & a & -a/5 \\ -a/5 & -a/5 & -a/5 & -a/5 & a \end{pmatrix}$$

Caption: $\|e_i\|^2 = a;\ i = 1,\cdots,5;\ \forall i = 1,\cdots,4,\ \forall j \neq i = 1,\cdots,4;\ e_i \cdot e_j = -a/5$ and $\cos(e_i e_j) = -1/5$

The ten point groups of the irreductible family N˚XXXII are listed **Table 6**, they act in five-dimensional space and they belong to eight isomorphism classes.

7. Conclusions

Thanks to the geometric approach, thanks to the study of the isomorphism classes of cr point groups in spaces E^2, E^3, E4 and E^5 and thanks to the Hermann-Mauguin or WPV symbols, we prove that substitutions groups, cr point groups, molecular or polytope symmetry groups are strongly correlated. Cayley's theorem anticipates this property. Let us verify these properties through **Table 7** and **Table 8**.

Table 7 lists some cr point groups isomorphic to mathematic groups C_n and D_n, from $n = 3$ to $n = 13$. The numbers of the families 60 and 61 are given in [6].

- [7] is the symbol of a cyclic group of order 7, in space E^6. It is generated by the triple rotation $7^1 7^2 7^3$ through angles 2π/7, 2 × 2π/7 and 3 × 2π/7; each of these rotations 7^1, 7^2 and 7^3 takes place into the planes (e_1, e_2), (e_3, e_4) and (e_5, e_6). The elements of group [7] are the following ones: $7^1 7^2 7^3$, $7^2 7^4 7^6$, $7^3 7^6 7^2$, $7^4 7^1 7^5$, $7^5 7^3 7^1$, $7^6 7^5 7^4$, and 1. All these elements, except for 1, are of order 7.
- [9] is the symbol of a cyclic group of order 9, in space E^6. It is generated by the triple rotation $9^1 9^2 9^4$ through angles 2π/9, 2 × 2π/9 and 4 × 2π/9; each of these rotations 9^1, 9^2 and 9^4 takes place into the planes (e_1, e_2), (e_3, e_4) and (e_5, e_6) of space E^6. The elements of group [9] are the following ones: $9^1 9^2 9^4$, $9^2 9^4 9^8$, $3^1 3^2 3^1$, $9^4 9^8 9^7$, $9^5 9^1 9^2$, $3^2 3^1 3^2$, $9^7 9^5 9^1$, $9^8 9^7 9^5$, and 1; two elements of this group are of order 3, and the other ones are of order 9, except for 1.

Table 6. Crystallographic point groups of the rhombotopic $\cos\alpha = -1/5$ crystal family.

Classes	Orders	Point groups and their WPV symbols	Arrangements
A_5	60	$[5] \times (2\ 33)$	24(5) 20(3) 15(2)
S_5	120	$[5] \times (42\ 33\ 2)$, $[5] \times (\bar{4}\ 33\ \bar{1})$	24(5) 20(6) 20(3) 30(4) 25(2)
$A_5 \times C_2$	120	$([5] \times (2\ 33)) \times \bar{1}_5$	24(10) 24(5) 20(6) 20(3) 31(2)
$S_5 \times C_2$	240	$([5] \times (42\ 33\ 2)) \times \bar{1}_5$	24(10) 24(5) 60(6) 20(3) 60(4) 51(2)
$A_5.D_3$	360	$[5] \times (2\ 33) \times (3\ 2)$	144(5) 80(3) 90(4) 45(2)
S_6	720	$[5] \times (42\ 3\ \bar{1}_4) \times 63$ $[5] \times \bar{4}3\ m \times \bar{\bar{63}}$	144(5) 240(6) 80(3) 180(4) 75(2)
$(A_5.D_3) \times C_2$	720	$[\bar{\bar{5}}] \times (2\ 33) \times (3\ 2$	144(10) 144(10) 80(6) 180(4) 80(3) 91(2)
$S_6 \times C_2$	1440	$[\bar{\bar{5}}] \times \bar{4}3\ m \times \bar{\bar{63}}$ (hol.)	144(10) 144(5) 560(6) 80(3) 360(4) 151(2)

Table caption: First column: Symbols of the isomorphism classes. Second column: Orders of these classes. Third column: WPV symbols of the cr point groups of the rhombotopic $\cos\alpha = -1/5$ family in space E^5. Fourth column: Lists of the symmetry elements with their number of every isomorphism class.

Table 7. Examples of crystallographic point groups isomorphic to mathematic groups C_n and D_n $(n = 3, \cdots, 13)$.

Classes	WPV symbol point groups	Generators	Classes	WPV symbol point groups	Crystal families
Space E^2					
C_4	4	4^1	D4	4 mm (hol.)	square
C_3	3	3^1	D3	3 m	hexagon
C_6	6	6^1	D6	6 mm (hol.)	hexagon
Space E^4					
C_8	[8]	$8^1 8^3$	D8	([8] 2 2) (hol.)	di iso squares
C_{12}	[12]	$12^1 12^5$	D12	([12] 2 2) (hol.)	di iso hexagons
C_{10}	[10]	$10^1 10^3$	D10	([10] 2 2) (hol.)	decadic
C_5	[5]	$5^1 5^2$	D5	([5]2)	decadic
Space E^6					
C_7	[7]	$7^1 7^2 7^3$	D7	$\left([7]\ \bar{1}\ \bar{1}\right)$ (hol.)	family n°60
C_9	[9]	$9^1 9^2 9^4$	D9	$\left([9]\ \bar{1}\ \bar{1}\right)$ (hol.)	family n°61
Space E^8 suggestion					
C_{11}	[11]	$(11^1 11^2 11^3 11^4)$	D11	$\left([11]\ \bar{1}_4\ \bar{1}_4\right)$	
C_{13}	[13]	$(13^1 13^2 13^3 13^4)$	D13	$\left([13]\ \bar{1}_4\ \bar{1}_4\right)$	

Table caption: First column and fourth column: Symbols of the isomorphism classes. Second column and fifth column: WPV symbols of the point groups. Third column: Generators of the point groups. Sixth column: Family names.

Table 8. Crystallographic point groups isomorphicto mathematic groups D_4, D_8, D_5 and D_{10} of spaces E^4 and E^5.

Classes	Arrangements	WPV symbol point groups		Crystal families (E^5)
		Space E^4	Space E^5	
D_8	4(8) 2(4) 9(2)	([8] 2 2)	$\left([8]\ \bar{1}\ \bar{1}\right), \left(\left[\bar{\bar{8}}\right]\ \bar{1}\ 2\right)$	(monoclinic di iso squares)-al
D_{12}	4(12) 2(6) 2(4) 2(3) 13(2)	([12] 2 2)	$\left([12]\ \bar{1}\ \bar{1}\right), \left(\left[\overline{\overline{12}}\right]\ \bar{1}\ 2\right)$	(monoclinic di iso hexagons)-al
		(34 2 2)	$\left([34]\ \bar{1}\ \bar{1}\right), \left(\left[\overline{\overline{34}}\right]\ \bar{1}\ 2\right)$	(hexagon square)-al
		(64 2 2)	$\left([64]\ \bar{1}\ \bar{1}\right), \left(\left[\overline{\overline{64}}\right]\ \bar{1}\ 2\right)$	(hexagon square)-al
D_5	4(5) 5(2)	([5] 2)	$\left([5]\ \bar{1}\right)$	decadic-al
D_{10}	4(10) 4(5) 11(2)	([10] 2 2)	$\left(\left[\bar{\bar{5}}\right]\ \bar{1}\ 2\right), \left([10]\ \bar{1}\ \bar{1}\right)$	decadic-al
$D_{10} = D_5 \times C_2$	4(10) 4(5) 11(2)		([5] 2 ⊥ m	decadic-al

Table caption: First column: Symbols of the isomorphism classes. Second column: Lists of the symmetry elements with their number of every isomorphism class. Third column: WPV symbols of the dihedralpoint groups in space E^4. Fourth column: WPV symbols of the dihedral cr point groups in space E^5. Fifth column: Names of the crystal families.

- [11] is the symbol of a cyclic group of order 11, in space E^8. It is generated by a «quadruple» rotation $11^1 11^2 11^3 11^4$ through angles 2π/11, 2 × 2π/11, 3 × 2π/11 and 4 × 2π/11; each of these rotations 11^1, 11^2, 11^3 and 11^4 takes place into the planes (e_1, e_2), (e_3, e_4), (e_5, e_6) and (e_7, e_8) of space E^8. The elements of group [11] are the following ones: $11^1 11^2 11^3 11^4$, $11^2 11^4 11^6 11^8$, $11^3 11^6 11^9 11^{12}$, $11^4 11^8 11^1 11^5$, $11^5 11^{10} 11^4 11^9$, $11^6 11^1 11^7 11^2$, $11^7 11^3 11^{10} 11^6$, $11^8 11^5 11^2 11^{10}$, $11^9 11^7 11^{10} 11^5$, $11^{10} 11^9 11^8 11^7$, and 1; all the elements of this group are of order 11, except for 1.

In the same way, [13] is the symbol of a cyclic group of order 13, in space E^8. It is generated by a «quadruple» rotation $13^1 13^2 13^3 13^4$ through angles $2\pi/13$, $2 \times 2\pi/13$, $3 \times 2\pi/13$ and $4 \times 2\pi/13$. The elements of this group are obtained as previously.

Table 8 gives the list of the cr point groups of the isomorphism classes D_8, D_{12}, D_5 and D_{10} of the spaces E^4 and E^5.

If you like mathematic crystallography, it is easy to prove and to generalize these results to space E^n whatever the dimension of space is.

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Annex

Some Additional Results about the Symmetry of the Polyhedrons, Molecules, Holohedries of the Crystal Families in Spaces E^2, E^3, E^4, E^5, ⋯

The definition of the simplexes with their properties has been given Paragraph 5-1.

Crystal family segment in space E^1: the holohedry m is a realization of mathematic group S_2 or D_2 with the list of element 1(2), in the set of the two cr point groups in E^1 (1, m). This symbol means that the segment is bounded by two points. The mirror, a point, is the middle of the segment.

Crystal family hexagon in space E^2: the hemihedry 3 m is a realization of the mathematic group D_3 of order 6 with the list of elements 2(3) 3(2), in the set of the ten cr point groups of space E^2. This symbol means that the equilateral triangle is bounded by three equal segments. As an example, we can cite the chemisorbed (in a mono molecular layer) molecule BF_3.

The holohedry 3 m × 2 = 6 mm of order 12 is a realization of the mathematic group $D_3 \times C_2$, with the list of elements 2(6) 2(3) 7(2). This symbol means that the regular hexagon in space E^2 is bounded by six equal segments. As example, we can cite the chemisorbed benzene molecule C_6H_6.

Crystal family cubic in space E^3: the hemihedry $\bar{4}3\ \mathrm{m}$ is a realization of the mathematic group S_4 of order 24, with the list of elements 6(4) 8(3) 9(2), in the set of the thirty-two cr point groups of space E^3. This symbol means that the regular tetrahedron in space E^3 is bounded by four equilateral triangles. Group $\bar{4}3\ \mathrm{m}$ is the symbol of the cr point group of the molecule SiH_4 (or Td Schöenflies symbol) and the one of the crystal ZnO (blende) which has $\mathrm{F}\ \bar{4}3\ \mathrm{m}$ as space group (International Tables of Crystallography) or T_d^2 Schöenflies symbol).

The holohedry $\bar{4}3\ \mathrm{m} \times \bar{1} = (4/\mathrm{m}\ \bar{3}\ 2/\mathrm{m}) = \mathrm{m}\bar{3}\mathrm{m}$ is a realization of the mathematic group $S_4 \times C_2$ of order 48 with the list of elements 8(6) 12(4) 8(3) 19(2), in same set of the point groups of space E^3. This symbol means that the cube in space E^3 is bounded by six equal squares. This symbol is the symmetry group of the regular octahedron, O_h or of the molecule $Co(OH)_6$.

Crystal family rhombotopic $\cos\alpha = -1/4$ in space E^4: the hemihedry $[5]\times\bar{4}3\ \mathrm{m}$ is a realization of the mathematic group S5 of order 120 with the list of elements 24(5) 20(6) 30(4) 20(3) 25(2) in the set of the 227 cr point groups of space E^4. The holohedry WPV symbol $[10]\times\bar{4}3\ \mathrm{m} = \left([5]\times\bar{4}3\ \mathrm{m}\right)\times\bar{1}_4$ is a realization of the mathematic group $S_5 \times C_2$ of order 240, with the list of elements 24(10) 24(5) 60(6) 60(4) 20(3) 51(2) in the set of the 227 cr point groups of space E^4. This symbol means that the cell of this family, in space E^4, is bounded by ten regular tetrahedrons.

Crystal family rhombotopic $\cos\alpha = -1/5$ in space E^5: the hemihedry $[5]\times\bar{4}3\ \mathrm{m}\times\bar{\bar{63}}$ is a realization of the mathematic group S_6 of order 720 with the list of elements 144(5) 240(6) 180(4) 80(3) 75(2) in the set of the 955 cr point groups of space E^5. The holohedry $\left([5]\times\bar{4}3\ \mathrm{m}\times\bar{\bar{63}}\right)\times\bar{1}_5 = \left[\bar{\bar{5}}\right]\times\bar{4}3\ \mathrm{m}\times\bar{\bar{63}}$ is a realization of the mathematic group $S_6 \times C_2$ of order 1440, with the list of elements 144(10) 144(5) 560(6) 360(4) 80(3) 151(2) among the set of the point groups of space E^5. The holohedry symbol means that the cell of this family is bounded by twelve regular rhombotopes of space E^4.