

Revised Mathematical Morphological Concepts

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Abstract

We revise some mathematical morphological operators such as Dilation, Erosion, Opening and Closing. We show proofs of our theorems for the above operators when the structural elements are partitioned. Our results show that structural elements can be partitioned before carrying out morphological operations.

Keywords

Mathematical Morphology, Dilation, Erosion, Opening, Closing

1. Introduction

Mathematical morphology is the theory and technique for the analysis and processing of geometrical structures, based on set theory, lattice theory, topology, and random functions. We consider classical mathematical morphology as a field of nonlinear geometric image analysis, developed initially by Matheron [1], Serra [2] and their collaborators, which is applied successfully to geological and biomedical problems of image analysis. The basic morphological operators were developed first for binary images based on set theory [1] [2] inspired by the work of Minkowski [3] and Hadwiger [4]. They were implemented for gray level images based on local min/max operators and level sets [2] [5] or on fuzzy sets [6] [7]. They were also implemented for gray level images with weighted min/max operators using a geometric interpretation based on the umbra approach of Sternberg [8] [9] which was algebraically equivalent to maxplus convolutions.

In the development of mathematical morphology in the mid-1960s by Georges Matheron and Jean Serra, they heavily stressed the mathematical formalism on mathematical morphology, and in the work of Haralick, Sternberg and Zhuang [10], the algebraic structure of morphological operators was given.

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In this paper, we outline in details the mathematical morphological operators and their algebraic structures when they are linked with union and intersection. We show that the partitioning of structural element before morphological operations is possible.

2. Preliminaries

2.1. Definitions

The following definitions are important for our purpose.

2.1.1. Dilation

Let the image set X and the structuring element B be subsets of the discrete space Z^2 : $X \subset Z^2$, $B \subset Z^2$. The dilation of X by B is defined as $X \oplus B = \{c = x + b \in Z^2 : x \in X; b \in B\}$; or the Dilation of a binary image A by structure element B , is $A \oplus B = \{a + b \mid a \in A \text{ and } b \in B\}$.

The dilation transform generally causes image objects to grow in size. From the definitions above, dilation is equivalent to a union of translates of the original image with respect to the structure element, that is, $A \oplus B = \bigcup_{b \in B} A_b$.

2.1.2. Erosion

Let the image set X and the structuring element B be subsets of the discrete space Z^2 : $X \subset Z^2$, $B \subset Z^2$. The erosion of X by B is defined as $X \ominus B = \{c \in Z^2 : \forall b \in B, \exists x \in X \text{ where } c = x - b\}$; or the Erosion of a binary image A by structure element B , is $A \ominus B = \{p \mid p + b \in A \forall b \in B\}$.

Similarly erosion transform allows image objects to shrink in size, that is, $A \ominus B = \bigcap_{b \in B} A_{-b}$.

2.1.3. Opening

Let the image set X and the structuring element B be subsets of the discrete space Z^2 : $X \subset Z^2$, $B \subset Z^2$. The Opening of X by B is defined as $X \circ B = (X \ominus B) \oplus B$; or the Opening of a binary image A by structure element B , is $A \circ B = (A \ominus B) \oplus B$.

2.1.4. Closing

Let the image set X and the structuring element B be subsets of the discrete space Z^2 : $X \subset Z^2$, $B \subset Z^2$. The Closing of X by B is defined as $X \bullet B = (X \oplus B) \ominus B$; or the Closing of a binary image A by structure element B , is $A \bullet B = (A \oplus B) \ominus B$.

2.2. Algebraic Properties of Dilation and Erosion

We note that Dilation is commutative and associative, that is, $A \oplus B = B \oplus A$ and $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, where as Erosion is non-commutative and non-associative, that is, $A \ominus B \neq B \ominus A$ and $(A \ominus B) \ominus C \neq A \ominus (B \ominus C)$ respectively.

Furthermore, Dilation and Erosion are both translation invariant, that is, if x is a vector belonging to A and B ($x \in A, x \in B$), then $A \oplus B_x = A_x \oplus B = (A \oplus B)_x$ and $A \ominus B_x = A_x \ominus B = (A \ominus B)_x$. Also both Dilation and Erosion are increasing in A , that is, if an image set A_1 is a subset of A_2 ($A_1 \subset A_2$), then $A_1 \oplus B \subset A_2 \oplus B$ and $A_1 \ominus B \subset A_2 \ominus B$. However, Erosion is decreasing in B , that is, if a structuring element B_1 is a subset of B_2 ($B_1 \subset B_2$), then $A \ominus B_1 \supset A \ominus B_2$. Dilation and Erosion transforms are duals of each other, that is, $(A \oplus B)^c = A^c \ominus \check{B}$ and $(A \ominus B)^c = A^c \oplus \check{B}$. Dilation and Erosion are also not the inverse of each other, that is, $(A \oplus B) \ominus B \neq A$ and $(A \ominus B) \oplus B \neq A$. Both the dilation and erosion transforms have an identity set, I , such

that $A \oplus I = A$ and $A \ominus I = A$. Dilation transform has an empty set, that is, \emptyset such that $A \oplus \emptyset = \emptyset$.

2.3. Algebraic Properties of Opening and Closing

We have Opening and Closing transforms as duals of each other, $(A \circ B)^c = A^c \bullet \check{B}$ and $(A \bullet B)^c = A^c \circ \check{B}$ but Opening and Closing are not the inverse of each other, $(A \circ B) \bullet B \neq A$ and $(A \bullet B) \circ B \neq A$. Also both Opening and Closing are translation invariant, if x is a vector belonging to A and B , then $A \circ B_x = A_x \circ B = (A \circ B)_x$ and $A \bullet B_x = A_x \bullet B = (A \bullet B)_x$. The opening transform is anti-extensive, the Opening of A by a structuring element B is always contained in A , regardless of B ($A \circ B \subseteq A$). The Closing transform is extensive, the Closing of A by a structuring element B always contains A , regardless of B ($A \subseteq A \bullet B$).

Furthermore, Opening and Closing are both increasing in A . If an image set A_1 is a subset of or equal to A_2 ($A_1 \subseteq A_2$), then $A_1 \circ B \subseteq A_2 \circ B$ and $A_1 \bullet B \subseteq A_2 \bullet B$ but only Opening is decreasing in B , if a structuring element B_1 is a subset of B_2 ($B_1 \subset B_2$), then $A \circ B_1 \supseteq A \circ B_2$. Finally, opening and Closing transforms are both idempotent, $A \circ B \circ B = A \circ B$ and $A \bullet B \bullet B = A \bullet B$. However if X is unchanged by opening with B , X is said to be open, whereas if X is unchanged by closing with B , X is said to be closed.

3. Results

In this section we present unions and intersections of Dilation, Erosion, Opening and Closing of two different sets and their extensions. The following theorems and their proofs will help us to describe the various results.

The morphological operators with n distinct sets

Theorem 1 (The union of Dilation with n different sets)

If $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ for $n \geq 2$

Then $A \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \dots \cup (A_n \oplus B)$

Proof.

If $A = A_1 \cup A_2$

Then

$$\begin{aligned} (A_1 \oplus B) \cup (A_2 \oplus B) &= \bigcup_{b \in B} A_{1_b} \cup \bigcup_{b \in B} A_{2_b} = \bigcup_{b \in B} \{A_{1_b} \cup A_{2_b}\} = \bigcup_{b \in B} (A_1 \cup A_2)_b \\ &= (A_1 \cup A_2) \oplus B = A \oplus B \end{aligned}$$

This implies $A \oplus B = (A_1 \cup A_2) \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B)$

Assume that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k$ for $k \geq 2$

Then $A \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \dots \cup (A_k \oplus B)$

Now we show that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}$

Then

$$\begin{aligned} A \oplus B &= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}) \oplus B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \cup A_{k+1}] \oplus B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \oplus B] \cup (A_{k+1} \oplus B) \\ &= (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \dots \cup (A_k \oplus B) \cup (A_{k+1} \oplus B) \end{aligned}$$

Theorem 2 (The intersection of Dilation with n different sets)

If $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ for $n \geq 2$

Then $A \oplus B = (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \dots \cap (A_n \oplus B)$

Proof.

If $A = A_1 \cap A_2$

Then

$$\begin{aligned} (A_1 \oplus B) \cap (A_2 \oplus B) &= \bigcup_{b \in B} A_{1_b} \cap \bigcup_{b \in B} A_{2_b} = \bigcup_{b \in B} \{A_{1_b} \cap A_{2_b}\} = \bigcup_{b \in B} (A_1 \cap A_2)_b \\ &= (A_1 \cap A_2) \oplus B = A \oplus B \end{aligned}$$

This implies $A \oplus B = (A_1 \cap A_2) \oplus B = (A_1 \oplus B) \cap (A_2 \oplus B)$

Assume that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k$ for $k \geq 2$

Then $A \oplus B = (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \dots \cap (A_k \oplus B)$

Now we show that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}$

Then

$$\begin{aligned} A \oplus B &= (A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}) \oplus B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \cap A_{k+1}] \oplus B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \oplus B] \cap (A_{k+1} \oplus B) \\ &= (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \dots \cap (A_k \oplus B) \cap (A_{k+1} \oplus B) \end{aligned}$$

Theorem 3 (The union of Erosion with n distinct sets)

If $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ for $n \geq 2$

Then $A \ominus B = (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \dots \cup (A_n \ominus B)$

Proof.

If $A = A_1 \cup A_2$

Then

$$\begin{aligned} (A_1 \ominus B) \cup (A_2 \ominus B) &= \bigcap_{b \in B} A_{1_b} \cup \bigcap_{b \in B} A_{2_b} = \bigcap_{b \in B} \{A_{1_b} \cup A_{2_b}\} = \bigcap_{b \in B} (A_1 \cup A_2)_b \\ &= (A_1 \cup A_2) \ominus B = A \ominus B \end{aligned}$$

This implies $A \ominus B = (A_1 \cup A_2) \ominus B = (A_1 \ominus B) \cup (A_2 \ominus B)$

Assume that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k$ for $k \geq 2$

Then $A \ominus B = (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \dots \cup (A_k \ominus B)$

Now we show that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}$

Then

$$\begin{aligned} A \ominus B &= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}) \ominus B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \cup A_{k+1}] \ominus B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \ominus B] \cup (A_{k+1} \ominus B) \\ &= (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \dots \cup (A_k \ominus B) \cup (A_{k+1} \ominus B) \end{aligned}$$

Theorem 4 (The intersection of Erosion with n distinct sets)

If $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ for $n \geq 2$

Then $A \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \dots \cap (A_n \ominus B)$

Proof.

If $A = A_1 \cap A_2$

Then

$$\begin{aligned} (A_1 \ominus B) \cap (A_2 \ominus B) &= \bigcap_{b \in B} A_{1-b} \cap \bigcap_{b \in B} A_{2-b} = \bigcap_{b \in B} \{A_{1-b} \cap A_{2-b}\} = \bigcap_{b \in B} (A_1 \cap A_2)_{-b} \\ &= (A_1 \cup A_2) \ominus B = A \ominus B \end{aligned}$$

This implies $(A_1 \cap A_2) \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B)$

Assume that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k$ for $k \geq 2$

Then $A \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \dots \cap (A_k \ominus B)$

Now we show that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}$

Then

$$\begin{aligned} A \ominus B &= (A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}) \ominus B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \cap A_{k+1}] \ominus B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \ominus B] \cap (A_{k+1} \ominus B) \\ &= (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \dots \cap (A_k \ominus B) \cap (A_{k+1} \ominus B) \end{aligned}$$

Theorem 5 (The union of Opening with n different sets)

If $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ for $n \geq 2$

Then $A \circ B = (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \dots \cup (A_n \circ B)$

Proof.

If $A = A_1 \cup A_2$

Then

$$\begin{aligned} (A_1 \circ B) \cup (A_2 \circ B) &= (A_1 \ominus B) \oplus B \cup (A_2 \ominus B) \oplus B = [(A_1 \ominus B) \cup (A_2 \ominus B)] \oplus B \\ &= [(A_1 \cup A_2) \ominus B] \oplus B = (A_1 \cup A_2) \circ B = A \circ B \end{aligned}$$

This implies $A \circ B = (A_1 \cup A_2) \circ B = (A_1 \circ B) \cup (A_2 \circ B)$

Assume that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k$ for $k \geq 2$

Then $A \circ B = (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \dots \cup (A_k \circ B)$

Now we show that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}$

Then

$$\begin{aligned} A \circ B &= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}) \circ B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \cup A_{k+1}] \circ B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \circ B] \cup (A_{k+1} \circ B) \\ &= (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \dots \cup (A_k \circ B) \cup (A_{k+1} \circ B) \end{aligned}$$

Theorem 6 (The intersection of Opening with n different sets)

If $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ for $n \geq 2$

Then $A \circ B = (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \dots \cap (A_n \circ B)$

Proof.

If $A = A_1 \cap A_2$

Then

$$\begin{aligned}(A_1 \circ B) \cap (A_2 \circ B) &= (A_1 \ominus B) \oplus B \cap (A_2 \ominus B) \oplus B = [(A_1 \ominus B) \cap (A_2 \ominus B)] \oplus B \\ &= [(A_1 \cap A_2) \ominus B] \oplus B = (A_1 \cap A_2) \circ B = A \circ B\end{aligned}$$

This implies $A \circ B = (A_1 \cap A_2) \circ B = (A_1 \circ B) \cap (A_2 \circ B)$

Assume that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k$ for $k \geq 2$

Then $A \circ B = (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \dots \cap (A_k \circ B)$

Now we show that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}$

Then

$$\begin{aligned}A \circ B &= (A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}) \circ B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \cap A_{k+1}] \circ B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \circ B] \cap (A_{k+1} \circ B) \\ &= (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \dots \cap (A_k \circ B) \cap (A_{k+1} \circ B)\end{aligned}$$

Theorem 7 (The union of Closing with n distinct sets)

If $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ for $n \geq 2$

Then $A \bullet B = (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \dots \cup (A_n \bullet B)$

Proof.

If $A = A_1 \cup A_2$

Then

$$\begin{aligned}(A_1 \bullet B) \cup (A_2 \bullet B) &= (A_1 \oplus B) \ominus B \cup (A_2 \oplus B) \ominus B = [(A_1 \oplus B) \cup (A_2 \oplus B)] \ominus B \\ &= [(A_1 \cup A_2) \oplus B] \ominus B = (A_1 \cup A_2) \bullet B = A \bullet B\end{aligned}$$

This implies $A \bullet B = (A_1 \cup A_2) \bullet B = (A_1 \bullet B) \cup (A_2 \bullet B)$

Assume that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k$ for $k \geq 2$

Then $A \bullet B = (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \dots \cup (A_k \bullet B)$

Now we show that if $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}$

Then

$$\begin{aligned}A \bullet B &= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1}) \bullet B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \cup A_{k+1}] \bullet B \\ &= [(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k) \bullet B] \cup (A_{k+1} \bullet B) \\ &= (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \dots \cup (A_k \bullet B) \cup (A_{k+1} \bullet B)\end{aligned}$$

Theorem 8 (The intersection of Closing with n distinct sets)

If $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ for $n \geq 2$

Then $A \bullet B = (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \dots \cap (A_n \bullet B)$

Proof.

If $A = A_1 \cap A_2$

Then

$$\begin{aligned}(A_1 \bullet B) \cap (A_2 \bullet B) &= (A_1 \oplus B) \ominus B \cap (A_2 \oplus B) \ominus B = [(A_1 \oplus B) \cap (A_2 \oplus B)] \ominus B \\ &= [(A_1 \cap A_2) \oplus B] \ominus B = (A_1 \cap A_2) \bullet B = A \bullet B\end{aligned}$$

This implies $A \bullet B = (A_1 \cap A_2) \bullet B = (A_1 \bullet B) \cap (A_2 \bullet B)$

Assume that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k$ for $k \geq 2$

Then $A \bullet B = (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \dots \cap (A_k \bullet B)$

Now we show that if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}$

Then

$$\begin{aligned} A \bullet B &= (A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k \cap A_{k+1}) \bullet B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \cap A_{k+1}] \bullet B \\ &= [(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) \bullet B] \cap (A_{k+1} \bullet B) \\ &= (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \dots \cap (A_k \bullet B) \cap (A_{k+1} \bullet B) \end{aligned}$$

4. Conclusion

We have shown that Dilation, Erosion, Opening and Closing of two or more sets with the same structural element and carrying out the union of the outcome is the same as taking the union of the two or more sets and operating the results with the structural element. The above result also holds for the intersection. These operators show ways of partitioning the structural element in order to carry out the morphological operation with ease. Furthermore, the results above give a simplification of morphological operations when dealing with lots of sets with the same structural element.

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