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On $k(D)$-Blocks

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Abstract

The objective of this research paper is to study numerical relationships between a block of a finite group and a defect group of such block. We define a new notion which is called a strongly $k(D)$-block and give a necessary and sufficient condition of a block with a cyclic defect group to be a $k(D)$-block in term of its inertial index. We believe that the notion and the results in this work will contribute to the developments of the theory of blocks of finite groups.

Keywords

Brauer's $k(B)$ Problem, Blocks with Cyclic Defect Groups

1. Introduction

Let p be a prime number and B a p-block of a finite group G with a defect group D of order p^d. Assuming Dade's projective conjecture, we prove in [1] that $k(B) \equiv k(D) \text{Mod}(p)$, where $k(B)$ is the number of the ordinary irreducible characters belonging to B and $k(D)$, is the number of the ordinary irreducible characters of the defect group D, which is an extra special p-group of order p^3 and exponent p, for an odd prime number p. In other words, we relate the number of the ordinary irreducible characters of a finite group which belong to a certain block and the number of the conjugacy classes of the defect group of that block under consideration.

This result led us to think about numerical relationships between a p-block and its defect group. In the present work, we are free from any condition about the prime number p. A question about the existence of a function on the natural numbers which relates some block-invariants under consideration is known as Brauer $k(B)$-conjecture (see [2] [3]). In fact, Brauer asks whether it is the case that $k(B) \le |D|$ in general.

For a bound of $k(B)$, it is well known that $k(B) \le p^{2d-2}$ for $d \ge 3$ and $k(B) \le p^d$ for $d \in \{0,1,2\}$. See [2]-[7] for more details and discussions in this direction.

However, we have arisen a question about which blocks and which conditions ensure the equality $k(B)=k(D)$ as well as ensure the congruency $k(B)\equiv k(D)\mathrm{Mod}(p)$. We have studied some general cases as well as some examples for small G. Then we try to characterize such blocks which have cyclic defect groups in terms of the order of the inertial subgroups.

However, as far as we know, we have not seen a similar relation in the literature. In fact, most of the examples have already been considered to satisfy the equality $k(B)=k(D)$. However, for $G=A_5$ with $p=5$, we find that $k(B_0(G))=4$, where $B_0(G)$ is the principal 5-block of G. But $k(D)=5$, where $D\in Syl_5(G)$. Since A_5 is a simple group and some unusual properties arise from such group, we think that there are some classes in group theory in such a way. p-blocks satisfy either the equality or the congruency relation.

Definition 1.1 Let p be a prime number, G a finite group and B a p-block of G with defect group D. Write $k(B)$ to mean the number of ordinary irreducible characters of B and write $k(D)$ to mean the number of ordinary irreducible characters of D. We call B a strongly $k(D)$-block if $k(B)=k(D)$.

Let us in the following definition consider an equality mod p.

Definition 1.2 Let p be a prime number, G a finite group and B a p-block of G with defect group D. Write $k(B)$ to mean the number of ordinary irreducible characters of B and write $k(D)$ to mean the number of ordinary irreducible characters of D. We call B a $k(D)$-block if $k(B)\equiv k(D)\mathrm{Mod}(p)$.

It is clear that a strongly $k(D)$-block is a $k(D)$-block. However, we shall see in Example 1.3 some $k(D)$-blocks which are not strongly $k(D)$-blocks.

Our main concern is to study finite groups and their blocks which satisfy Definitions 1.1 and 1.2. Note that it is well known that $k(D)$ is the number of the conjugacy classes of D. It is well known that blocks with cyclic defect groups are well understood. This theory is rich and has many applications. So, we shall start by doing some sort of characterization of strongly $k(D)$-blocks with cyclic defect groups. Our main tool is Dade's theorem for the number of irreducible characters of a block with a cyclic defect group (see [8] and ([9], p. 420])).

At the end of the paper, we use the computations and the results in [10] [11] to see that such phenomena do occur quite often in block theory.

1.1. Examples of Strongly $k(D)$-Blocks, $k(D)$-Blocks and Non $k(D)$-Blocks

We shall start with some examples which illustrate the phenomenon of $k(D)$-blocks.

Example 1.3 For $G=S_n$; the symmetric group of n letters and $n\in\{2,3,4,5,6,7,8\}$, it happens that $k(B_0(G))=k(D)$, for $D\in Syl_p(G)$ and $p\in\{2,3,5,7\}$. However, for any prime number $p\geq 5$, the defect group of the principal p-block of the symmetric group S_{2p} is an abelain p-group of order p^2 and $k(B_0)=\dfrac{p^2+3p}{2}$. In this situation, we obtain $k(D)$-blocks which are not strongly $k(D)$-blocks. A similar conclusion holds for the principal p-block of the symmetric group S_{3p}, with $p\geq 7$. However, when $p=3$ and $G=S_9$, then $22=k(B_0(S_9))\neq k(D)=17$, where $D\in Syl_3(S_9)$.

Example 1.4 Let G be the dihedral group of order 8. It has a unique 2-block with five ordinary irreducible characters which obviously coincide with the number of the conjugacy classes of G.

Example 1.5 Let G be the alternating group A_4. Then for $p=2$, G has a unique 2-block with four ordinary irreducible characters which is the same as the number of the conjugacy classes of the defect group. For $p=3$, we see that G has one 3-block of defect zero and the principal 3-block with three ordinary irreducible characters with the same number as the number of the conjugacy classes of a Sylow 3-subgroup of G.

Example 1.6 For $G=SL(2,3)$; the special linear group, we have for $p=2$, the principal 2-block has the quaternion group Q_8 as a defect group and indeed, $k(B_0(SL(2,3)))=7$ and $5=k(Q_8)$. Then $k(B_0(SL(2,3)))$ is a $k(D)$-block.

Example 1.7 Now, we have faced the first example which does not obey our speculation. It is the first non abelian simple group: A_5. Although, for $p \in \{2,3\}$, $k(B_0(A_5)) = k(D)$, where $D \in Syl_p(A_5)$, we observed that for $p = 5$, $k(B_0(A_5)) = 4 \neq k(D) = 5$, where, $D \in Syl_5(A_5)$. The same obstacle we have faced for the group $G = GL(3,2)$, since $k(B_0(G)) \neq k(D)$ when $D \in Syl_7(G)$.

Example 1.8 The principal 3-block for $G = PSL(3,3)$; the projective special linear group, satisfies $11 = k(B_0(G)) = k(D) = 11$, where D is an extra special 3-group of order 27 and exponent 3.

1.2. General Cases for the Notion of $k(D)$-Blocks

1) Let p be a prime number and G a finite group. Assume that the prime number p does not divide the order of the group G. Then each block of G has defect zero, (see Theorem 6.29 Page 247 in [12]). Hence such block is a strongly $k(D)$-block.

2) It is well known that if $G = DO_{p'}(G)$ is a p-nilpotent group with a Sylow p-subgroup D and the maximal normal p-subgroup of G then $k(B) = k(D)$. Certainly, a nilpotent p-block is a strongly $k(D)$-block (see Problem 13 in Chapter 5 Page 389 in [12]).

3) We know that if G is a p-group then it has a unique p-block, namely, the principal block and such block is an strongly $k(D)$-block.

4) For p-blocks with dihedral defect groups, we have $k(B) = k(D)$. Hence such blocks are examples of strongly $k(D)$-blocks.

2. $k(D)$-Blocks with Cyclic Defect Groups

In this section, we discuss p-blocks with cyclic defect groups. Recall that a root of a p-block B of a finite group G with defect group D is a p-block b of the subgroup $DC_G(D)$ such that $b^G = B$ (see Chapter 5 Page 348 in [12]), where $C_G(D)$ is the centralizer subgroup of D in G. Now for the root b, we define the inertial index of B to be the natural number $e := e(B) = [I_G(b) : DC_G(D)]$, where $I_G(b) = \{g \in G : b^g = b\}$. It is clear that $I_G(b)$ is a subgroup of G which contains $DC_G(D)$ and the index $[I_G(b) : DC_G(D)]$ is well-defined. The number e above is crucial to investigate some fundamental results in block theory.

Let us restate the following well known result which was established by Dade regarding the number of irreducible characters in a block with a cyclic defect group. For more detail, the reader can see the proof and other constructions in [8] [13] [14].

Lemma 2.1 Let B be a p-block of a finite group G with a cyclic defect group D of order p^d. Then B has $e + \frac{p^d - 1}{e}$ ordinary irreducible characters, where $|D| = p^d$.

With the above notation, we characterize strongly $k(D)$-blocks in the term of the inertial index for blocks with cyclic defect groups. Also, we believe that it is worth looking for some positive theorems regarding the notion of $k(D)$-blocks.

Theorem 2.2 Let B be a p-block of a finite group G with a cyclic defect group D of order p^d. Then B is a strongly $k(D)$-block if and only if $e = 1$ or ($e = p - 1$ and $d = 1$).

Proof: Assuming that B is a strongly $k(D)$-block and using Lemma 2.1, we can write $p^d = e + \frac{p^d - 1}{e}$. Then we have $e^2 - p^d e + (p^d - 1) = 0$. Letting e be the variable, we see that the only solution we have is that $e = 1$ or $e = p^d - 1$. The result follows as e divides $p - 1$. The converse is clear and the main result follows.

Remark 2.3 We get an analogue result of Theorem 2.2 for $k(D)$-blocks with cyclic defect groups, by solving the congruency equation $p^d \equiv_p e + \frac{p^d - 1}{e}$.

3. The Interplay with Fundamental Results

There are fundamental progress in solving Brauer problems. We recast the following result which is due to Kessar and Malle [11, HZC1]. This result can be used to see an strongly block B with abelian defect group D of order p^d as such that $k_0(B)=k(D)$, where $k_0(B)$ is the number of ordinary irreducible characters of height zero belonging to B.

Lemma 3.1 Let G be a finite group, and B be a p-block of G with defect group D. If D is abelian, then every ordinary irreducible character of B has height zero.

Let us conclude this paper by mentioning the following lemma in such a way that we rely on the computation in [10, Proposition 2.1] by Kulshammer and Sambale. These computations guarantee that the phenomena of strongly $k(D)$-block occur quite often in the theory of blocks.

Lemma 3.2 Let G be a finite group, and B be a 2-block of G with a defect group D. If D is an elementary abelian of order 16, then B is a $k(D)$-block.

We would like to mention that the origin of the concept of block theory is due to Brauer (see [15]-[21]). For the case for $p=2$, see [18]. He dealt with some elements in the center of the defect groups. In our case, we shall assume that the defect groups are abelian groups. In fact, Lemma 3.2 can be replaced by the following much stronger result.

Theorem 3.3 Let G be a finite group and $p=2$ or 3. Then, each p-block of G with abelian defect group is a $k(D)$-block.

Proof: Using Lemma 3.1, we have that every ordinary irreducible character of B has height zero. Then, the result is followed by elementary observations in [18].

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