Scientific Research Publishing

Retraction Notice

Title of retracted article: **Elementary Operations on L-R Fuzzy Number**

Author(s): Abdul Alim*, Fatema Tuj Johora, Shohel Babu, Abeda Sultana

* Corresponding author. abdulalim@buft.edu.bd

Journal: Advances in Pure Mathematics
Year: 2015
Volume: 5
Number: 3
Pages (from - to): 131 - 136
DOI (to PDF): http://dx.doi.org/10.4236/apm.2015.53016
Article page: http://www.scirp.org/journal/PaperInformation.aspx?PaperID=54814

Retraction date: 2015-08-24

Retraction initiative (multiple responses allowed; mark with **X**):
☐ All authors
☐ Some of the authors:
X Editor with hints from Journal owner (publisher)
Institution:
X Reader: Michael Hanss
Other:
Date initiative is launched: yyyy-mm-dd

Retraction type (multiple responses allowed):
X Unreliable findings
Lab error Inconsistent data Analytical error Biased interpretation
X Other:
☐ Irreproducible results
☐ Failure to disclose a major competing interest likely to influence interpretations or recommendations
X Unethical research

☐ Fraud
Data fabrication Fake publication Other:
☐ Plagiarism **X** Self plagiarism ☐ Overlap ☐ Redundant publication *
☐ Copyright infringement ☐ Other legal concern:

☐ Editorial reasons
Handling error Unreliable review(s) Decision error Other:

☐ Other:

Results of publication (only one response allowed):

☐ are still valid.
X we're found to be overall invalid.

Author's conduct (only one response allowed):
☐ honest error
X academic misconduct
☐ none (not applicable in this case – e.g. in case of editorial reasons)

* Also called duplicate or repetitive publication. Definition: "Publishing or attempting to publish substantially the same work more than once."

History
Expression of Concern:
☐ yes, date: yyyy-mm-dd
☐ no

Correction:
☐ yes, date: yyyy-mm-dd
☐ no

Comment:

Free style text with summary of information from above and more details that can not be expressed by ticking boxes.

This article has been retracted to straighten the academic record. In making this decision the Editorial Board follows COPE's Retraction Guidelines. Aim is to promote the circulation of scientific research by offering an ideal research publication platform with due consideration of internationally accepted standards on publication ethics. The Editorial Board would like to extend its sincere apologies for any inconvenience this retraction may have caused.

Editor guiding this retraction: Prof. Dexing Kong (EiC of APM)

Advances in Pure Mathematics, 2015, 5, 131-136
Published Online March 2015 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2015.53016

Scientific Research Publishing

Elementary Operations on L-R Fuzzy Number

Abdul Alim[1], Fatema Tuj Johora[2], Shohel Babu[2], Abeda Sultana[3]

[1]Mathematics, BGMEA University of Fashion and Technology, Dhaka, Bangladesh
[2]Mathematics, IUBAT—International University of Business Agriculture and Technology, Dhaka, Bangladesh
[3]Department of Mathematics, Jahanginagar University, Dhaka, Bangladesh
Email: abdulalim@buft.edu.bd

Received 26 February 2015; accepted 13 March 2015; published 19 March 2015

Abstract

The aim of this paper is to find the formula for the elementary operations on L-R fuzzy number. In this paper we suggest and describe addition, subtraction, multiplication and division of two L-R fuzzy numbers in a brief.

Keywords

Fuzzy Number, L-R Fuzzy Number, Membership Function

1. Introduction

A fuzzy set [1] A on $\mathbb{R}$, set of real numbers is called a *fuzzy number* [2] which satisfies at least the following three properties:

1) A must be a normal fuzzy set [3].
2) A^α must be a closed interval for every $\alpha \in (0,1]$.
3) The support [1] of A, A^{0+} must be bounded.

The fundamental idea of the L-R representation of fuzzy numbers is to split the membership function $\mu_{\tilde{p}_i}(x_i)$ of a fuzzy number $\tilde{p}_i$ into two curves $\mu_{l_i}(x_i)$ and $\mu_{r_i}(x_i)$, left and right of the modal value $\bar{x}_i$. The membership function $\mu_{\tilde{p}_i}(x_i)$ can be expressed through parameterized reference functions or shape function L and R in the form

$$\mu_{\tilde{p}_i}(x_i) = \begin{cases} \mu_{l_i}(x_i) = L\left[\dfrac{\bar{x}_i - x_i}{\alpha_i}\right] & \text{for } x_i < \bar{x}_i \\ \mu_{r_i}(x_i) = R\left[\dfrac{x_i - \bar{x}_i}{\beta_i}\right] & \text{for } x_i \ge \bar{x}_i \end{cases} \tag{1}$$

How to cite this paper: Alim, A., Johora, F.T., Babu, S. and Sultana, A. (2015) Elementary Operations on L-R Fuzzy Number. *Advances in Pure Mathematics*, **5**, 131-136. http://dx.doi.org/10.4236/apm.2015.53016

where $\bar{x}_i$ is the modal value of the membership function and α_i and β_i are the spreads corresponding to the left-hand and right-hand curve of the membership function [4] respectively.

As an abbreviated notation, we can define an L-R fuzzy number $\tilde{p}_i$ with the membership function $\mu_{\tilde{p}_i}(x_i)$ in (1) by

$$\tilde{p}_i = \langle \bar{x}_i, \alpha_i, \beta_i \rangle_{L,R} \tag{2}$$

where the subscripts *L* and *R* specify the reference functions [5].

2. Operations on L-R Fuzzy Number

In this section, the formulas for the elementary operations (addition, subtraction, multiplication, division) [5] between L-R fuzzy numbers [5] will be presented.

2.1. Addition of L-R Fuzzy Number

Suppose two fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$, represented as L-R fuzzy numbers of the form

$$\tilde{p}_1 = \langle \bar{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \text{ and } \tilde{p}_2 = \langle \bar{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \tag{3}$$

The sum $E_a(\tilde{p}_1, \tilde{p}_2) = \tilde{q} = \tilde{p}_1 + \tilde{p}_2$ is again an L-R fuzzy number of the form

$$\tilde{q} = \langle \bar{z}, \alpha, \beta \rangle_{L,R} \tag{4}$$

with the modal value

$$\bar{z} = \bar{x}_1 + \bar{x}_2 \tag{5}$$

and the spreads

$$\alpha = \alpha_1 + \alpha_2 \text{ and } \beta = \beta_1 + \beta_2 \tag{6}$$

In short we can write

$$\langle \bar{x}_1, \alpha_1, \beta_1 \rangle_{L,R} + \langle \bar{x}_2, \alpha_2, \beta_2 \rangle_{L,R} = \langle \bar{x}_1 + \bar{x}_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 \rangle_{L,R} \tag{7}$$

The left-hand reference functions of both fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ have to be given by *L*, and the right-hand reference functions by *R*.

The formula of the L-R addition in (7) is motivated by the following ways:

We first consider the right-hand curves $\mu_{r_1}(x_1)$ and $\mu_{r_2}(x_2)$ of the L-R fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ with

$$\mu_{r_1}(x_1) = R\left[\frac{x_1 - \bar{x}_1}{\beta_1}\right] \text{ and } \mu_{r_2}(x_2) = R\left[\frac{x_2 - \bar{x}_2}{\beta_2}\right] \tag{8}$$

The degree of membership $\mu^* \in [0,1]$ is taken on for the argument values

$$x_1^* = \bar{x}_1 + \beta_1 R^{-1}(\mu^*) \text{ and } x_2^* = \bar{x}_2 + \beta_2 R^{-1}(\mu^*) \tag{9}$$

This implies

$$z^* = x_1^* + x_2^* = \bar{x}_1 + \bar{x}_2 + (\beta_1 + \beta_2) R^{-1}(\mu^*) \tag{10}$$

and we obtain for the right-hand curve $\mu_r(z)$ of the fuzzy number $\tilde{q}$

$$\mu_r(z^*) = \mu^* = R\left[\frac{z^* - \bar{z}}{\beta}\right] \text{ with } \bar{z} = \bar{x}_1 + \bar{x}_2 \text{ and } \beta = \beta_1 + \beta_2 \tag{11}$$

The same reasoning holds for the left-hand curves of $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{q}$, and we get

$$\mu_l(z) = L\left[\frac{\bar{z} - z}{\alpha}\right] \text{ with } \bar{z} = \bar{x}_1 + \bar{x}_2 \text{ and } \alpha = \alpha_1 + \alpha_2 \tag{12}$$

2.2. Subtraction of L-R Fuzzy Number

Suppose two fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$, represented as L-R fuzzy numbers of the form

$$\tilde{p}_1 = \langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \text{ and } \tilde{p}_2 = \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \tag{13}$$

The opposite $-\tilde{p}$ of the L-R fuzzy number is defined as

$$-\tilde{p}_1 = -\langle \overline{x}, \alpha, \beta \rangle_{L,R} = -\langle \overline{x}, \beta, \alpha \rangle_{R,L} \tag{14}$$

Now by using (7) we can deduce the following formula for the subtraction $\tilde{q} = E_s(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 + \tilde{p}_2$ of the L-R fuzzy numbers:

$$\langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} - \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} = \langle \overline{x}_1 - \overline{x}_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2 \rangle_{L,R} \tag{15}$$

2.3. Multiplication of L-R Fuzzy Number

Let us consider two positive fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ of the same L-R type given by the L-R representations

$$\tilde{p}_1 = \langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \text{ and } \tilde{p}_2 = \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \tag{16}$$

We can construct the right-hand curve $\mu_r(z)$ of the product $\tilde{q} = E_m(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1\tilde{p}_2$ on the basis of the right-hand curves

$$\mu_{r_1}(x_1) = R\left[\frac{x_1 - \overline{x}_1}{\beta_1}\right] \text{ and } \mu_{r_2}(x_2) = R\left[\frac{x_2 - \overline{x}_2}{\beta_2}\right] \tag{17}$$

of L-R fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$. In accordance with the deduction of the formula for the L-R addition, the degree of membership $\mu^* \in [0,1]$ is taken on for the argument values

$$x_1^* = \overline{x}_1 + \beta_1 R^{-1}(\mu^*) \text{ and } x_2^* = \overline{x}_2 + \beta_2 R^{-1}(\mu^*) \tag{18}$$

This implies

$$z^* = x_1^* x_2^* = \overline{x}_1\overline{x}_2 + (\overline{x}_1\beta_2 + \overline{x}_2\beta_1) R^{-1}(\mu^*) + \beta_1\beta_2 \left[R^{-1}(\mu^*)\right]^2 \tag{19}$$

Two approximations have been proposed, which is referred to as tangent approximation and secant approximation in the following:

2.3.1. Tangent Approximation

Let α_1 and α_2 are small compared to $\overline{x}_1$ and $\overline{x}_2$ and μ^* is in the neighborhood of 1. Then we can neglect the quadratic term $\left[R^{-1}(\mu^*)\right]^2$ in (19) and we obtain for the right-hand curve $\mu_r(z)$ of the approximated product $\tilde{q}_t$ an expression of the form

$$\mu_r(z^*) = \mu^* = R\left[\frac{z^* - \overline{z}}{\beta}\right] \text{ with } \overline{z} = \overline{x}_1\overline{x}_2 \text{ and } \beta = \overline{x}_1\beta_2 + \overline{x}_2\beta_1 \tag{20}$$

Using the same reasoning for the left-hand curves of $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{q}_t$, we deduce the following formula for the multiplication of L-R fuzzy numbers

$$\langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \cdot \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \approx \langle \overline{x}_1\overline{x}_2, \overline{x}_1\alpha_2 + \overline{x}_2\alpha_1, \overline{x}_1\beta_2 + \overline{x}_2\beta_1 \rangle_{L,R} \tag{21}$$

2.3.2. Secant Approximation

If the spreads are not negligible compared to the modal values $\overline{x}_1$ and $\overline{x}_2$, the rough shape of the product $\tilde{q} = \tilde{p}_1\tilde{p}_2$ can be estimated by approximating quadratic term $\left[R^{-1}(\mu^*)\right]^2$ in (19) by the linear term $\left[R^{-1}(\mu^*)\right]$. This gives the right-hand curve $\mu_r(z)$ of the approximated product $\tilde{q}_s$ in the form

$$\mu_r(z^*) = \mu^* = R\left[\frac{z^* - \overline{z}}{\beta}\right] \text{ with } \overline{z} = \overline{x}_1\overline{x}_2 \text{ and } \beta = \overline{x}_1\beta_2 + \overline{x}_2\beta_1 + \beta_1\beta_2 \tag{22}$$

With the same reasoning for the left-hand curves of $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{q}_s$, the overall formula for the multiplication of L-R fuzzy numbers results in

$$\langle \bar{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \cdot \langle \bar{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \approx \langle \bar{x}_1 \bar{x}_2, \bar{x}_1 \alpha_2 + \bar{x}_2 \alpha_1 - \alpha_1 \alpha_2, \bar{x}_1 \beta_2 + \bar{x}_2 \beta_1 + \beta_1 \beta_2 \rangle_{L,R} \tag{23}$$

2.4. Division of L-R Fuzzy Number

An appropriate formulation for the quotient $\tilde{q} = E_d(\tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 / \tilde{p}_2$ of two L-R fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ can be obtained by reducing the division of the fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$ to the multiplication of the dividend $\tilde{p}_1$ with the inverse $\tilde{p}_2^{-1} = 1/\tilde{p}_2$ of the divisor $\tilde{p}_2$.

When we consider a fuzzy number $\tilde{p}$ which is either positive or negative, *i.e.*, $0 \notin supp(\tilde{p})$, given by the L-R representation

$$\tilde{p} = \langle \bar{x}, \alpha, \beta \rangle_{L,R}$$

the tangent approximation $(\tilde{p}^{-1})_t$ for the inverse $\tilde{p}^{-1}$ is defined by

$$(\tilde{p}^{-1})_t = \left\langle \frac{1}{\bar{x}}, \frac{\beta}{\bar{x}^2}, \frac{\alpha}{\bar{x}^2} \right\rangle_{R,L} \approx \tilde{p}^{-1}$$

and the secant approximation $(\tilde{p}^{-1})_s$ by

$$(\tilde{p}^{-1})_s = \left\langle \frac{1}{\bar{x}}, \frac{\beta}{\bar{x}(\bar{x}+\beta)}, \frac{\alpha}{\bar{x}(\bar{x}-\alpha)} \right\rangle_{R,L} \approx \tilde{p}^{-1}$$

Using the above mentioned identity $\tilde{p}_1 / \tilde{p}_2 = \tilde{p}_1 \tilde{p}_2^{-1}$ as well as the approximation formulas for the multiplication of L-R fuzzy numbers on one side and those for the inverse of an L-R fuzzy number on the other, a number of different approximated L-R representations for the quotient $\tilde{p}_1 / \tilde{p}_2$ can be formulated.

3. Example

We consider two L-R fuzzy number

$$\tilde{p}_1 = \langle 2,1,1 \rangle_{l,l} \text{ and } \tilde{p}_2 = \langle 4,2,4 \rangle_{l,l}$$

Then using Equation (7) we get

$$\tilde{p}_1 + \tilde{p}_2 = \tilde{q} = \langle 2+4, 1+2, 1+4 \rangle_{l,l} = \langle 6,3,5 \rangle_{l,l}$$

Also can be written in the form

$$\tilde{q} = \begin{cases} 0; & x \le 3 \\ \frac{x-3}{3}; & 3 < x < 6 \\ \frac{11-x}{5}; & 6 \le x < 11 \\ 0; & x \ge 11 \end{cases} = \text{tfn}(6,3,5).$$

Using (15) we get

$$\tilde{p}_1 - \tilde{p}_2 = \tilde{q} = \langle 2-4, 1+4, 1+2 \rangle_{l,l} = \langle -2,5,3 \rangle_{l,l}$$

Also can be written in the form

$$\tilde{q} = \begin{cases} 0; & x \le -7 \\ \frac{x+7}{5}; & -7 < x < -2 \\ \frac{1-x}{3}; & -2 \le x < 1 \\ 0; & x \ge 1 \end{cases} = \text{tfn}(-2,5,3).$$

If we use the tangent approximation the product $\tilde{q}=\tilde{p}_1\tilde{p}_2$ is approximated by the triangular L-R fuzzy number

$$\tilde{q}_t=\langle 8,8,12\rangle_{l,l}=\text{tnf}\,(8,8,12)=\begin{cases}0; & x\le 0\\ \dfrac{x}{8}; & 0<x<8\\ \dfrac{20-x}{12}; & 8\le x<20\\ 0; & x\ge 20\end{cases}$$

Again in the case of secant approximation the result $\tilde{q}=\tilde{p}_1\tilde{p}_2$ is approximated by

$$\tilde{q}_s=\langle 8,6,16\rangle_{l,l}=\text{tnf}\,(8,6,16)=\begin{cases}0; & x\le 2\\ \dfrac{x-2}{6}; & 2<x<8\\ \dfrac{24-x}{16}; & 8\le x<24\\ 0; & x\ge 24\end{cases}$$

If we use the tangent approximation the inverse p_2^{-1} is approximated by the triangular L-R fuzzy number

$$\left(p_2^{-1}\right)_t=\left\langle\frac{1}{4},\frac{4}{16},\frac{2}{16}\right\rangle_{l,l}=\left\langle\frac{1}{4},\frac{1}{4},\frac{1}{8}\right\rangle_{l,l}$$

Thus

$$\frac{\tilde{p}_1}{\tilde{p}_2}=\tilde{q}_{tt}=\langle 2,1,1\rangle_{l,l}\cdot\left\langle\frac{1}{4},\frac{1}{4},\frac{1}{8}\right\rangle_{l,l}=\left\langle 2\cdot\frac{1}{4},2\cdot\frac{1}{4}+\frac{1}{4}\cdot 1,2\cdot\frac{1}{8}+\frac{1}{4}\cdot 1\right\rangle_{l,l}$$

$$=\left\langle\frac{1}{2},\frac{3}{4},\frac{1}{2}\right\rangle_{l,l}=\text{tfn}\left(\frac{1}{2},\frac{3}{4},\frac{1}{2}\right)=\begin{cases}0; & x\le -\dfrac{1}{4}\\ \dfrac{4x+1}{3}; & -\dfrac{1}{4}<x<\dfrac{1}{2}\\ 2(1-x); & \dfrac{1}{2}\le x<1\\ 0; & x\ge 1\end{cases}$$

But if we use the secant approximation the inverse p_2^{-1} is approximated by the triangular L-R fuzzy number

$$\left(p_2^{-1}\right)_s=\left\langle\frac{1}{4},\frac{4}{4(4+4)},\frac{2}{4(4+2)}\right\rangle_{l,l}=\left\langle\frac{1}{4},\frac{1}{8},\frac{1}{4}\right\rangle_{l,l}$$

Thus

$$\frac{\tilde{p}_1}{\tilde{p}_2}=\tilde{q}_{ss}=\langle 2,1,1\rangle_{l,l}\cdot\left\langle\frac{1}{4},\frac{1}{8},\frac{1}{4}\right\rangle_{l,l}=\left\langle 2\cdot\frac{1}{4},2\cdot\frac{1}{8}+\frac{1}{4}\cdot 1-1\cdot\frac{1}{8},2\cdot\frac{1}{4}+\frac{1}{4}\cdot 1+1\cdot\frac{1}{4}\right\rangle_{l,l}$$

$$=\left\langle\frac{1}{2},\frac{3}{8},1\right\rangle_{l,l}=\text{tfn}\left(\frac{1}{2},\frac{3}{8},1\right)=\begin{cases}0; & x\le 1/8\\ \dfrac{8x-1}{3}; & 1/8<x<1/2\\ \dfrac{3-2x}{2}; & 1/2\le x<3/2\\ 0; & x\ge 3/2\end{cases}$$

4. Conclusion

In this paper we have presented exact calculation formulas for addition, subtraction, multiplication and division of two L-R fuzzy numbers. Finally we have taken two L-R fuzzy numbers as an example and obtained results of addition, subtraction, multiplication and division. We have reviewed some research papers with proper references.

References

[1] Klir, G.J. and Yuan, B. (1995) Fuzzy Sets and Fuzzy Logic, Theory and Applications. Prentice Hall Inc., Upper Saddle River.

[2] Dubis, D. and Prade, H. (1980) Fuzzy Sets and Systems, Theory and Applications. Academic Press, New York. http://www.jstor.org/stable/2030767

[3] Zadeh, L.A. (1965) Fuzzy Sets. *Information and Control*, **8**, 338-353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X

[4] Hanss, M. (2005) Applied Fuzzy Arithmetic—An Introduction with Engineering Applications. Springer-Verlag, Berlin Heidelberg.

[5] Zimmermann, H.J. (1995) Fuzzy Set Theory and its Applications. Third Edition, Kluwer Academic Publishers, Boston.