Advances in Pure Mathematics, 2015, 5, 71-81
Published Online February 2015 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2015.52009

Oscillatory and Asymptotic Behavior of Solutions of Second Order Neutral Delay Difference Equations with "Maxima"

Ramalingam Arul, Manvel Angayarkanni

Department of Mathematics, Kandaswami Kandar's College, Namakkal, India
Email: drrarul@gmail.com, angayarkanni66@rediffmail.com

Received 14 January 2015; accepted 2 February 2015; published 6 February 2015

Abstract

In this paper, we study the oscillatory and asymptotic behavior of second order neutral delay difference equation with "maxima" of the form

$$\Delta\left(a_n\Delta\left(x_n+p_nx_{n-\tau}\right)\right)+q_n\max_{[n-\sigma,n]}x_s^{\alpha}=0,\quad n\in\mathbb{N}\left(n_0\right).$$

Examples are given to illustrate the main result.

Keywords

Second Order, Oscillatory, Asymptotic Behavior, Neutral Delay Difference Equations with "Maxima"

1. Introduction

Consider the oscillatory and asymptotic behavior of second order neutral delay difference equation with "maxima" of the form

$$\Delta\left(a_n\Delta\left(x_n+p_nx_{n-\tau}\right)\right)+q_n\max_{[n-\sigma,n]}x_s^{\alpha}=0,\quad n\in\mathbb{N}\left(n_0\right),\tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$ and $\mathbb{N}\left(n_0\right)=\left\{n_0,n_0+1,n_0+2,\cdots\right\}$ and n_0 is a nonnegative integer subject to the following conditions:

(C_1) τ and σ are positive integers;
(C_2) α is a ratio of odd positive integers;
(C_3) $\left\{p_n\right\}$ and $\left\{q_n\right\}$ are nonnegative real sequences with $\Delta p_n\ge 0$ and $0\le p_n\le p<1$ for all $n\ge n_0$;

How to cite this paper: Arul, R. and Angayarkanni, M. (2015) Oscillatory and Asymptotic Behavior of Solutions of Second Order Neutral Delay Difference Equations with "Maxima". *Advances in Pure Mathematics*, **5**, 71-81.
http://dx.doi.org/10.4236/apm.2015.52009

(C_4) $\{a_n\}$ is a positive real sequence such that $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$.

Let $\theta=\max\{\tau,\sigma\}$. By a solution of Equation (1), we mean a real sequence $\{x_n\}$ satisfying Equation (1) for all $n\ge n_0-\theta$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

From the review of literature it is well known that there is a lot of results available on the oscillatory and asymptotic behavior of solutions of neutral difference equations, see [1]-[5], and the references cited therein. But very few results are available in the literature dealing with the oscillatory and asymptotic behavior of solutions of neutral difference equations with "maxima", see [6]-[9], and the references cited therein. Therefore, in this paper, we investigate the oscillatory and asymptotic behavior of all solutions of Equation (1). The results obtained in this paper extend that in [4] for equation without "maxima".

In Section 2, we obtain some sufficient conditions for the oscillation of all solutions of Equation (1). In Section 3, we present some sufficient conditions for the existence of nonoscillatory solutions for the Equation (1) using contraction mapping principle. In Section 4, we present some examples to illustrate the main results.

2. Oscillation Results

In this section, we present some new sufficient conditions for the oscillation of all solutions of Equation (1). Throughout this section we use the following notation without further mention:

$$z_n=x_n+p_nx_{n-\tau},$$

$$A_n=\sum_{s=n_0}^{n-1}\frac{1}{a_s},$$

$$B_n=\sum_{s=n-\tau}^{\infty}\frac{1}{a_s},$$

and

$$C_n=a_n\sum_{s=n_0}^{n-1}\frac{1}{a_{s-\sigma}}.$$

Lemma 2.1. *Let* $\{x_n\}$ *be an eventually positive solution of Equation* (1). *Then one of the following holds*

(I) $z_n>0,\Delta z_n>0$ and $\Delta(a_n\Delta z_n)\le 0$;

(II) $z_n>0,\Delta z_n<0$ and $\Delta(a_n\Delta z_n)\le 0$.

Proof. Let $\{x_n\}$ be an eventually positive solution of Equation (1). Then we may assume that $x_{n-\sigma}>0$, $x_{n-\tau}>0$ for all $n\ge n_0$. Then inview of (C_3) we have $z_n>0$ for all $n\ge n_0$. From the Equation (1), we obtain

$$\Delta(a_n\Delta z_n)=-q_n\max_{[n-\sigma,n]}x_s^{\alpha}\le 0.$$

Hence $a_n\Delta z_n$ and z_n are of eventually of one sign. This completes the proof. □

Lemma 2.2. *Let* $\{x_n\}$ *be an eventually negative solution of Equation* (1). *Then one of the following holds*

(I) $z_n<0,\Delta z_n<0$ and $\Delta(a_n\Delta z_n)\ge 0$;

(II) $z_n<0,\Delta z_n>0$ and $\Delta(a_n\Delta z_n)\ge 0$.

Proof. The proof is similar to that of Lemma 2.1. □

Lemma 2.3. *The sequence* $\{x_n\}$ *is an eventually negative solution of Equation* (1) *if and only if* $\{-x_n\}$ *is an eventually positive solution of the equation*

$$\Delta\left(a_n\Delta\left(x_n+p_nx_{n-\tau}\right)\right)+q_n\max_{[n-\sigma,n]}x_s^{\alpha}=0.$$

The assertion of Lemma 2.3 can be verified easily.

Lemma 2.4. *Let* $\{x_n\}$ *be an eventually positive solution of Equation* (1) *and suppose Case* (*I*) *of Lemma* 2.1 *holds. Then there exists* $N\in\mathbb{N}(n_0)$ *such that*

$$(1-p_n)z_n\le x_n\le z_n \text{ for all } n\ge N.$$

Proof. From the definition of z_n and condition (C_3), we have $z_n \le x_n$. Further $x_n = z_n - p_n x_{n-k} \ge z_n - p_n z_{n-k} \ge (1-p_n) z_n$, since $\{z_n\}$ is nondecreasing. This completes the proof. □

Lemma 2.5. *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1) *and suppose Case* (I) *of Lemma* 2.1 *holds. Then there exists* $N \in \mathbb{N}(n_0)$ *such that*

$$z_{n-\sigma} \ge C_n \Delta z_n \text{ for all } n \ge N.$$

Proof. Since $\Delta(a_n \Delta z_n) \le 0$, we see that

$$z_{n-\sigma} = z_{N-\sigma} + \sum_{s=N}^{n-1} \Delta z_{s-\sigma} \ge a_{n-\sigma} \Delta z_{n-\sigma} \sum_{s=N}^{n-1} \frac{1}{a_{s-\sigma}}$$

or

$$z_{n-\sigma} \ge a_n \Delta z_n \sum_{s=N}^{n-1} \frac{1}{a_{s-\sigma}}.$$

The proof is now complete. □

Lemma 2.6. *Let* $\{x_n\}$ *be an eventually positive solution of Equation* (1) *and suppose Case* (II) *of Lemma* 2.1 *holds. Then there exists* $N \in \mathbb{N}(n_0)$ *such that* $\{x_n\}$ *is nonincreasing for all* $n \ge N$.

Proof. Since $\Delta p_n \ge 0$ and $\Delta z_n = \Delta x_{n-k} + \Delta p_n x_{n-k} + p_{n+1} \Delta x_n < 0$ then we have $\Delta x_n \le 0$ for $n \ge N$. This completes the proof.

Theorem 2.1. *Assume that* $\alpha \ge 1$, *and there exists a positive integer k such that* $\sigma \ge \tau - k$. *If for all sufficiently large* $n_1 \in \mathbb{N}(n_0)$ *and for all constants* $M > 0$, $L > 0$. *One has*

$$\sum_{n=n_1}^{\infty} A_{n+1} q_n \max_{[n-\sigma,n]} (1-p_s)^{\alpha} = \infty, \tag{2}$$

and

$$\sum_{n=n_1}^{\infty} \left[B_{n+1}^{\alpha} q_n \max_{[n-\sigma,n]} \left(\frac{1}{1+p_{n-\tau}} \right)^{\alpha} - \frac{\alpha}{L^{\alpha-1} B_n a_{n-\tau}} \right] = \infty, \tag{3}$$

then every solution of Equation (1) *is oscillatory*.

Proof. Assume to the contrary that there exists a nonoscillatory solution $\{x_n\}$ of Equation (1). Without loss of generality we may assume that $x_{n-\theta} > 0$ for all $n \ge N \in \mathbb{N}(n_0)$, where N is chosen so that both the cases of Lemma 2.1 hold for all $n \ge N$. We shall show that in each case we are led to a contradiction.

Case(I). From Lemma 2.4 and Equation (1), we have

$$\Delta\left(a_n (\Delta z_n)\right) + q_n z_n^{\alpha} \max_{[n-\sigma,n]} (1-p_s)^{\alpha} \le 0$$

or

$$\frac{\Delta\left(a_n (\Delta z_n)\right)}{z_n^{\alpha}} \le -q_n \max_{[n-\sigma,n]} (1-p_s)^{\alpha}. \tag{4}$$

Define $w_n = A_n \frac{a_n \Delta z_n}{z_n^{\alpha}}$, $n \ge N \ge n_0 + k$, then we have

$$\begin{aligned}
\Delta w_n &= A_{n+1} \frac{\Delta(a_n \Delta z_n)}{z_n^{\alpha}} + \frac{\Delta A_n a_n \Delta z_n}{z_n^{\alpha}} - \frac{A_{n+1} a_{n+1} \Delta z_{n+1}}{z_n^{\alpha} z_{n+1}^{\alpha}} \Delta z_n^{\alpha} \\
&\le -A_{n+1} q_n \max_{[n-\sigma,n]} (1-p_s)^{\alpha} + \frac{\Delta A_n a_n \Delta z_n}{z_n^{\alpha}} \\
&\le -A_{n+1} q_n \max_{[n-\sigma,n]} (1-p_s)^{\alpha} + \frac{\Delta z_n}{z_n^{\alpha}}
\end{aligned}$$

or

$$\Delta w_n \le -A_{n+1} q_n \max_{[n-\sigma,n]} \left(1-p_s\right)^{\alpha}. \tag{5}$$

Summing the last inequality from $n_1 \ge N$ to $n-1$, we have

$$0 < w_n \le w_{n_1} - \sum_{s=n_1}^{n-1} A_{s+1} q_s \max_{[s-\sigma,s]} \left(1-p_t\right)^{\alpha}.$$

Letting $n \to \infty$, we get a contradictions to (2).

Case(II). Define

$$v_n = \frac{a_n \Delta z_n}{z_{n-\tau}^{\alpha}}, \ n \ge N. \tag{6}$$

Then $v_n < 0$ for $n \ge N$. Since $\{a_n \Delta z_n\}$ is nonincreasing, we have

$$\Delta z_s \le \frac{a_n \Delta z_n}{a_s}, \ s \ge n.$$

Summing the last inequality from $n-\tau$ to $\ell-1$, we obtain

$$z_\ell \le z_{n-\tau} + a_n \Delta z_n \sum_{s=n-\tau}^{\ell-1} \frac{1}{a_s}.$$

Since $z_\ell > 0$ and $\Delta z_\ell < 0$ by letting $\ell \to \infty$, in the last inequality we obtain

$$0 \le b \le z_{n-\tau} + a_n \Delta z_n B_n, \ n \ge N$$

or

$$0 \le z_{n-\tau} + a_n \Delta z_n B_n, \ n \ge N$$

or

$$B_n \frac{a_n \Delta z_n}{z_{n-\tau}} \ge -1, \ n \ge N.$$

Thus

$$-\frac{a_n \Delta z_n \left(-a_n \Delta z_n\right)^{-1}}{z_{n-\tau}^{\alpha}} B_n^{\alpha} \le 1.$$

So, by $\Delta\left(-a_n \Delta z_n\right) > 0$ and (6), we have

$$-\frac{1}{L^{\alpha-1}} \le v_n B_n^{\alpha} \le 0, \ n \ge N, \tag{7}$$

where $L = -a_n \Delta z_N$. From (6), we obtain

$$\Delta v_n = \frac{\Delta\left(a_n \Delta z_n\right)}{z_{n-\tau}^{\alpha}} - \frac{a_{n+1} \Delta z_{n+1}}{z_{n-\tau}^{\alpha} z_{n+1-\tau}^{\alpha}} \Delta z_{n-\tau}^{\alpha}.$$

By Mean Value Theorem,

$$\Delta z_{n-\tau}^{\alpha} = z_{n+1-\tau}^{\alpha} - z_{n-\tau}^{\alpha} = \alpha t^{\alpha-1} \Delta z_{n-\tau}$$

where $z_{n+1-\tau} \le t \le z_{n-\tau}$. Since $\alpha \ge 1$ and $\Delta z_{\tau(n)} < 0$, we have

$$\Delta z_{n-\tau}^{\alpha} \le \alpha z_{n+1-\tau}^{\alpha-1} \Delta z_{n-\tau}.$$

Therefore,

$$\Delta v_n \le -q_n \max_{[n-\sigma,n]} \frac{x_s^{\alpha}}{z_{s-\tau}^{\alpha}} - \frac{\alpha a_{n+1} \Delta z_{n+1} z_{n+1-\tau}^{\alpha-1} \Delta z_{n-\tau}}{z_{n-\tau}^{\alpha} z_{n+1-\tau}^{\alpha}}.$$

Since $z_{n+1-\tau} \le z_{n-\tau}$, we have

$$\Delta v_n \le -q_n \max_{[n-\sigma,n]}\left(\frac{x_s}{z_{s-\tau}}\right)^{\alpha} - \frac{\alpha a_{n+1}\Delta z_{n+1}}{z_{n-\tau}^{\alpha+1}}\Delta z_{n-\tau}. \tag{8}$$

From Lemma 2.6, $\Delta x_n \le 0$ for $n \ge N$, we have

$$\left(\frac{x_n}{z_{n-\tau}}\right)^{\alpha} = \left(\frac{x_n}{x_{n-\tau} + p_{n-\tau}x_{n-\tau-k}}\right)^{\alpha} \ge \left(\frac{1}{1+p_{n-\tau}}\right)^{\alpha}. \tag{9}$$

From (8) and (9), we have

$$\Delta v_n + q_n \max_{[n-\sigma,n]}\left(\frac{1}{1+p_{s-\tau}}\right)^{\alpha} \le 0,\ n \ge N. \tag{10}$$

Multiply (10) by B_{n+1}^{α} and summing it from $n_1 \ge N$ to $n-1$, we have

$$\sum_{s=n_1}^{n-1} B_{s+1}^{\alpha}\Delta v_s + \sum_{s=n_1}^{n-1} B_{s+1}^{\alpha} q_s \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \le 0.$$

Summation by parts formula yields

$$\sum_{s=n_1}^{n-1} B_{s+1}^{\alpha}\Delta v_s = B_n^{\alpha} v_n - B_{n_1}^{\alpha} v_{n_1} - \sum_{s=n_1}^{n-1} v_s \Delta B_s^{\alpha}.$$

Using Mean Value Theorem, we obtain

$$\Delta B_s^{\alpha} \ge -\frac{\alpha B_s^{\alpha-1}}{a_{s-\tau}}.$$

Since $v_n < 0$, we have

$$\sum_{s=n_1}^{n-1} B_{s+1}^{\alpha}\Delta v_s \ge B_n^{\alpha} v_n - B_{n_1}^{\alpha} v_{n_1} - \sum_{s=n_1}^{n-1} \frac{\alpha v_s B_s^{\alpha-1}}{a_{s-\tau}}.$$

or

$$B_n^{\alpha} v_n - B_{n_1}^{\alpha} v_{n_1} + \sum_{s=n_1}^{n-1} \frac{\alpha v_s B_s^{\alpha-1}}{a_{s-\tau}} + \sum_{s=n_1}^{n-1} B_{s+1}^{\alpha} q_s \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \le 0 \tag{11}$$

Therefore, from (7) and (11), we have

$$B_n^{\alpha} v_n \le B_{n_1}^{\alpha} v_{n_1} - \sum_{s=n_1}^{n-1}\left[B_{s+1}^{\alpha} q_s \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} - \frac{\alpha}{L^{\alpha-1} B_s a_{s-\tau}}\right].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3). This completes the proof. □

Theorem 2.2. *Assume that* $\alpha \ge 1$, *and there exists a positive integer k such that* $\sigma \ge \tau - k$. *If for all sufficiently large* $n_1 \in \mathbb{N}(n_0)$ *and for every constant* $M > 0$, (2) *holds, and*

$$\sum_{n=n_1}^{\infty} B_{n+1}^{\alpha+1} q_n \max_{[n-\sigma,n]}\left(\frac{1}{1+p_{s-\tau}}\right)^{\alpha} = \infty, \tag{12}$$

hold, then every solution of equation (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we see that Lemma 2.1 holds for $n \ge N \in \mathbb{N}(n_0)$.

Case(I). Proceeding as in the proof of Theorem 2.1 (Case(I)) we obtain a contradiction to (12).

Case(II). Proceeding as in the proof of Theorem 2.1 (Case(II)) we obtain (7) and (10). Multiplying (10) by $B_{n+1}^{\alpha+1}$ and summing it from $n_1 \ge N$ to $n-1$ we have

$$\sum_{n=n_1}^{\infty} B_{s+1}^{\alpha+1}\Delta v_s + \sum_{s=n_1}^{n-1} B_{s+1}^{\alpha+1} q_s \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \le 0.$$

Using the summation by parts formula in the first term of the last inequality and rearranging, we obtain

$$B_n^{\alpha+1}v_n - B_{n_1}^{\alpha+1}v_{n_1} + \sum_{s=n_1}^{n-1}\frac{(\alpha+1)v_s B_s^{\alpha}}{a_{s-\tau}} + \sum_{s=n_1}^{n-1} q_s B_s^{\alpha+1}\max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \le 0 \tag{13}$$

Inview of (7), we have $-v_n B_n^{\alpha+1} \le \frac{1}{L^{\alpha-1}}B_n < \infty$ as $n\to\infty$ and

$$\sum_{s=n_1}^{n-1} q_s B_{s+1}^{\alpha+1}\max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \le B_{n_1}^{\alpha+1}v_{n_1} - B_n^{\alpha+1}v_n + \frac{\alpha+1}{L^{\alpha-1}}\sum_{s=n_1}^{n-1}\frac{1}{a_{s-\tau}}.$$

As $n\to\infty$ in the last inequality, we obtain a contradiction to (12). This completes the proof.

Theorem 2.3. *Assume that* $\alpha \ge 1$, *and there exists a positive integer* k *such that* $\sigma \ge \tau - k$. *If for all sufficiently large* $n_1 \in \mathbb{N}(n_0)$ *and for every constant* $M > 0$, (2) *holds, and*

$$\sum_{n=n_1}^{\infty}\frac{1}{a_n}\sum_{s=n_1}^{n-1} q_s B_s^{\alpha}\max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} = \infty, \tag{14}$$

then every solution of equation (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we see that Lemma 2.1 holds and Case(I) is eliminated by the condition (2).

Case(II). Proceeding as in the proof of Theorem 2.1 (Case(II)) we have

$$z_{\tau(n)} \ge -a_n\Delta z_n B_n \ge -a_{n_1}\Delta z_{n_1} B_n = cB_n$$

where $c = -a_{n_1}\Delta z_{n_1}$. From Equation (1), we have

$$\Delta(-a_n\Delta z_n) = q_n \max_{[n-\sigma,n]} x_s^{\alpha},$$

and

$$\frac{x_n}{z_{n-\tau}} \ge \frac{1}{1+p_{n-\tau}}. \tag{15}$$

Hence

$$\Delta(-a_n\Delta z_n) \ge c^{\alpha} q_n B_n^{\alpha}\max_{[n-\sigma,n]}\left(\frac{1}{1+p_{n-\tau}}\right)^{\alpha}.$$

Summing the last inequality from $n_1 \ge N$ to $n-1$, we obtain

$$-a_n\Delta z_n \ge -a_{n_1}\Delta z_{n_1} + c^{\alpha}\sum_{s=n_1}^{n-1} q_s B_s^{\alpha}\max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha} \ge c^{\alpha}\sum_{s=n_1}^{n-1} q_s B_s^{\alpha}\max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right)^{\alpha}.$$

Again summing the last inequality from $n_1 \ge N$ to $n-1$, we have

$$z_{n_1} \ge z_{n_1} - z_n \ge c^{\alpha}\sum_{s=n_1}^{n-1}\frac{1}{a_s}\sum_{t=s_1}^{s-1} q_t B_t^{\alpha}\max_{[t-\sigma,t]}\left(\frac{1}{1+p_{i-\tau}}\right)^{\alpha}.$$

Letting $n\to\infty$ in the above inequality, we obatin

$$c^{\alpha}\sum_{s=n_1}^{n-1}\frac{1}{a_s}\sum_{t=s_1}^{s-1} q_t B_t^{\alpha}\max_{[t-\sigma,t]}\left(\frac{1}{1+p_{i-\tau}}\right)^{\alpha} \le z_{n_1}$$

a contradiction to (14). This completes the proof.

Next, we obtain sufficient conditions for the oscillation of all solutions of Equation (1) when $0 < \alpha \le 1$.

Theorem 2.4. *Assume that* $0 < \alpha \le 1$, *and there exists a positive integer* k *such that* $\sigma \ge \tau - k$. *If for all sufficiently large* $n_1 \in \mathbb{N}(n_0)$ *and for every constant* $M_1 > 0, L > 0$, *one has*

$$\sum_{n=n_1}^{\infty}\left[A_{n+1}q_n \max_{[n-\sigma,n]}(1-p_s)^{\alpha}-\frac{(M_1A_n)^{1-\alpha}}{C_n}\right]=\infty, \tag{16}$$

and

$$\sum_{n=n_1}^{\infty}\left[B_{n+1}q_nL^{1-\alpha}\max_{[n-\sigma,n]}\left(\frac{1}{1+p_{t-\tau}}\right)-\frac{1}{4a_{n-\tau}B_{n+1}}\right]=\infty, \tag{17}$$

then every solution of equation (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we see that Lemma 2.4 holds for $n\ge N\in\mathbb{N}(n_0)$.

Case(I). Define w_n by

$$w_n=A_n\frac{a_n\Delta z_n}{z_n^{\alpha}},\ n\ge N.$$

Then $w_n>0$ and from Equation (1) and Lemma 2.2, we have

$$\Delta w_n\le -A_{n+1}q_n\max_{[n-\sigma,n]}(1-p_s)^{\alpha}+\frac{a_n\Delta z_n}{a_nz_n^{\alpha}}. \tag{18}$$

Using Lemma 2.5 in (18), we obtain

$$\Delta w_n\le -A_{n+1}q_n\max_{[n-\sigma,n]}(1-p_s)^{\alpha}+\frac{z_n^{1-\alpha}}{C_n}. \tag{19}$$

From the monotoncity of $\{a_n\Delta z_n\}$, we have

$$z_n=z_N+\sum_{s=N}^{n-1}\frac{a_s\Delta z_s}{a_s}\le z_N+a_N\Delta z_NR_n.$$

and hence

$$z_n\le M_1R_n \tag{20}$$

for some constant $M_1>0$ for all large *n*. Using (20) in (19) and then summing the resulting inequality from $n_1\ge N$ to $n-1$, we have

$$0<w_n\le w_{n_1}-\sum_{s=n_1}^{n-1}\left[A_{s+1}q_s\max_{[s-\sigma,s]}(1-p_t)^{\alpha}-\frac{(M_1A_n)^{1-\alpha}}{C_n}\right]. \tag{21}$$

Letting $n\to\infty$ in (21), we obtain a contradiction to (16).

Case(II). Define a function v_n by

$$v_n=\frac{a_n\Delta z_n}{z_{n-\tau}},\ n\ge N.$$

Then $v_n<0$ for $n\ge N$, we have

$$\Delta v_n=\frac{\Delta(a_n\Delta z_n)}{z_{n-\tau}}-\frac{a_{n+1}\Delta z_{n+1}}{z_{n-\tau}z_{n+1-\tau}}\Delta z_{n-\tau}\le -q_n\max_{[n-\sigma,n]}\left(\frac{x_s^{\alpha}}{z_{s-\tau}}\right)-\frac{a_n\Delta z_n}{z_{n-\tau}z_{n+1-\tau}}\Delta z_{n-\tau}.$$

Since $\tau(n)\ge n$, and $a_n\Delta z_n$ is negative and decreasing we have

$$a_{n-\tau}\Delta z_{n-\tau}\le a_n\Delta z_n.$$

Therefore

$$\Delta v_n\le -q_n\max_{[n-\sigma,n]}\left(\frac{x_s^{\alpha}}{z_{s-\tau}}\right)-\frac{(a_n\Delta z_n)^2}{a_{n-\tau}z_{n-\tau}z_{n+1-\tau}}.$$

Since z_n is a positive and decreasing, we have $z_{n+1-\tau}\le z_{n-\tau}$. Combining the last two inequalities, we have

$$\Delta v_n \le -q_n \max_{[n-\sigma,n]}\left(\frac{x_s^\alpha}{z_{s-\tau}}\right) - \frac{v_n^2}{a_{n-\tau}}. \tag{22}$$

Now using (15) in (22), we obtain

$$\Delta v_n \le -q_n L^{\alpha-1} \max_{[n-\sigma,n]}\left(\frac{1}{1+p_{s-\tau}}\right) - \frac{v_n^2}{a_{n-\tau}}$$

for some constant $L > 0$. That is

$$\Delta v_n + q_n L^{\alpha-1} \max_{[n-\sigma,n]}\left(\frac{1}{1+p_{s-\tau}}\right) + \frac{v_n^2}{a_{n-\tau}} \le 0, \ n \ge N.$$

Multiplying the last inequality by B_{n+1}, and then summing it from $n_1 \ge N$ to $n-1$, we have

$$\sum_{s=n_1}^{n-1} B_{s+1}\Delta v_s + \sum_{s=n_1}^{n-1} q_s L^{\alpha-1} B_{s+1} \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right) + \sum_{s=n_1}^{n-1} \frac{B_{s+1}}{a_{s-\tau}} v_s^2 \le 0.$$

Using the summation by parts formula in the first term of the above inequality and rearranging we obtain

$$B_n v_n - B_{n_1} v_{n_1} + \sum_{s=n_1}^{n-1} q_s B_{s+1} L^{\alpha-1} \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right) + \sum_{s=n_1}^{n-1}\left(\frac{v_s}{a_{s-\tau}} + \frac{B_{s+1}v_s^2}{a_{s-\tau}}\right) \le 0.$$

Using completing the square in the las term of the left hand side of the last inequality, we obtain

$$B_n v_n - B_{n_1} v_{n_1} + \sum_{s=n_1}^{n-1} q_s B_{s+1} L^{\alpha-1} \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right) + \sum_{s=n_1}^{n-1} \frac{B_{s+1}}{a_{s-\tau}}\left(v_s + \frac{1}{2B_{s+1}}\right)^2 - \sum_{s=n_1}^{n-1} \frac{1}{4a_{s-\tau}B_{s+1}} \le 0$$

or

$$B_n v_n \le B_{n_1} v_{n_1} - \sum_{s=n_1}^{n-1}\left[q_s B_{s+1} L^{\alpha-1} \max_{[s-\sigma,s]}\left(\frac{1}{1+p_{t-\tau}}\right) - \frac{1}{4a_{s-\tau}B_{s+1}}\right].$$

Letting $n \to \infty$ in the above inequality, we obtain a contradiction to (17). The proof is now complete. □

3. Existence of Nonoscillatory Solutions

In this section, we provide sufficient conditions for the existence of nonoscillatory solutions of Equation (1) in case $\alpha > 1$ or $0 < \alpha < 1$. Note that in this section we do not require $p_n \equiv p$.

Theorem 3.1. *Assume that* $\alpha > 1$. *If*

$$\sum_{n=n_0}^{\infty} q_n A_{n-\sigma}^\alpha < \infty \tag{23}$$

and

$$\lim_{n\to\infty} \frac{A_{n-\tau}}{A_n} = 1, \tag{24}$$

then Equation (1) has a bounded nonoscillatory solution.

Proof. Choose $N \ge n_0$ sufficiently large so that

$$p_n \frac{A_{n-\tau}}{A_n} \le \frac{3p+1}{4} \tag{25}$$

and

$$\sum_{s=N}^{\infty} q_s A_{s-\sigma}^\alpha \le \frac{1-p}{8\alpha} \tag{26}$$

for $n \ge N$. Let χ be the set of all bounded real sequences defined for all $n \ge n_0$ with norm

$$\|x_n\| = \sup_{n\ge n_0}\left\{\frac{x_n}{A_n}\right\},$$

and let

$$S = \left\{x \in \chi; \frac{3(1-p)}{8} \le x_n \le 1, n \ge n_0\right\}.$$

Define a mapping $T : S \to \chi$ by

$$(Tx)_n = \begin{cases} \frac{3p+5}{8}A_n - p_n x_{n-\tau} + A_n \sum_{s=N}^{n} q_s \max_{[s-\sigma,s]} x_t^\alpha + \sum_{s=n}^{\infty} q_s A_s \max_{[s-\sigma,s]} x_t^\alpha, & n \ge N \\ (Tx)_N, & n_0 \le n < N. \end{cases}$$

Clearly, T is continuous. Now for every $x \in S$ and $n \ge N$, (25) implies

$$(Tx)_n \ge A_n \frac{3p+5}{8} - p_n x_{n-\tau} \ge A_n\left(\frac{3p+5}{8} - p_n \frac{A_{n-\tau}}{A_n}\right)$$
$$\ge A_n\left(\frac{3p+5}{8} - \frac{3p+1}{4}\right) \ge \frac{3(1-p)}{8} A_n.$$

Also, from (26) we have

$$(Tx)_n \le A_n \frac{3p+5}{8} + A_n \sum_{s=N}^{n} q_s \max_{[s-\sigma,s]} x_t^\alpha + \sum_{s=n}^{\infty} q_s A_s \max_{[s-\sigma,s]} x_t^\alpha$$
$$\le A_n \frac{3p+5}{8} + A_n \sum_{s=N}^{\infty} q_s \max_{[s-\sigma,s]} A_t^\alpha$$
$$\le A_n\left(\frac{3p+5}{8} + \frac{1-p}{8\alpha}\right) < A_n.$$

Thus, we have that $TS \subset S$. Since S is bounded, closed and convex subset of χ, we only need to show that T is contraction mapping on S in order to apply the contraction mapping principle. For $x, y \in S$ and $n \ge N$, we have

$$\frac{1}{A_n}\left|(Tx)_n - (Ty)_n\right| \le \frac{p_n}{A_n} \max_{[n-\sigma,n]}\left|x_{s-\tau+\sigma} - y_{s-\tau+\sigma}\right| + \sum_{s=N}^{n} q_s \max_{[s-\sigma,s]}\left|x_t^\alpha - y_t^\alpha\right| + \frac{1}{A_n}\sum_{s=n}^{\infty} q_s A_s \max_{[s-\sigma,s]}\left|x_t^\alpha - y_t^\alpha\right|$$
$$\le p_n \frac{A_{n-\tau}}{A_n} \max_{[n-\sigma,n]}\left|\frac{x_{s-\tau}}{A_{s-\tau}} - \frac{y_{s-\tau}}{A_{s-\tau}}\right| + \sum_{s=N}^{\infty} q_s A_{s-\sigma}^\alpha \max_{[s-\sigma,s]}\left|\left(\frac{x_{t-\sigma}}{A_{t-\sigma}}\right)^\alpha - \left(\frac{y_{t-\sigma}}{A_{t-\sigma}}\right)^\alpha\right|.$$

By the Mean Value Theorem applied to the function $f(u) = u^\alpha, \alpha > 1$, we see that for any $x, y \in S$, we have $\left|x^\alpha - y^\alpha\right| \le 2\alpha\left|x - y\right|$ for all $n \ge N$. Hence

$$\|Tx - Ty\| \le \frac{3p+1}{4}\|x - y\| + 2\alpha \sum_{s=N}^{\infty} q_s A_{s-\sigma}^\alpha \|x - y\|$$
$$\le \frac{3p+1}{4}\|x - y\| + 2\alpha \frac{1-p}{8\alpha}\|x - y\|$$
$$= \frac{p+1}{2}\|x - y\| < \|x - y\|.$$

Thus, T is a contraction mapping, so T has a unique fixed point $x \in S$ such that $Tx = x$. It is easy to see that $x = \{x_n\}$ is a positive solution of Equation (1). This complete the proof of the theorem. □

Theorem 3.2. *Assume that* $0 < \alpha < 1$. *If*

$$\sum_{n=n_0}^{\infty} A_n q_n < \infty \tag{27}$$

then Equation (1) has a bounded nonoscillatory solution.

Proof. Choose $N \ge n_0$ sufficiently large so that

$$\sum_{s=N}^{\infty} q_s A_s \le \frac{(1-p)^2}{8}.$$

Let χ be the set of all bounded real sequences defined for all $n \ge n_0$ with norm

$$\|x\| = \sup_{n \ge n_0} \{x_n\},$$

and let

$$S = \left\{ x \in \chi : \frac{3(1-p)}{8} \le x_n \le 1, n \ge n_0 \right\}.$$

Define a mapping $T : S \to \chi$ by

$$(Tx)_n = \begin{cases} \dfrac{3p+5}{8} - p_n x_{n-\tau} + A_n \sum_{s=N}^{n} q_s \max_{[s-\sigma,s]} x_t^{\alpha} + \sum_{s=n}^{\infty} q_s A_s \max_{[s-\sigma,s]} x_t^{\alpha}, & n \ge N \\ (Tx)_N, & n_0 \le n < N. \end{cases}$$

It is easy to see that T is continuous, $TS \subset S$, and for any $x, y \in S$ and $n \ge N$, we have

$$\left|(Tx)_n - (Ty)_n\right| \le p_n \max_{[n-\sigma,n]} \left|x_{s-\tau} - y_{s-\tau}\right| + \sum_{s=N}^{\infty} q_s A_s \max_{[s-\sigma,s]} \left|x_t^{\alpha} - y_t^{\alpha}\right|.$$

By the Mean Value Theorem applied to the function $f(u) = u^{\alpha}, 0 < \alpha < 1$, we see that for any $x, y \in S$, we have $\left|x^{\alpha} - y^{\alpha}\right| \le \frac{8\alpha}{3(1-p)} |x - y|$ for all $n \ge N$. Hence

$$\|Tx - Ty\| \le \|x - y\| \left(p + \frac{8\alpha}{3(1-p)} \frac{(1-p)^2}{8} \right) < \|x - y\|,$$

and we see that T is a contraction on S. Hence, T has a unique fixed point which is clearly a positive solution of Equation (1). This completes the proof of the theorem.

4. Examples

In this section we present some examples to illustrate the main results.

Example 4.1. Consider the difference equations

$$\Delta\left(2^{n+1} \Delta\left(x_n + \frac{1}{3} x_{n-3}\right)\right) + 2^{4n} \max_{[n-1,n]} x_s^3 = 0, \ n \ge 0. \tag{28}$$

Here $\alpha = 3, a_n = 2^{n+1}, p_n = \frac{1}{3}, q_n = 2^{4n}$ and $\tau = 3, \sigma = 1$. Then $A_n = \frac{2^n - 1}{2^n}, B_n = \frac{1}{2^{n-3}}$. Choosing $k = 3$, we see that $\sigma \ge \tau - k$. Further it is easy to verify that all other conditions of Theorem 2.1 are satisfied. Therefore every solution of Equation (28) is oscillatory.

Example 4.2. Consider the difference equations

$$\Delta\left(2^{n+1} \Delta\left(x_n + \frac{1}{4} x_{n-3}\right)\right) + 2^{2n} \max_{[n-2,n]} x_s^{\frac{1}{3}} = 0, \ n \ge 0. \tag{29}$$

Here $\alpha = \frac{1}{3}, a_n = 2^{n+1}, p_n = \frac{1}{4}, q_n = 2^{4n}$ and $\tau = 3, \sigma = 2$. Then $A_n = \frac{2^n - 1}{2^n}, B_n = \frac{1}{2^{n-3}}$ and $C_n = 2(2^n - 1)$. Choosing $k = 2$, we see that $\sigma \ge \tau - k$. Further it is easy to verify that all other conditions of Theorem 2.4 are satisfied. Therefore every solution of Equation (29) is oscillatory.

Example 4.3. Consider the difference equations

$$\Delta\left(n(n+1)\Delta\left(x_n+\frac{1}{4}x_{n-3}\right)\right)+n^2\max_{[n-1,n]}x_s^3=0,\ n\geq 1. \tag{30}$$

Here $\alpha=3, a_n=n(n+1), p_n=\frac{1}{4}, q_n=n^2$ and $\tau=3,\sigma=1$. By talking $A_n=\frac{1}{n}$, we see that all conditions of Theorem 3.1 are satisfied and hence Equation (30) has a bounded nonoscillatory solution.

Example 4.4. Consider the difference equations

$$\Delta\left(n(n+1)\Delta\left(x_n+\frac{1}{2}x_{n-3}\right)\right)+\frac{1}{n}\max_{[n-2,n]}x_s^{\frac{1}{3}}=0,\ n\geq 1. \tag{31}$$

Here $\alpha=\frac{1}{3}, a_n=n(n+1), p_n=\frac{1}{2}, q_n=\frac{1}{n}$ and $\tau=3,\sigma=2$. By talking $A_n=\frac{1}{n}$, we see that all conditions of Theorem 3.2 are satisfied and hence Equation (31) has a bounded nonoscillatory solution.

References

[1] Agarwal, R.P. (2000) Difference Equations and Inequalities. 2nd Edition, Marcel Dekker, New York.

[2] Agarwal, R.P., Bohner, M., Grace, S.R. and O'Regan, D. (2005) Discrete Oscillation Theory. Hindawi Publ. Corp., New York. http://dx.doi.org/10.1155/9789775945198

[3] Kelley, W.G. and Peterson, A.C. (2001) Difference Equations: An Introduction with Applications. 2nd Edition, Academic Press, New York.

[4] Thandapani, E. and Selvarangam, S. (2012) Oscillation of Second Emden-Fowler Type Neutral Difference Equations. *Dynamics Continous Discrete Impulise System*, **19**, 453-469.

[5] Zhang, G. and Geo, Y. (2001) Oscillation Theory for Difference Equations. Publishing House of Higher Education, Beijing.

[6] Arul, R. and Angayarkanni, M. (2013) Asymptotic Behavior of Second Order Nonlinear Neutral Difference Equations with "Maxima". *Far East Journal of Mathematical Science*, **82**, 79-92.

[7] Arul, R. and Angayarkanni, M. (2014) Oscillatory and Asymptotic Behavior of Second Order Neutral Difference Equations with "Maxima". *Journal of Advances in Mathematics*, 1916-1924.

[8] Luo, J.W. and Bainov, D.D. (2001) Oscillatory and Asymptotic Behavior of Second-Order Neutral Difference Equations with "Maxima". *Journal of Computational and Applied Mathematics*, **131**, 333-341. http://dx.doi.org/10.1016/S0377-0427(00)00264-8

[9] Luo, J.W. and Petrov, V.A. (1999) Oscillation of Second Order Neutral Difference Equations with "Maxima". *Journal of Mathematical Sciences Research. Hot-Line*, **3**, 17-22.