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# Argument Estimates of Multivalent Functions Involving a Certain Fractional Derivative Operator

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## Abstract

**The object of the present paper is to investigate various argument results of analytic and multivalent functions which are defined by using a certain fractional derivative operator. Some interesting applications are also considered.**

## Keywords

**Multivalent Analytic Functions, Argument, Integral Operator, Fractional Derivative Operator**

## 1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}, \quad \left(p\in\mathbb{N}:=\{1,2,3,\cdots\}\right), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$. Also let $\mathcal{A}(1)=\mathcal{A}$ denote the class of all analytic functions $p(z)$ with $p(0)=1$ which are defined on $\mathbb{U}$.

Let $a$, $b$ and $c$ be complex numbers with $c\neq 0,-1,-2,\cdots$. Then the *Gaussian hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined by

$$ {}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.2}$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

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$$(\eta)_k = \frac{\Gamma(\eta+k)}{\Gamma(\eta)} = \begin{cases} 1, & (k=0); \\ \eta(\eta+1)\cdots(\eta+k-1), & (k\in\mathbb{N}). \end{cases}$$

The hypergeometric function ${}_2F_1(a,b;c;z)$ is analytic in $\mathbb{U}$ and if $a$ or $b$ is a negative integer, then it reduces to a polynomial.

There are a number of definitions for fractional calculus operators in the literature (cf., e.g., [1] and [2]). We use here the Saigo type fractional derivative operator defined as follows ([3]; see also [4]):

**Definition 1.** Let $0\le\lambda<1$ and $\mu,\ \nu\in\mathbb{R}$. Then the generalized fractional derivative operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ of a function $f(z)$ is defined by

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-\zeta)^{-\lambda}\ {}_2F_1\left(\mu-\lambda,1-\nu;1-\lambda;1-\frac{\zeta}{z}\right) f(\zeta)\mathrm{d}\zeta\right). \tag{1.3}$$

The function $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, with the order

$$f(z) = O\left(|z|^{\epsilon}\right), \quad (z\to 0)$$

for $\epsilon > \max\{0,\mu-\nu\}-1$, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $\log(z-\zeta)$ to be real when $z-\zeta>0$.

**Definition 2.** Under the hypotheses of Definition 1, the fractional derivative operator $\mathcal{J}_{0,z}^{\lambda+m,\mu+m,\nu+m}$ of a function $f(z)$ is defined by

$$\mathcal{J}_{0,z}^{\lambda+m,\mu+m,\nu+m} f(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z), \quad \left(z\in\mathbb{U}; m\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}\right) \tag{1.4}$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ by

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^{\mu}\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \tag{1.5}$$

for $f(z)\in\mathcal{A}(p)$ and $\mu-\nu-p<1$. Then it is observed that $\Delta_{z,p}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}(p)$ onto itself as follows:

$$\begin{gathered}\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty}\frac{(p+1)_k(p+1-\mu+\nu)_k}{(p+1-\mu)_k(p+1-\lambda+\nu)_k} a_{k+p} z^{k+p}, \\ \left(z\in\mathbb{U}; 0\le\lambda<1; \mu-\nu-p<1; f\in\mathcal{A}(p)\right).\end{gathered} \tag{1.6}$$

It is easily verified from (1.6) that

$$z\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)' = (p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,\nu} f(z). \tag{1.7}$$

Note that $\Delta_{z,p}^{0,0,\nu} f = f$, $\Delta_{z,p}^{1,1,\nu} f = zf'/p$ and $\Delta_{z,p}^{\lambda,\lambda,\nu} f = \Omega_z^{(\lambda,p)} f$, where $\Omega_z^{(\lambda,p)}$ is the fractional derivative operator defined by Srivastava and Aouf [5].

In this manuscript, we drive interesting argument results of multivalent functions defined by fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$.

## 2. Main Results

In order to establish our results, we require the following lemma due to Lashin [6].

**Lemma 1** [6]. *Let* $h(z)$ *be analytic in* $\mathbb{U}$, *with* $h(0)=1$ *and* $h(z)\ne 0$ $(z\in\mathbb{U})$. *Further suppose that* $\alpha,\beta\in\mathbb{R}^+ = (0,\infty)$ *and*

$$\left|\arg\left(h(z)+\beta z h'(z)\right)\right| < \frac{\pi}{2}\left(\alpha+\frac{2}{\pi}\arctan(\beta\alpha)\right) \quad (\alpha>0;\beta>0) \tag{2.1}$$

*then*

$$\left|\arg h(z)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}). \tag{2.2}$$

We begin by proving the following result.

**Theorem 1.** *Let* $\lambda \ge 0$, $\mu < \min\{\nu + p + 1, p\}$ *and* $\alpha, \gamma, \delta \in \mathbb{R}^+$, *and let* $g(z) \in \mathcal{A}(p)$. *Suppose that* $f(z) \in \mathcal{A}(p)$ *satisfies the condition*

$$\begin{aligned} &\left|\arg\left(\left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right\}^{\gamma}\left\{1+\delta\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}-\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} g(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right)\right\}\right)\right| \\ &< \frac{\pi}{2}\left(\alpha+\frac{2}{\pi}\arctan\left[\frac{\delta}{\gamma(p-\mu)}\alpha\right]\right), \end{aligned} \tag{2.3}$$

*then*

$$\left|\arg\left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right\}^{\gamma}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}). \tag{2.4}$$

*Proof.* If we define the function $h(z)$ by

$$h(z) = \left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right\}^{\gamma}, \quad (\gamma \ne 0), \tag{2.5}$$

then $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$, with $h(0) = 1$ and $h'(0) \ne 0$. Making use of the logarithmic differentiation on both sides of (2.5), we have

$$\frac{1}{\gamma}\frac{zh'(z)}{h(z)} = \frac{z\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} - \frac{z\left(\Delta_{z,p}^{\lambda,\mu,\nu} g(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}. \tag{2.6}$$

By applying the identity (1.7) in (2.6), we observe that

$$h(z) + \frac{\delta}{\gamma(p-\mu)} zh'(z) = \left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right\}^{\gamma}\left\{1+\delta\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}-\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} g(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} g(z)}\right)\right\}.$$

Hence, by using Lemma 1, we conclude that

$$\left|\arg h(z)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 1.

**Remark 1.** Putting $\lambda = \mu = 0$, $\delta = p = 1$ and $g(z) = z$ in Theorem 1, we obtain the result due to Lashin ([6], Theorem 2.2).

Taking $\gamma = 1$ and $g(z) = z^p$ in Theorem 1, we have the following corollary.

**Corollary 1.** *Let* $\lambda \ge 0$, $\mu < \min\{\nu + p + 1, p\}$ *and* $\alpha, \delta \in \mathbb{R}^+$. *Suppose that* $f(z) \in \mathcal{A}(p)$ *satisfies the condition*

$$\left|\arg\left\{(1-\delta)\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} + \delta\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^p}\right\}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\alpha}{p-\mu}\right]\right),$$

*then*

$$\left|\arg\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}\right)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

**Theorem 2.** *Let* $\lambda \ge 0$, $\mu < \min\{\nu + p + 1, p\}$, $0 < \delta \le 1$ *and* $\alpha, \delta \in \mathbb{R}^+$. *Suppose that* $f(z) \in \mathcal{A}(p)$ *sa-*

*tisfies the condition*

$$\left|\arg\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{\gamma(p-\mu)}\alpha\right]\right), \quad (z \in \mathbb{U}). \tag{2.7}$$

*then*

$$\left|\arg\left(\frac{\gamma(p-\mu)}{\delta} z^{-\frac{\gamma(p-\mu)}{\delta}} \int_0^z t^{\frac{\gamma(p-\mu)-\delta(p+1)}{\delta}} \Delta_{z,p}^{\lambda,\mu,\nu} f(t)\mathrm{d}t\right)\right| < \frac{\pi}{2}\alpha. \tag{2.8}$$

*Proof*. If we set

$$h(z) = \frac{\gamma(p-\mu)}{\delta} z^{-\frac{\gamma(p-\mu)}{\delta}} \int_0^z t^{\frac{\gamma(p-\mu)-\delta(p+1)}{\delta}} \Delta_{z,p}^{\lambda,\mu,\nu} f(t)\mathrm{d}t, \tag{2.9}$$

then $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$, with $h(0) = 1$ and $h'(0) \neq 0$. By using the logarithmic differentiation on both sides of (2.9), we obtain

$$h(z) + \frac{\delta}{\gamma(p-\mu)} zh'(z) = \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}.$$

Thus, in view of Lemma 1, we have

$$\left|\arg h(z)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}),$$

which evidently proves Theorem 2.

**Remark 2.** Setting $\lambda = \mu = 0$ and $\gamma = \delta = p = 1$ in Theorem 2, we get the result obtained by Goyal and Goswami ([7], Corollary 3.6).

Putting $\lambda = \mu = \gamma = \delta = 1$ in Theorem 2, we obtain the following result.

**Corollary 2.** *Let* $\alpha \in \mathbb{R}^+$. *Suppose that* $f(z) \in \mathcal{A}(p)$ $(p \neq 1)$ *satisfies the condition*

$$\left|\arg\left(\frac{f'(z)}{pz^{p-1}}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left(\frac{\alpha}{p-1}\right)\right),$$

*then*

$$\left|\arg\left(\frac{p-1}{pz^{p-1}} \int_0^z \frac{f'(t)}{t}\mathrm{d}t\right)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Finally, we consider the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_\sigma(f)$ $(\sigma > -p)$ defined by (*cf.* [8] [9] and [10])

$$\mathcal{L}_\sigma(f) \equiv \mathcal{L}_\sigma(f)(z) := \frac{\sigma+p}{z^\sigma}\int_0^z t^{\sigma-1} f(t)\mathrm{d}t, \quad \left(f \in \mathcal{A}(p); \sigma > -p\right). \tag{2.10}$$

**Theorem 3.** *Let* $\lambda \geq 0$, $\mu < \min\{\nu + p + 1, p\}$, $\sigma > -p$ *and* $\alpha, \gamma, \delta \in \mathbb{R}^+$, *and let* $g(z) \in \mathcal{A}(p)$. *Suppose that* $f(z) \in \mathcal{A}(p)$ *satisfies the condition*

$$\left|\arg\left(\left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right\}^\gamma \left\{1 + \delta\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)} - \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} g(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)\right\}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{\gamma(\sigma+p)}\alpha\right]\right), \tag{2.11}$$

*then*

$$\left|\arg\left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right\}^\gamma\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}). \tag{2.12}$$

*Proof*. From (2.10) we observe that

$$z\left(\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)\right)' = (\sigma+p)\Delta_{z,p}^{\lambda,\mu,\nu}f(z) - \sigma\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z). \tag{2.13}$$

If we let

$$h(z) = \left\{\frac{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right\}^\gamma, \quad (\gamma \neq 0), \tag{2.14}$$

then $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$, with $h(0) = 1$ and $h'(0) \neq 0$. Differentiating both sides of (2.14) logarithmically, it follows that

$$\frac{1}{\gamma}\frac{zh'(z)}{h(z)} = \frac{z\left(\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)} - \frac{z\left(\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}. \tag{2.15}$$

Hence, by applying the same arguments as in the proof of Theorem 1 with (2.13) and (2.15), we obtain

$$\left|\arg h(z)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}),$$

which proves Theorem 3.

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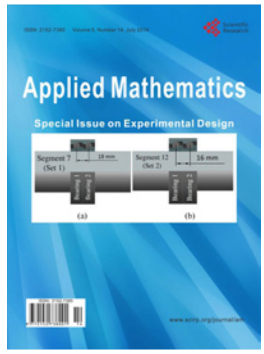
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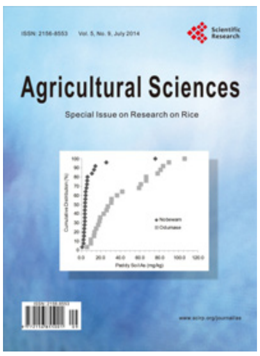
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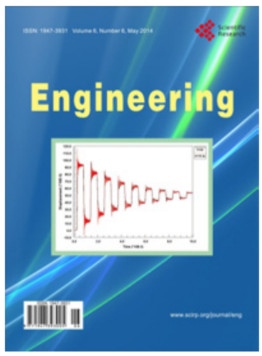
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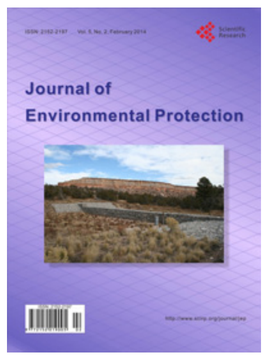
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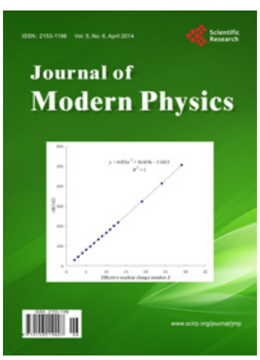
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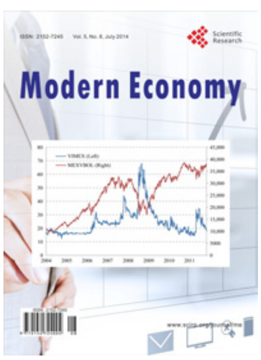
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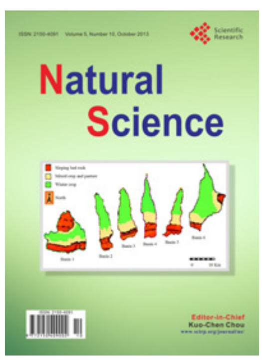
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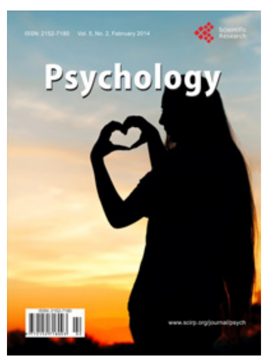
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