

A Note on the Structure of Affine Subspaces of $L^2(\mathbb{R}^d)$

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Abstract

This paper investigates the structure of general affine subspaces of $L^2(\mathbb{R}^d)$. For a $d \times d$ expansive matrix A , it shows that every affine subspace can be decomposed as an orthogonal sum of spaces each of which is generated by dilating some shift invariant space in this affine subspace, and every non-zero and non-reducing affine subspace is the orthogonal direct sum of a reducing subspace and a purely non-reducing subspace, and every affine subspace is the orthogonal direct sum of at most three purely non-reducing subspaces when $|\det A| = 2$.

Keywords

Affine Subspace, Reducing Subspace, Shift Invariant Subspace, Orthogonal Sum

1. Introduction

Let A be a $d \times d$ expansive matrix. Define the dilation operator D and the shift operator T_k , $k \in \mathbb{Z}^d$, by

$$Df(\cdot) = |\det A|^{\frac{1}{2}} f(A \cdot) \text{ and } T_k f(\cdot) = f(\cdot - k), \quad f \in L^2(\mathbb{R}^d),$$

respectively. It is easy to check that they are both unitary operators on $L^2(\mathbb{R}^d)$. Given a closed subspace X of $L^2(\mathbb{R}^d)$, X is called a *shift invariant subspace* if $T_k X = X$ for every $k \in \mathbb{Z}^d$; X is called a *reducing subspace* of $L^2(\mathbb{R}^d)$ if $DX = X$ and $T_k X = X$ for every $k \in \mathbb{Z}^d$; X is called an *affine subspace* of $L^2(\mathbb{R}^d)$ if there exists an at most countable subset Φ of $L^2(\mathbb{R}^d)$ such that

$$X = \overline{\text{span}}\{D^j T_k \phi : \phi \in \Phi, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

In this case, we say that Φ generates the affine subspace X . An affine subspace, which does not contain any non-zero reducing subspace, is called purely non-reducing. By Theorem 3.1 in [1], a closed subspace X of $L^2(\mathbb{R}^d)$ is an affine subspace if and only if $X = \overline{\text{span}}\{D^j f : f \in M, j \in \mathbb{Z}\}$ for some shift invariant subspace M . Therefore an affine subspace X of $L^2(\mathbb{R}^d)$ is a reducing subspace if and only if it is shift invariant. So far, the study of reducing subspaces has achieved fruitful results. The existence and construction of wavelet frames for an arbitrary reducing subspace can be seen in [2]-[7]. For one-dimensional case $A = 2$, Gu and Han investigated the existence of Parseval wavelet frames for singly generated affine subspaces in [8] and the structural properties of affine subspaces in [9]. For a given $d \times d$ expansive matrix A , Zhou and Li studied the construction of wavelet frames in the setting of finitely generated affine subspaces of $L^2(\mathbb{R}^d)$ in [10]. For a general $d \times d$ expansive matrix A , this paper focuses on the structure of affine subspaces of $L^2(\mathbb{R}^d)$, which is a continuation of the literature [10] and has not been investigated yet.

2. Main Results

Lemma 1. *Let X and Y be closed subspaces of a Hilbert space H and P_{X^\perp} be the orthogonal projection onto X^\perp . Then*

- 1) $(X \cap Y)^\perp = \overline{\text{span}}\{X^\perp, Y^\perp\}$;
- 2) $\overline{P_{X^\perp} Y} = (\overline{\text{span}}\{X, Y\}) \ominus X$.

Proof. 1) Obviously, $(X \cap Y)^\perp = X^\perp \cup Y^\perp \subset \overline{\text{span}}\{X^\perp, Y^\perp\}$. For the other direction, note that

$$X^\perp \subset \overline{\text{span}}\{X^\perp, Y^\perp\} \text{ and } Y^\perp \subset \overline{\text{span}}\{X^\perp, Y^\perp\},$$

then

$$\left(\overline{\text{span}}\{X^\perp, Y^\perp\}\right)^\perp \subset X \text{ and } \left(\overline{\text{span}}\{X^\perp, Y^\perp\}\right)^\perp \subset Y.$$

So $\left(\overline{\text{span}}\{X^\perp, Y^\perp\}\right)^\perp \subset X \cap Y$. Therefore, $(X \cap Y)^\perp \subset \overline{\text{span}}\{X^\perp, Y^\perp\}$. Thus 1) holds.

2) For $f \in P_{X^\perp} Y \subset P_{X^\perp} \overline{\text{span}}\{X, Y\}$, there exists some $g \in \overline{\text{span}}\{X, Y\}$ such that $f = P_{X^\perp} g$. So $g = P_{X^\perp} g + P_X g = f + P_X g$, or $f = g - P_X g \in \overline{\text{span}}\{X, Y\}$, which shows $f \in \overline{\text{span}}\{X, Y\} \ominus X$ due to the fact that $f \in X^\perp$. For $f \in \overline{\text{span}}\{X, Y\} \ominus X$, we have $f \in \overline{\text{span}}\{X, Y\}$ and $f \perp X$. Thus for any $\epsilon > 0$, there is g with $\|g\| < \epsilon$ and $h_1 \in X$ and $h_2 \in Y$ such that $f = h_1 + h_2 + g$. Consequently,

$$f = P_{X^\perp} f = P_{X^\perp} (h_1 + h_2 + g) = P_{X^\perp} g + P_{X^\perp} h_2 \in \overline{P_{X^\perp} Y}$$

since $\|P_{X^\perp} g\| \leq \|g\| < \epsilon$. The proof is completed. \square

Lemma 2. *Let $\{X_n : n \in \mathbb{Z}\}$ be a monotone sequence of subspaces in a Hilbert space \mathcal{H} .*

1) *If $\{X_n : n \in \mathbb{Z}\}$ is increasing, then*

$$\overline{\text{span}}\{X_n : n \in \mathbb{Z}\} = \overline{\bigcup_{n \in \mathbb{Z}} X_n} = \left(\bigcap_{n \in \mathbb{Z}} X_n\right) \oplus \left(\bigoplus_{n \in \mathbb{Z}} (X_{n+1} \ominus X_n)\right).$$

2) *If $\{X_n : n \in \mathbb{Z}\}$ is decreasing, then*

$$\overline{\text{span}}\{X_n : n \in \mathbb{Z}\} = \overline{\bigcup_{n \in \mathbb{Z}} X_n} = \left(\bigcap_{n \in \mathbb{Z}} X_n\right) \oplus \left(\bigoplus_{n \in \mathbb{Z}} (X_n \ominus X_{n+1})\right).$$

Proof. We only prove 1) since 2) can be obtained similarly. Since $\{X_n : n \in \mathbb{Z}\}$ is increasing, the first equality is obvious and

$$\left(\bigcap_{n \in \mathbb{Z}} X_n\right) \oplus \left(\bigoplus_{n \in \mathbb{Z}} (X_{n+1} \ominus X_n)\right) \subset \overline{\bigcup_{n \in \mathbb{Z}} X_n}.$$

If $f \in \overline{\bigcup_{n \in \mathbb{Z}} X_n}$, then for any $\epsilon > 0$, there exists $g \in \mathcal{H}$, $n_0 \in \mathbb{Z}$ and $h \in X_{n_0}$ such that $\|g\| < \epsilon$ and $f = g + h$. For such h , there is a unique sequence $\{h_n\}_{n=-\infty}^{n_0}$ and a unique \tilde{h} such that $h_n \in X_n \ominus X_{n-1}$ for each $n \leq n_0$, $\tilde{h} \in \bigcap_{n \in \mathbb{Z}} X_n$ and $h = \tilde{h} + \sum_{n=-\infty}^{n_0} h_n$. This means that

$$f \in \overline{\left(\bigcap_{n \in \mathbb{Z}} X_n \right) \oplus \left(\bigoplus_{n \in \mathbb{Z}} (X_{n+1} \ominus X_n) \right)} = \left(\bigcap_{n \in \mathbb{Z}} X_n \right) \oplus \left(\bigoplus_{n \in \mathbb{Z}} (X_{n+1} \ominus X_n) \right).$$

The proof is completed. \square

Proposition 1. *Suppose that X is an affine subspace of $L^2(\mathbb{R}^d)$ with M being its generating shift invariant subspace. Then there exist a shift invariant subspace M_1 in X and a reducing subspace Y of $L^2(\mathbb{R}^d)$ contained in X such that the length of M_1 is no more than that of M and $X = \left(\bigoplus_{n \in \mathbb{Z}} M_1 \right) \oplus Y$.*

Proof. For each $j \in \mathbb{Z}$, define

$$Y_j = \overline{\text{span}\{D^n M : n \in \mathbb{Z}, n > j\}}.$$

Obviously, $Y_{j+1} \subset Y_j$ for $j \in \mathbb{Z}$ and $X = \overline{\bigcup_{j \in \mathbb{Z}} Y_j}$. Let $Y = \bigcap_{j \in \mathbb{Z}} Y_j$. Similarly to the proof of Proposition 2.2 in [10], we know that Y is a reducing subspace. Now define $M_1 = Y_{-1} \ominus Y_0$. Then $Y_j \ominus Y_{j+1} = D^{j+1} M_1$ by Lemma 2 and

$$X = \overline{\bigcup_{j \in \mathbb{Z}} Y_j} = \overline{\left(\bigcap_{j \in \mathbb{Z}} Y_j \right) \oplus \left(\bigoplus_{j \in \mathbb{Z}} (Y_j \ominus Y_{j+1}) \right)} = Y \oplus \left(\bigoplus_{j \in \mathbb{Z}} D^{j+1} M_1 \right) = Y \oplus \left(\bigoplus_{j \in \mathbb{Z}} D^j M_1 \right).$$

Suppose that for some subset $\Psi \subset L^2(\mathbb{R}^d)$ such that

$$M = \overline{\text{span}\{T_k \psi : k \in \mathbb{Z}^d, \psi \in \Psi\}}.$$

Since Y_0 is a shift invariant subspace, so is Y_0^\perp . Thus for each $k \in \mathbb{Z}^d$, $T_k P_{Y_0^\perp} = P_{Y_0^\perp} T_k$. Also note that $Y_{-1} = \overline{\text{span}\{D^j M, M : j \in \mathbb{Z}, j > 0\}} = \overline{\text{span}\{Y_0, M\}}$. Therefore, by Lemma 1,

$$\begin{aligned} M_1 &= Y_{-1} \ominus Y_0 = \left(\overline{\text{span}\{M, Y_0\}} \right) \ominus Y_0 \\ &= \overline{P_{Y_0^\perp} M} = \overline{\text{span}\{P_{Y_0^\perp} T_k \psi : k \in \mathbb{Z}^d, \psi \in \Psi\}} \\ &= \overline{\text{span}\{T_k P_{Y_0^\perp} \psi : k \in \mathbb{Z}^d, \psi \in \Psi\}}, \end{aligned}$$

which shows that M_1 is a shift invariant subspace of length no more than the length of M . The proof is completed. \square

Proposition 2. *Suppose that X is a non-zero affine subspace of $L^2(\mathbb{R}^d)$ and Q is the maximal shift invariant subspace contained in X . Then the following hold:*

- 1) $DQ \subset Q$ and $Q \ominus DQ$ is a shift invariant subspace contained in X ;
- 2) $X = \bigoplus_{n \in \mathbb{Z}} D(Q \ominus DQ)$ if and only if X is purely non-reducing subspace of $L^2(\mathbb{R}^d)$.

Proof. 1): Obviously, $DQ \subset X$ is shift invariant space since Q is shift invariant. So $DQ \subset Q$ due to the fact that Q is the maximal shift invariant subspace contained in X . Thus $Q \ominus DQ$ is a shift invariant subspace contained in X .

2): By 1) and Lemma 2, it follows that

$$X = \overline{\text{span}\{D^n Q : n \in \mathbb{Z}\}} = \overline{\bigcup_{n \in \mathbb{Z}} D^n Q} = \left(\bigoplus_{n \in \mathbb{Z}} (Q \ominus DQ) \right) \oplus \left(\bigcap_{n \in \mathbb{Z}} D^n Q \right).$$

If X is purely non-reducing, then $\bigcap_{n \in \mathbb{Z}} D^n Q = \{0\}$ since $\bigcap_{n \in \mathbb{Z}} D^n Q = \bigcap_{n \in \mathbb{N}} D^n Q$ is a reducing subspace. So $X = \bigoplus_{n \in \mathbb{Z}} D(Q \ominus DQ)$. Suppose $X = \bigoplus_{n \in \mathbb{Z}} D(Q \ominus DQ)$ and X contains a reducing subspace Y . Next we only need to show $Y = \{0\}$. Since Y is reducing, we have $Y \subset Q$ and $DY = Y \subset DQ$, i.e., $Y \subset Q \cap DQ$. Also note that $Q = (Q \ominus DQ) \oplus (Q \cap DQ)$. Hence $Y \perp (Q \ominus DQ)$. Thus for each $n \in \mathbb{Z}$, $Y = D^n Y \perp D^n (Q \cap DQ)$. Therefore $Y \perp X$, which shows that $Y = \{0\}$. The proof is completed. \square

Proposition 3. Let X be an affine subspace of $L^2(\mathbb{R}^d)$, and define $L := \overline{\text{span}\{T_k X : k \in \mathbb{Z}^d\}} \ominus X$. The $X \cap \left(\overline{\text{span}\{T_k L : k \in \mathbb{Z}^d\}}\right)^\perp$ is the maximal shift invariant subspace contained in X .

Proof. We first show that $X \cap \left(\overline{\text{span}\{T_k L : k \in \mathbb{Z}^d\}}\right)^\perp$ is shift invariant. For $k \in \mathbb{Z}^d$ and $f \in X \cap \left(\overline{\text{span}\{T_l L : l \in \mathbb{Z}^d\}}\right)^\perp$, it follows that

$$f \in X, T_k f \perp \overline{\text{span}\{T_l L : l \in \mathbb{Z}^d\}}.$$

Next we will show that $T_k f \in X$ by contradiction. If there exists some $k_0 \in \mathbb{Z}^d$ such that $T_{k_0} f \notin X$, then

$$T_{k_0} f = \eta + \zeta \text{ for } \eta \in X \text{ and } 0 \neq \zeta \in X^\perp.$$

So $\zeta = T_{k_0} f - \eta \in \overline{\text{span}\{T_l X : l \in \mathbb{Z}^d\}} \ominus X = L$. Therefore $T_{-k_0} \zeta \in \overline{\text{span}\{T_l L : l \in \mathbb{Z}^d\}}$, which implies that $T_{-k_0} \zeta \perp f$ since $f \in \left(\overline{\text{span}\{T_k L : k \in \mathbb{Z}^d\}}\right)^\perp$. Consequently, $\|f\|^2 = \langle f, T_{-k_0} \eta + T_{-k_0} \zeta \rangle = \langle f, T_{-k_0} \eta \rangle \leq \|f\| \|\eta\|$. This leads to a contradiction since $\|f\| > \|\eta\|$. Assume that M is a shift invariant subspace contained in X . Obviously $M \perp \overline{\text{span}\{T_l X : l \in \mathbb{Z}^d\}} \ominus X = L$. Thus, $M \perp \overline{\text{span}\{T_l L : l \in \mathbb{Z}^d\}}$. So $M \subset X \cap \left(\overline{\text{span}\{T_l L : l \in \mathbb{Z}^d\}}\right)^\perp$. The result follows. The proof is completed. \square

Lemma 3. Let X and Y be affine subspaces of $L^2(\mathbb{R}^d)$ with $X \perp Y$. Let M and N be generating shift invariant subspaces for X and Y respectively. Then $X \oplus Y$ is an affine subspace of $L^2(\mathbb{R}^d)$ with $M \oplus N$ as a generating shift invariant subspace.

Proof. Since $X = \overline{\text{span}\{D^j M : j \in \mathbb{Z}\}}$ and $Y = \overline{\text{span}\{D^j N : j \in \mathbb{Z}\}}$, it follows that

$$\overline{\text{span}\{D^j (M \oplus N) : j \in \mathbb{Z}\}} = \overline{\text{span}\{D^j M : j \in \mathbb{Z}\}} \oplus \overline{\text{span}\{D^j N : j \in \mathbb{Z}\}} = X \oplus Y.$$

The proof is completed. \square

Lemma 4. Assume $\{X_n : n \in \mathbb{Z}\}$ is a monotone sequence of subspaces in a Hilbert space \mathcal{H} and give a subspace $Y \subset \mathcal{H}$ satisfying $X_n \subset Y$ for each $n \in \mathbb{Z}$. Then

$$\overline{\text{span}\{Y \ominus X_n : n \in \mathbb{Z}\}} = \overline{\bigcup_{n \in \mathbb{Z}} Y \ominus X_n} = Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n\right).$$

Proof. Since $\{X_n : n \in \mathbb{Z}\}$ is a monotone sequence of subspaces and $X_n \subset Y$, $n \in \mathbb{Z}$, we have $\{Y \ominus X_n : n \in \mathbb{Z}\}$ is also a monotone sequence. Then the first equality follows by Lemma 2. For $x \in \bigcup_{n \in \mathbb{Z}} Y \ominus X_n$, there exists some $n_0 \in \mathbb{Z}$ such that $x \in Y \ominus X_{n_0}$, namely $x \in Y$ and $x \perp X_{n_0}$. Then $x \in Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n\right)$. Thus

$\bigcup_{n \in \mathbb{Z}} Y \ominus X_n \subset Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n\right)$. So $\overline{\bigcup_{n \in \mathbb{Z}} Y \ominus X_n} \subset Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n\right)$. For the other direction, without loss of generality, assume that $\{X_n : n \in \mathbb{Z}\}$ is increasing. By Lemma 2,

$$Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n \right) = \left(Y \ominus \overline{\bigcup_{n \in \mathbb{Z}} X_n} \right) \oplus \left(\overline{\bigcup_{n \in \mathbb{Z}} X_n} \ominus \left(\bigcap_{k \in \mathbb{Z}} X_k \right) \right) = Y \ominus \overline{\bigcup_{n \in \mathbb{Z}} X_n} \oplus \left(\bigoplus_{k \in \mathbb{Z}} (X_{k-1} \ominus X_k) \right),$$

which shows that $Y \ominus \left(\bigcap_{n \in \mathbb{Z}} X_n \right) \subset \overline{\bigcup_{k \in \mathbb{Z}} (Y \ominus X_k)}$. The proof is completed. \square

Lemma 5. Let X be an affine subspace of $L^2(\mathbb{R}^d)$ and Q be the maximal shift invariant subspace contained in X . Define $V := \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}}$. Then the following hold:

- 1) $D^j Q \subset D^{j-1} Q$ and $D^j V \subset D^{j+1} V$ for $j \in \mathbb{Z}$;
- 2) $Q \perp V$, $V = \{0\}$ if and only if X is a reducing subspace of $L^2(\mathbb{R}^d)$;
- 3) $X \perp \bigcap_{j \in \mathbb{Z}} D^j V$, $\bigcap_{j \in \mathbb{Z}} D^j V$ is in any reducing subspace of $L^2(\mathbb{R}^d)$ containing X .

Proof. 1): Note that we only need to show $DQ \subset Q$ and $D^{-1}V \subset V$. While $DQ \subset Q$ follows by Proposition 2. So $Q \subset D^{-1}Q$. Thus we have

$$\begin{aligned} D^{-1}V &= \overline{\text{span}\{D^{-1}T_k(X \ominus Q) : k \in \mathbb{Z}^d\}} = \overline{\text{span}\{T_{Ak}(X \ominus D^{-1}Q) : k \in \mathbb{Z}^d\}} \\ &\subset \overline{\text{span}\{T_{Ak}(X \ominus Q) : k \in \mathbb{Z}^d\}} \subset \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}} = V. \end{aligned}$$

2): Since Q is shift invariant and $Q \perp (X \ominus Q)$, it follows that

$$Q \perp \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}} = V.$$

If X is a reducing subspace, then $Q = X$. By the definition of V , we have $V = \{0\}$. If $V = \{0\}$, then $X = Q$, which shows that X is shift invariant. Thus X is a reducing subspace.

3): By 1) and 2), we have $D^j Q \perp D^l V$ for all $j, l \in \mathbb{Z}$ with $j \geq l$. Thus for each $j \in \mathbb{Z}$, $D^j Q \perp \bigcap_{l \leq j} D^l V = \bigcap_{l \in \mathbb{Z}} D^l V$. Therefore $X = \overline{\text{span}\{D^j Q : j \in \mathbb{Z}\}} \perp \bigcap_{l \in \mathbb{Z}} D^l V$. Let M be a reducing subspace containing X . Then $V = \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}} \subset M$. So for each $j \in \mathbb{Z}$, $D^j V \subset D^j M = M$. Hence $\bigcap_{j \in \mathbb{Z}} D^j V \subset M$.

The proof is completed. \square

Proposition 4. Let X and Y be affine subspaces of $L^2(\mathbb{R}^d)$ satisfying $X \subset Y$. Let Q and S be the maximal shift invariant subspaces contained in X and Y respectively. Define $V := \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}}$. Then $(S \cap V^\perp) \ominus Q$ is the maximal shift invariant subspace contained in $Y \ominus X$.

Proof. Let M be a shift invariant subspace contained in $Y \ominus X$. By Lemma 3 and the maximality of S as a shift invariant subspace in Y , we have $M \oplus Q \subset S$. Note that $Q \perp (X \ominus Q)$ and $M \perp X$. Then $(M \oplus Q) \perp (X \ominus Q)$. So $(M \oplus Q) \perp \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}} = V$. Hence $(M \oplus Q) \subset (S \cap V^\perp)$. Therefore $M \subset (S \cap V^\perp) \ominus Q$. The proof is completed. \square

Proposition 5. Let X and Y be affine subspaces of $L^2(\mathbb{R}^d)$ satisfying $X \subset Y$. Let Q and S be the maximal shift invariant subspaces contained in X and Y respectively. Define $V := \overline{\text{span}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}}$. Then $Y \ominus X$ is an affine subspace of $L^2(\mathbb{R}^d)$ if and only if $Y = \overline{\text{span}\{D^l(S \cap V^\perp) : l \in \mathbb{Z}\}}$.

Proof. According to Proposition 4, $(S \cap V^\perp) \ominus Q$ is the maximal shift invariant subspace in $Y \ominus X$. If $Y \ominus X$ is an affine subspace, then by Lemma 3, $S \cap V^\perp$ is a generating shift invariant subspace for Y , i.e., $Y = \overline{\text{span}\{D^l(S \cap V^\perp) : l \in \mathbb{Z}\}}$. Now suppose $Y = \overline{\text{span}\{D^l(S \cap V^\perp) : l \in \mathbb{Z}\}}$. Since $D^{l+1}S \subset D^l S$ and $D^l V \subset D^{l+1}V$ by Lemma 5 for $l \in \mathbb{Z}$, we have $D^{l+1}(S \cap V^\perp) \subset D^l(S \cap V^\perp)$ for $l \in \mathbb{Z}$. Thus by Lemma 4,

$$Y = \overline{\text{span}\{D^l(S \cap V^\perp) : l \in \mathbb{Z}\}} = \overline{\bigcup_{l \in \mathbb{Z}} D^l(S \cap V^\perp)}.$$

Write $M := (S \cap V^\perp) \ominus Q$ and $\tilde{M} := (Y \ominus X) \ominus ((Y \ominus X) \cap (S \cap V^\perp))$. Then $\tilde{M} \subset M$. In fact,

$$\begin{aligned} (Y \ominus X) \ominus (S \cap V^\perp \cap Q^\perp) &= (Y \ominus X) \cap ((S \cap V^\perp) \cap Q^\perp)^\perp \\ &= (Y \ominus X) \cap ((S \cap V^\perp)^\perp \cup Q) \\ &= ((Y \ominus X) \cap (S \cap V^\perp)^\perp) \cup ((Y \ominus X) \cap Q) \\ &= (Y \ominus X) \cap (S \cap V^\perp)^\perp. \end{aligned}$$

Hence

$$(Y \ominus X) \ominus ((Y \ominus X) \cap (S \cap V^\perp)^\perp) \subset ((S \cap V^\perp) \ominus Q)$$

due to the fact that $\mathcal{H} \ominus \mathcal{M} \subset \mathcal{L}$ is equivalent to $\mathcal{H} \ominus \mathcal{L} \subset \mathcal{M}$ for a given Hilbert space \mathcal{H} with its two subspaces \mathcal{L} and \mathcal{M} . Also by Lemma 4, we have

$$\begin{aligned} \overline{\text{span}}\{D^l \tilde{M} : l \in \mathbb{Z}\} &= \overline{\bigcup_{l \in \mathbb{Z}} D^l \tilde{M}} \\ &= (Y \ominus X) \ominus \left((Y \ominus X) \cap \bigcap_{l \in \mathbb{Z}} (D^l (S \cap V^\perp))^\perp \right) \\ &= (Y \ominus X) \ominus \left((Y \ominus X) \cap \left(\overline{\bigcup_{l \in \mathbb{Z}} D^l (S \cap V^\perp)} \right)^\perp \right) \\ &= (Y \ominus X) \ominus ((Y \ominus X) \cap Y^\perp) = Y \ominus X. \end{aligned}$$

So $Y \ominus X = \overline{\text{span}}\{D^l \tilde{M} : l \in \mathbb{Z}\} = \overline{\text{span}}\{D^l M : l \in \mathbb{Z}\}$. The proof is completed. \square

Proposition 6. Let X and Y be two affine subspaces of $L^2(\mathbb{R}^d)$ with $X \subset Y$. Then the following holds.

- 1) $Y \ominus X$ is affine if X is reducing;
- 2) $\bigcap_{n \in \mathbb{Z}} D^n V = \{0\}$ if Y is reducing and $Y \ominus X$ is affine, where $V = \overline{\text{span}}\{T_k(X \ominus Q) : k \in \mathbb{Z}^d\}$ and Q is the

maximal shift invariant subspace in X .

Proof. 1): By Lemma 5, $V = \{0\}$ with X being a reducing subspace. Then $S \ominus Q$ is the maximal shift invariant subspace for $Y \ominus X$ by Proposition 4. Now we only need to show that

$Y \ominus X = \overline{\text{span}}\{D^j(S \ominus Q) : j \in \mathbb{Z}\}$. Note that

$$(S \ominus Q) \oplus X = \overline{\text{span}}\{(S \ominus Q) \oplus Q, X\}$$

due to the facts that $(S \ominus Q) \perp X$ and $Q \subset X$. So by Lemma 1,

$$\overline{P_{X^\perp} S} = \overline{\text{span}}\{S, X\} \ominus X = \overline{\text{span}}\{(S \ominus Q) \oplus Q, X\} \ominus X = (S \ominus Q) \oplus X \ominus X = S \ominus Q.$$

Observe that $P_{X^\perp} D^l = D^l P_{X^\perp}$ since X^\perp is invariant under D^l for $l \in \mathbb{Z}$. Therefore,

$$\begin{aligned} &\overline{\text{span}}\{D^j(S \ominus Q) : j \in \mathbb{Z}\} \\ &= \overline{\text{span}}\{D^j P_{X^\perp} S : j \in \mathbb{Z}\} = \overline{\text{span}}\{D^j P_{X^\perp} S : j \in \mathbb{Z}\} = \overline{\text{span}}\{P_{X^\perp} D^j S : j \in \mathbb{Z}\} \\ &= \overline{P_{X^\perp} \overline{\text{span}}\{D^j S : j \in \mathbb{Z}\}} = \overline{P_{X^\perp} Y} = \overline{\text{span}}\{X, Y\} \ominus X = Y \ominus X. \end{aligned}$$

- 2): According to Proposition 5, it follows that $Y = \overline{\bigcup_{l \in \mathbb{Z}} D^l (Y \cap V^\perp)}$. By Lemma 4,

$$\begin{aligned} Y &= \overline{\bigcup_{l \in \mathbb{Z}} D^l (Y \cap V^\perp)} = \overline{\bigcup_{l \in \mathbb{Z}} (Y \cap (D^l V^\perp))} = \overline{Y \cap \left(\bigcup_{l \in \mathbb{Z}} (D^l V^\perp) \right)} \\ &\subset Y \cap \overline{\bigcup_{l \in \mathbb{Z}} (D^l V^\perp)} = Y \cap \left(\bigcap_{l \in \mathbb{Z}} (D^l V) \right)^\perp, \end{aligned}$$

which shows that $Y \subset \left(\bigcap_{l \in \mathbb{Z}} (D^l V) \right)^\perp$, i.e., $\bigcap_{l \in \mathbb{Z}} (D^l V) \subset Y^\perp$. Since $\bigcap_{l \in \mathbb{Z}} (D^l V)$ is contained in any reducing space containing X by Lemma 4, $\bigcap_{l \in \mathbb{Z}} (D^l V) \subset Y$. Consequently $\bigcap_{l \in \mathbb{Z}} (D^l V) = \{0\}$. The proof is completed. \square

Theorem 7. Let X be an affine subspace of $L^2(\mathbb{R}^d)$. Then the following holds.

- 1) There exist a shift invariant subspace M in X such that $D^n M \perp D^m M$ for $n, m \in \mathbb{Z}$ with $n \neq m$, and $X = \bigoplus_{n \in \mathbb{Z}} D^n M$;
- 2) If X is a non-zero reducing subspace and $|\det A| = 2$, then there exist two purely non-reducing affine subspaces X_1 and X_2 such that $X = X_1 \oplus X_2$;
- 3) If X is non-zero and not reducing, then there exists a unique decomposition $X = X_1 \oplus X_2$ with X_1 be reducing and X_2 being purely non-reducing;
- 4) If X is non-zero and $|\det A| = 2$, then X is the orthogonal direct sum of at most three purely non-reducing affine subspace.

Proof. 1): By Proposition 1, it follows that $X = \left(\bigoplus_{n \in \mathbb{Z}} D^n M_1 \right) \oplus Y$, where M_1 is some shift invariant subspace in X and Y is a reducing subspace. If $Y = \{0\}$, then the result follows. Otherwise, there is a $\psi \in L^2(\mathbb{R}^d)$ such that $\{D^n T_k \psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is an orthonormal basis for Y . Let $M_2 = \overline{\text{span}\{T_k \psi : k \in \mathbb{Z}^d\}}$ and define $M = M_1 \oplus M_2$. Note that by the definition of M_1 in the proof of Proposition 1, it follows that $D^n M_1 \perp D^m M_1$ for $n, m \in \mathbb{Z}$ with $n \neq m$. So $X = \bigoplus_{n \in \mathbb{Z}} D^n M$ with $D^n M \perp D^m M$ when $n, m \in \mathbb{Z}$ and $n \neq m$.

2): Let ψ be an orthonormal wavelet for X . Choose $k, l \in \mathbb{N}$ such that $k = 2^{l-1}$ and $n_0 \in \mathbb{Z}^d \setminus A\mathbb{Z}^d$. Let $E_1 := \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1}\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d / A^{l-1}\mathbb{Z}^d$. Then $F := \{A^{l-1}n_0 + \varepsilon_0, A^{l-1}n_0 + \varepsilon_1, \dots, A^{l-1}n_0 + \varepsilon_{k-1}\} \cup E_1$ is a set of representatives of distinct cosets in $\mathbb{Z}^d / A^l\mathbb{Z}^d$. Indeed, for $0 \leq i, j \leq k-1$, clearly $A^{l-1}n_0 + \varepsilon_i - \varepsilon_j = A^{l-1}n_0 \notin A^l\mathbb{Z}^d$ if $i = j$. Now we consider the case $i \neq j$. Observe that $A^{l-1}n_0 + \varepsilon_i - \varepsilon_j \notin A^l\mathbb{Z}^d$ equals to $n_0 + A^{-(l-1)}(\varepsilon_i - \varepsilon_j) \notin A\mathbb{Z}^d$. Note that $\varepsilon_i - \varepsilon_j \notin A^{l-1}\mathbb{Z}^d$. So $A^{l-1}(\varepsilon_i - \varepsilon_j) \notin \mathbb{Z}^d$. Define two subsets Ψ and Φ of X and two shift invariant subspaces P and M as follows:

$$\begin{aligned} \Psi &= \{D^l T_{\varepsilon_0} \psi, D^l T_{\varepsilon_1} \psi, \dots, D^l T_{\varepsilon_{k-1}} \psi\}, \\ \Phi &= \{D^l T_{A^{l-1}n_0 + \varepsilon_0} \psi, \dots, D^l T_{A^{l-1}n_0 + \varepsilon_{k-1}} \psi\}, \\ P &= \overline{\text{span}\{T_n f : f \in \Psi, n \in \mathbb{Z}^d\}}, \\ M &= \overline{\text{span}\{T_n g : g \in \Phi, n \in \mathbb{Z}^d\}}. \end{aligned}$$

Then $\{T_n f : f \in \Psi, n \in \mathbb{Z}^d\}$ forms an orthonormal basis for P due to the fact that ψ is an orthonormal wavelet for X . The same to M . Define

$$X_1 = \bigoplus_{j \in \mathbb{Z}} D^j P \quad \text{and} \quad X_2 = \bigoplus_{j \in \mathbb{Z}} D^j M.$$

Then $X = X_1 \oplus X_2$. Next, we will show X_1 is a purely non-reducing affine subspace. Write

$$Q := \bigoplus_{j=0}^{\infty} D^j P.$$

Obviously Q is a shift invariant subspace contained in X_1 and $P = Q \ominus DQ$. According to Proposition 2, it suffices to show that Q is the maximal shift invariant subspace contained in X_1 . Also by Proposition 3, it is enough to show $X_1 \cap \left(\overline{\text{span}}\{T_n L_1 : n \in \mathbb{Z}^d\} \right)^\perp \subset Q$, where $L_1 = \overline{\text{span}}\{T_n X_1 : n \in \mathbb{Z}^d\} \ominus X_1$. Observe that for each $j \in \mathbb{N}$,

$$\begin{aligned} T_{A^{j-1}n_0} D^{-j} \Psi &= \left\{ T_{A^{j-1}n_0} D^{l-j} T_{\varepsilon_0} \psi, T_{A^{j-1}n_0} D^{l-j} T_{\varepsilon_1} \psi, \dots, T_{A^{j-1}n_0} D^{l-j} T_{\varepsilon_{k-1}} \psi \right\} \\ &= \left\{ D^{l-j} T_{A^{l-1}n_0 + \varepsilon_0} \psi, D^{l-j} T_{A^{l-1}n_0 + \varepsilon_1} \psi, \dots, D^{l-j} T_{A^{l-1}n_0 + \varepsilon_{k-1}} \psi \right\} \\ &\subset D^{-j} M. \end{aligned}$$

Then for each $j \in \mathbb{N}$,

$$T_{A^{j-1}n_0} D^{-j} \Psi \subset \overline{\text{span}}\{T_n X_1 : n \in \mathbb{Z}^d\} \ominus X_1 = L_1$$

since $D^m M \perp X_1$ for all $m \in \mathbb{Z}$. Therefore, for each $j \in \mathbb{N}$,

$$D^{-j} \Psi \subset \overline{\text{span}}\{T_n L_1 : n \in \mathbb{Z}^d\}.$$

Hence

$$\begin{aligned} \bigoplus_{j \in \mathbb{N}} D^{-j} P &= \overline{\text{span}}\{D^{-j} T_n D^l T_{\varepsilon_0} \psi, D^{-j} T_n D^l T_{\varepsilon_1} \psi, \dots, D^{-j} T_n D^l T_{\varepsilon_{k-1}} \psi : j \in \mathbb{N}, n \in \mathbb{Z}^d\} \\ &= \overline{\text{span}}\{D^{-j} T_{A^{-j}n} D^l T_{\varepsilon_0} \psi, D^{-j} T_{A^{-j}n} D^l T_{\varepsilon_1} \psi, \dots, D^{-j} T_{A^{-j}n} D^l T_{\varepsilon_{k-1}} \psi : j \in \mathbb{N}, n \in \mathbb{Z}^d\} \\ &= \overline{\text{span}}\{T_n D^{l-j} T_{\varepsilon_0} \psi, T_n D^{l-j} T_{\varepsilon_1} \psi, \dots, T_n D^{l-j} T_{\varepsilon_{k-1}} \psi : j \in \mathbb{N}, n \in \mathbb{Z}^d\} \\ &= \overline{\text{span}}\{T_n h : h \in D^{-j} \Psi, j \in \mathbb{N}, n \in \mathbb{Z}^d\} \subset \overline{\text{span}}\{T_n L_1 : n \in \mathbb{Z}^d\}. \end{aligned}$$

Thus $X_1 \cap \left(\overline{\text{span}}\{T_n L_1 : n \in \mathbb{Z}^d\} \right)^\perp \subset \bigoplus_{j=0}^{\infty} D^j P = Q$. So X_1 is a purely non-reducing affine subspace. Similarly to

X_2 .

3): Let X be a non-reducing affine subspace of $L^2(\mathbb{R}^d)$ and X_1 be the maximal reducing subspace contained in X . Write $X_2 = X \ominus X_1$. Then X_2 is affine by Proposition 6 and X_2 is purely non-reducing since X_1 is the maximal reducing subspace in X . Also note that the orthogonal complement of a reducing space within another reducing space is always reducing. Then the uniqueness follows.

4): 4) follows after 2) and 3). The proof is completed. \square

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