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Relation between Two Operator Inequalities $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\geq B^{-1}$ and $A^{-1}\geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)$

Mohammad Ilyas[1], Reyaz Ahmad[2], Shadab Ilyas[3]

[1]Department of Mathematics, Gaya College, Gaya, India
[2]Al-Ain University of Science and Technology, Al Ain, UAE
[3]Department of Information Technology, Gaya College, Gaya, India
Email: milyas347@gmail.com, reyaz56@hotmail.com, shadabilyas@gmail.com

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Abstract

We shall show relation between two operator inequalities $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\geq B^{-1}$ and $A^{-1}\geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)$ for positive, invertible operators *A* and *B*, where *f* and *g* are non-negative continuous invertible functions on $(0,\infty)$ satisfying $f(t)g(t)=t^{-1}$.

Keywords

Operator Inequality, Orthoprojection, Representing Function

1. Introduction

We denote by capital letter *A*, *B et al.* the bounded linear operators on a complex Hilbert space *H*. An operator *T* on *H* is said to be positive, denoted by $T\geq 0$ if $(Tx,x)\geq 0$ for all $x\in H$.

M. Ito and T. Yamazaki [1] obtained relations between two inequalities

$$\left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{r}{p+r}}\geq B^{r} \quad \text{and} \quad A^{p}\geq\left(A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}}\right)^{\frac{p}{p+r}}, \tag{1.1}$$

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and Yamazaki and Yanagida [2] obtained relation between two inequalities

$$\frac{p}{p+r}I+\frac{r}{p+r}B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\geq B^{r} \text{ and } A^{p}\geq A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}}\left(\frac{p}{p+r}I+\frac{r}{p+r}A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}}\right)^{-1}, \tag{1.2}$$

for (not necessarily invertible) positive operators *A* and *B* and for fixed $p\geq 0$ and $r\geq 0$. These results led M. Ito [3] to obtain relation between two operator inequalities

$$f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\geq B \text{ and } A\geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right), \tag{1.3}$$

for (not necessarily invertible) positive operators *A* and *B*, where *f* and *g* are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$.

Remarks (1.1): The two inequalities in (1.1) are closely related to Furuta inequalities [4].

The inequalities in (1.1) and (1.2) are equivalent, respectively, if *A* and *B* are invertibles; but they are not always equivalent. Their equivalence for invertible case was shown in [5].

Motivated by the result (1.3) of M. Ito [3], we obtain the results taking representing functions *f* and *g* as non-negative continuous invertible functions on $(0,\infty)$ satisfying $f(t)g(t)=t^{-1}$.

2. Main Results

We denote by $N(T)$ the kernel of an operator *T*.

Theorem 1: Let *A* and *B* be positive invertible operators, and let *f* and *g* be non-negative invertible continuous functions on $(0,\infty)$ satisfying $f(t)g(t)=t^{-1}$. Then the following hold:

1) $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\geq B^{-1}$ ensures $A^{-1}-g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)\geq A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}-g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$

2) $B^{-1}\geq f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)$ ensures $g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)-A^{-1}\geq g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}-A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}$.

Here $E_{B^{-1}}$ and $E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$ denote orthoprojections to $N(B^{-1})$ and $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)$ respectively.

The following Lemma is helpful in proving our results:

Lemma 2: If $h(t)$ is a continuous function on $(0,r^2)$ and *T* is an invertible operator with $\|T\|\leq r$, then $\frac{1}{T^*T}h(T^*T)=\frac{1}{T}h(TT^*)\frac{1}{T^*}$.

Proof of Lemma: Since $h(t)$ is a continuous function on $[0,r^2]$, it can be uniformly approximated by a sequence of polynomials on $[0,r^2]$. We may assume that $h(t)$ itself is a polynomial $h(T)=\sum_{k=0}^{n}\alpha_k t^k$. Then

$$\begin{aligned}h(T^*T)\cdot T^*T&=\sum_{k=0}^{n}\alpha_k(T^*T)^k\cdot T^*T\\&=T^*\left[\sum_{k=0}^{n}\alpha_k(TT^*)^k\right]\cdot T\\&=T^*h(TT^*)\cdot T\end{aligned}$$

$$\Rightarrow (T^*T)^{-1}\left[h(T^*T)\cdot T^*T\right](T^*T)^{-1}=(T^*T)^{-1}T^*h(TT^*)\cdot T(T^*T)^{-1}$$

$$\Rightarrow \frac{1}{T^*T}h(T^*T)=\frac{1}{T}h(TT^*)\frac{1}{T^*}.$$

Hence the result.

Proof of Theorem 1: For $\in>0$, let $f_{\in}(t)=f(t)+\in$ and $g_{\in}(t)=\frac{1}{tf_{\in}(t)}=\frac{1}{t\left[f(t)+\in\right]}$; $0<t<\infty$

1) We suppose that $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) \geq B^{-1}$. Then

$$f_{\in}\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) = f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) + \in I \geq B^{-1} + \in I.$$

Let $h_{\in}(t) = \frac{1}{f_{\in}(t)}$ and $T = B^{\frac{1}{2}}A^{\frac{1}{2}}$ then

$$\begin{aligned}
A^{-1} - g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) &= A^{-1} - \frac{1}{\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) f_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)} \\
&= A^{-1} - \frac{1}{T^{*}Tf_{\in}\left(T^{*}T\right)} \\
&= A^{-1} - \frac{1}{T} h_{\in}\left(TT^{*}\right) \cdot \frac{1}{T^{*}} \\
&= A^{-1} - \frac{1}{B^{\frac{1}{2}}A^{\frac{1}{2}}} \cdot \frac{1}{f_{\in}\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)} \cdot \frac{1}{A^{\frac{1}{2}}B^{\frac{1}{2}}} \\
&\geq A^{-1} - \frac{1}{B^{\frac{1}{2}}A^{\frac{1}{2}}} \frac{I}{B^{-1} + \in I} \cdot \frac{1}{A^{\frac{1}{2}}B^{\frac{1}{2}}} \\
&= A^{-\frac{1}{2}}\left[I - B^{-\frac{1}{2}} \cdot \frac{B}{I + \in B} \cdot B^{-\frac{1}{2}}\right] A^{-\frac{1}{2}} \\
&= A^{-\frac{1}{2}} \cdot \frac{\in B}{I + \in B} \cdot A^{-\frac{1}{2}} \\
&= A^{-\frac{1}{2}} \cdot \frac{\in}{B^{-1} + \in I} \cdot A^{-\frac{1}{2}} \\
&= A^{-\frac{1}{2}} \cdot \in \left(B^{-1} + \in I\right)^{-1} \cdot A^{-\frac{1}{2}}.
\end{aligned}$$

We have $\underset{\in \to 0}{\mathrm{lt}} \in \left(B^{-1} + \in I\right)^{-1} = E_{B^{-1}}$.

Further since $g_{\in}(t)$ increases as $\in$ decreases and

$$\underset{\in \to 0}{\mathrm{lt}}\, g_{\in}(t) = \begin{cases} g(t), & \text{when } t \neq 0 \\ 0. & \text{when } t \to \infty, \end{cases}$$

we have

$$\underset{\in \to 0}{\mathrm{lt}} \left\{A^{-1} - g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)\right\} = A^{-1} - \left\{g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) - g(\infty) E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}\right\}.$$

Then

$$A^{-1} - \left\{g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) - g(\infty) E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}\right\} = \underset{\in \to 0}{\mathrm{lt}} \left\{A^{-1} - g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)\right\} \geq \underset{\in \to 0}{\mathrm{lt}} \in A^{-\frac{1}{2}}\left(B^{-1} + \in I\right)^{-1} A^{-\frac{1}{2}}$$

i.e.

$$A^{-1} - g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) \geq A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}} - g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$$
$$= A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}} - g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}.$$

2) We suppose that $B^{-1} \geq f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)$; *i.e.* $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) \leq B^{-1}$, then

$$f_{\in}\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) = f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) + \in I \leq B^{-1} + \in I.$$

With $h_{\in}(t) = \dfrac{1}{f_{\in}(t)}$ and $T = B^{\frac{1}{2}}A^{\frac{1}{2}}$, we have by Lemma 2

$$g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) - A^{-1} = g_{\in}\left(T^*T\right) - A^{-1}$$
$$= \frac{1}{T^*Tf_{\in}\left(T^*T\right)} - A^{-1}$$
$$= \frac{1}{T^*T}h_{\in}\left(T^*T\right) - A^{-1}$$
$$= \frac{1}{T}h_{\in}\left(TT^*\right)\cdot\frac{1}{T^*} - A^{-1}$$
$$= \frac{1}{B^{\frac{1}{2}}A^{\frac{1}{2}}}\cdot h_{\in}\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\frac{1}{A^{\frac{1}{2}}B^{\frac{1}{2}}} - A^{-1}$$
$$= \frac{1}{B^{\frac{1}{2}}A^{\frac{1}{2}}}\cdot\frac{I}{f_{\in}\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)}\cdot\frac{1}{A^{\frac{1}{2}}B^{\frac{1}{2}}} - A^{-1}$$
$$\geq \frac{1}{B^{\frac{1}{2}}A^{\frac{1}{2}}}\cdot\frac{I}{\left(B^{-1} + \in I\right)}\cdot\frac{1}{A^{\frac{1}{2}}B^{\frac{1}{2}}} - A^{-1}$$
$$= A^{-\frac{1}{2}}\left[B^{-\frac{1}{2}}\frac{B}{I + \in B}B^{-\frac{1}{2}} - I\right]A^{-\frac{1}{2}}$$
$$= -A^{-\frac{1}{2}}\frac{\in}{B^{-1} + \in I}A^{-\frac{1}{2}}$$
$$= -\in A^{-\frac{1}{2}}\left(B^{-1} + \in I\right)^{-1}A^{-\frac{1}{2}}.$$

Now as $\operatorname{lt}_{\in \to 0} \in\left(B^{-1} + \in I\right) = E_{B^{-1}}$ and since

$$\operatorname{lt}_{\in \to 0} g_{\in}(t) = \begin{cases} g(t), & \text{when } t \neq 0 \\ 0. & \text{when } t \to \infty, \end{cases}$$

we have

$$\operatorname{lt}_{\in \to 0}\left\{g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) - A^{-1}\right\} = g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right) - g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}} - A^{-1}.$$

Then

$$g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)-g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}-A^{-1}=\underset{\in\to 0}{\text{lt}}\left\{g_{\in}\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)-A^{-1}\right\}$$
$$\geq \underset{\in\to 0}{\text{lt}}-\in A^{-\frac{1}{2}}\left(B^{-1}+\in I\right)^{-1}A^{-\frac{1}{2}}$$
$$=-A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}$$
$$\Rightarrow g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)-A^{-1}\geq g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}-A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}.$$

thus completing the proof of 2.

Corollary 3. Let A and B be positive invertible operators, and let f and g be non-negative continuous invertible functions on $(0,\infty)$ satisfying $f(t)g(t)=t^{-1}$.

1) If $g(\infty)=0$ or $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)=\{0\}$, then $f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)\geq B^{-1}$ ensures $A^{-1}\geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)$.

2) If $N\left(B^{-1}\right)\subseteq N\left(A^{-1}\right)$, then $B^{-1}\geq f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)$ ensures $g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)\geq A^{-1}$.

Proof 1) This result follows from 1) of Theorem 1 because each of the conditions $g(\infty)=0$ and $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)=\{0\}$ implies $g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}=0$, so that

$$A^{-1}-g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)\geq A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}-g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}=A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}\geq 0$$
$$\Rightarrow A^{-1}\geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right).$$

2) This result follows from 2) of Theorem (1) because $N\left(B^{-1}\right)\subseteq N\left(A^{-1}\right)\Rightarrow A^{-\frac{1}{2}}E_{B^{-1}}=0$, so that

$$g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)-A^{-1}\geq g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}-A^{-\frac{1}{2}}E_{B^{-1}}A^{-\frac{1}{2}}$$
$$=g(\infty)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}\geq 0.$$

Hence the proof is complete.

Remark (3.1) 1) If $f(\infty)>0$, then automatically $g(\infty)=0$ since $f(\infty)g(\infty)=\frac{1}{\infty}=0$, so 1) of corollary 3 holds without any conditions.

2) The invertibility of positive operators A and B is necessary condition.

3) We have considered $(0,\infty)$ instead of $[0,\infty)$ because the requirement of the limit.

$\underset{\in\to 0}{\text{lt}}\, g_{\in}(t)=0$ when $t=0$ is not fulfilled, rather it is fulfilled when $t\to\infty$ because $g_{\in}(t)=\frac{1}{tf_{\in}(t)}$.

We have the following results as a consequence of corollary 3.

Theorem 4: Let A and B be positive invertible operators. Then for each $p\geq 0$ and $r>0$, the following hold

1) If $\left(B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\right)^{\frac{-r}{p+r}}\geq B^{r}$ then $A^{-p}\geq\left(A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\right)^{\frac{-p}{p+r}}$.

2) If $A^{-p}\ge\left(A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\right)^{\frac{-p}{p+r}}$ and $N\left(A^{-1}\right)\subseteq N\left(B^{-1}\right)$ then $\left(B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\right)^{\frac{-r}{p+r}}\ge B^{r}$.

In Theorem 4 we consider that $t^0=1$ for $t>0$ or $t^0=0$ when $t\to\infty$ and we define $T^0=I-E_T$ for a positive invertible operator *T*.

Theorem 5: Let *A* and *B* be positive invertible operators. Then for each $p>0$ and $r>0$, the following hold:

1) If $\frac{-p}{p+r}I-\frac{r}{p+r}B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\ge B^{r}$, then $A^{-p}\ge A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\left(\frac{-r}{p+r}A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}-\frac{p}{p+r}I\right)^{-1}$.

2) If $A^{-p}\ge A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\left(\frac{-r}{p+r}A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}-\frac{p}{p+r}I\right)^{-1}$ and $N\left(A^{-1}\right)\subseteq N\left(B^{-1}\right)$, then

$\frac{-p}{p+r}I-\frac{r}{p+r}B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\ge B^{r}$.

Proof of Theorem 4: 1) First we consider the case when $p>0$ and $r>0$. Replacing *A* with A^p and *B* with B^{-r} and putting $f(t)=t^{\frac{-r}{p+r}}$ and $g(t)=t^{\frac{-p}{p+r}}$ in 1) of Corollary 3 so that $f(t)g(t)=t^{-1}$, we have

$$\text{if }\left(B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\right)^{\frac{-r}{p+r}}\ge B^{r}\text{ then }A^{-p}\ge\left(A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\right)^{\frac{-p}{p+r}}. \tag{5.1}$$

If $p=0$ and $r>0$ (5.1) means that

if $\left[B^{-\frac{r}{2}}\left(I-E_A\right)B^{-\frac{r}{2}}\right]^{-1}\ge B^{r}$ then $I-E_A\ge I-E_{(I-E_A)B^{-r}(I-E_A)}$

i.e., if $\left[B^{-r}-B^{-\frac{r}{2}}E_AB^{-\frac{r}{2}}\right]^{-1}\ge B^{r}$ then $I-E_A\ge I-E_{(I-E_A)B^{-r}(I-E_A)}$

i.e., if $\left(I-E_A\right)^{-1}\ge I$ then $I-E_A\ge I-E_{(I-E_A)B^{-r}(I-E_A)}$

i.e., if $\left(I-E_A\right)\le I$ then $I-E_A\ge I-E_{(I-E_A)B^{-r}(I-E_A)}$

or in other words, $B^{-\frac{r}{2}}E_AB^{-\frac{r}{2}}=0$ ensures $E_{(I-E_A)B^{-r}(I-E_A)}\ge E_A$.

But, since $B^{-\frac{r}{2}}E_AB^{-\frac{r}{2}}=0$ implies $\left(I-E_A\right)B^{-r}\left(I-E_A\right)=B^{-r}$, it follows an equivalent assertion $B^{-\frac{r}{2}}E_AB^{-\frac{r}{2}}=0$ ensures $E_{B^{-r}}\ge E_A$, *i.e.*, $E_{B^{-1}}=E_{B^{-r}}\ge E_A$ which is further equivalent to the trivial assertion $N(A)\subseteq N\left(B^{-1}\right)$ ensures $N(A)\subseteq N\left(B^{-1}\right)$.

2) Again first we consider the case $p>0$ and $r>0$. Replacing *A* with B^{-r} and *B* with A^p and putting $f(t)=t^{\frac{-p}{p+r}}$ and $g(t)=t^{\frac{-r}{p+r}}$ in 2) of Corollary 3.

Since $N\left(A^{-p}\right)=N\left(A^{-1}\right)\subseteq N\left(B^{-1}\right)=N\left(B^{-r}\right)$, we have

$$A^{-p}\ge\left(A^{\frac{p}{2}}B^{-r}A^{\frac{p}{2}}\right)^{\frac{-p}{p+r}}\text{ ensures }\left(B^{-\frac{r}{2}}A^{p}B^{-\frac{r}{2}}\right)^{\frac{-r}{p+r}}\ge B^{r}. \tag{5.2}$$

If $p=0$ and $r>0$, (5.2) means that $\left(I-E_A\right)\ge I-E_{(I-E_A)B^{-r}(I-E_A)}$ ensures $\left[B^{-\frac{r}{2}}\left(I-E_A\right)B^{-\frac{r}{2}}\right]^{-1}\ge B^{r}$ *i.e.*,

$\left(I-E_A\right) \geq I-E_{\left(I-E_A\right)B^{-r}\left(I-E_A\right)}$

$$\text{ensures } B^{-\frac{r}{2}}E_A B^{-\frac{r}{2}}=0, \tag{5.3}$$

which implies that $\left(I-E_A\right)B^{-r}\left(I-E_A\right)=B^{-r}$.

Hence (5.3) means that $E_{B^{-1}}=E_{B^{-r}} \geq E_A$ ensures $B^{-\frac{r}{2}}E_A B^{-\frac{r}{2}}=0$, *i.e.* $N\left(A\right) \subseteq N\left(B^{-1}\right)$ ensures $N\left(A\right) \subseteq N\left(B^{-1}\right)$.

Hence the result.

Proof of Theorem 5: We can prove by the similar way to Theorem 4 for $p>0$ and $r>0$, replacing A with A^p and B with B^{-r} and putting $f\left(t\right)=-\frac{p}{p+r}-\frac{r}{p+r}t$ and $g\left(t\right)=t\left(-\frac{r}{p+r}t-\frac{p}{p+r}\right)^{-1}$ for 1) in 1) of Corollary 3 and replacing A with B^{-r} and B with A^p and putting $f\left(t\right)=t\left(-\frac{r}{p+r}t-\frac{p}{p+r}\right)^{-1}$ and $g\left(t\right)=-\frac{p}{p+r}-\frac{r}{p+r}t$ for 2) in 2) of Corollary 3.

Corollary 4: Let A and B be positive invertible operators, and let f and g be non-negative continuous invertible functions on $\left(0,\infty\right)$ satisfying $f\left(t\right)g\left(t\right)=t^{-1}$. If $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)=\{0\}$, then

$$f\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right) \geq B^{-1} \Rightarrow A^{-1} \geq g\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right).$$

Proof: The proof $\left(\Rightarrow\right)$ follows directly by applying the condition $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)=\{0\}$, in 1) of Corollary 3 and for the proof $\left(\Leftarrow\right)$ we have only to interchange the roles of A and B and those of f and g in 2) of Corollary 3, Since $\{0\}=N\left(A^{-1}\right) \subseteq N\left(B^{-1}\right)$ if $N\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)=\{0\}$.

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