

Asymptotic Expansion of Wavelet Transform

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Received 14 October 2014; revised 30 October 2014; accepted 14 November 2014

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Abstract

In the present paper, we obtain asymptotic expansion of the wavelet transform for large value of dilation parameter a by using López technique. Asymptotic expansion of Shannon wavelet, Morlet wavelet and Mexican Hat wavelet transform are obtained as special cases.

Keywords

Asymptotic Expansion, Wavelet Transform, Mellin Convolution, Integral Transform

1. Introduction

The continuous wavelet transform of a function h with respect to the wavelet ϕ is defined as

$$(W_{\phi}h)(b, a) = a^{-\frac{1}{2}} \int_{\mathbb{R}} h(x) \overline{\phi\left(\frac{x-b}{a}\right)} dx, \quad a > 0, \quad b \in \mathbb{R}, \quad (1)$$

provided the integral exists [1]. The asymptotic expansion for Mellin convolution

$$I(\mu) = \int_0^{\infty} h(x) g(\mu x) dx; \quad \text{as } \mu \rightarrow 0+, \quad (2)$$

was proposed by López [2], under dyadic conditions on g and h . Let us remind earlier results from [2], which will be used in present study. We assume that $g(x)$ and $h(x)$ have asymptotic expansions of the form:

$$g(x) = \sum_{i=0}^{n-1} a_i x^{i-p} + g_n(x); \quad \text{as } x \rightarrow 0+, \quad (3)$$

and

$$h(x) = \sum_{i=0}^{n-1} b_i x^{-i-q} + h_n(x); \quad \text{as } x \rightarrow \infty. \quad (4)$$

Also assume that

$$g(x) = O(x^{-\lambda_1}); \text{ when } x \rightarrow \infty \text{ and } \lambda_1 \in \mathbb{R}, \quad (5)$$

and

$$h(x) = O(x^{-\lambda_2}); \text{ when } x \rightarrow 0+ \text{ and } \lambda_2 \in \mathbb{R}, \quad (6)$$

with the parameters p, q, λ_1 and λ_2 satisfying the following conditions:

$$p + \lambda_1 < 1 < q + \lambda_2; \lambda_1 < q \text{ and } p < \lambda_2. \quad (7)$$

The asymptotic expansion of (2) at the origin is given by the following Theorem ([2], pp. 631, 633, 634).

Theorem 1 Assume that (i) $g(x)$ and $h(x)$ are locally integrable on $(0, \infty)$; (ii) $g(x)$ and $h(x)$ have expansions of the form (3), (5) and (4), (6) respectively and (iii) p, q, λ_1 and λ_2 satisfy (7), then the asymptotic expansion of (2) as $\mu \rightarrow 0+$ are given by

Case I: For any $n = 1, 2, 3, \dots$ and $m = n + \lfloor p + q \rfloor$ with $p + q \notin \mathbb{Z}$, we have

$$\int_0^\infty h(x) g(\mu x) dx = \sum_{i=0}^{n-1} b_i M[g; 1-i-q] \mu^{i+q-1} + \sum_{i=0}^{m-1} a_i M[h; 1+i-p] \mu^{i-p} + O(\mu^{n+q-1}). \quad (8)$$

Case II: For any $n = 1, 2, 3, \dots$ and $m = n + p + q - 1$ with $p + q \in \mathbb{N}$, we have

$$\begin{aligned} \int_0^\infty h(x) g(\mu x) dx &= \sum_{i=0}^{p+q-2} a_i M[h; 1+i-p] \mu^{i-p} \\ &+ \sum_{i=0}^{n-1} \mu^{i+q-1} \left\{ -b_i a_{i+p+q-1} \log \mu + \lim_{z \rightarrow 0} [b_i M[g; z+1-i-q] + a_{i+p+q-1} M[h; z+i+q]] \right\} \\ &+ O(\mu^{m-p} \log \mu). \end{aligned} \quad (9)$$

Case III: For any $n = 1, 2, 3, \dots$ and $n = m + 1 - p - q$ with $1 - p - q \in \mathbb{N}$, we have

$$\begin{aligned} \int_0^\infty h(x) g(\mu x) dx &= \sum_{i=0}^{-p-q} b_i M[g; 1-i-q] \mu^{i+q-1} \\ &+ \sum_{i=0}^{m-1} \mu^{i-p} \left\{ -a_i b_{i+1-p-q} \log \mu + \lim_{z \rightarrow 0} [b_{i+1-p-q} M[g; z+p-i] + a_i M[h; z+i+1-p]] \right\} \\ &+ O(\mu^{m-p} \log \mu). \end{aligned} \quad (10)$$

By using Wong technique, the asymptotic expansions of wavelet transform (1) for large and small values of dilation parameters and translation, parameters were obtained by Pathak and Pathak 2009 [3]-[5].

The main aim of the present paper is to derive asymptotic expansion of the wavelet transform for large value of a , by using Theorem 1. We also obtain asymptotic expansions for the special transforms corresponding to Shannon wavelet, Morlet wavelet and Mexican hat wavelet.

2. Asymptotic Expansion of the Wavelet Transform for Large Value of a

In this section, we obtain asymptotic expansion of the wavelet transform (1), when $a \rightarrow \infty$.

Now, let us rewrite (1) in the form:

$$\begin{aligned} (W_\phi h)(b, a) &= \mathcal{G}^2 \int_{-\infty}^\infty h(x+b) \overline{\phi(\mathcal{G}x)} dx \\ &= \mathcal{G}^2 \left\{ \int_0^\infty h(x+b) \overline{\phi(\mathcal{G}x)} dx + \int_0^\infty h(-x+b) \overline{\phi(-\mathcal{G}x)} dx \right\} \\ &= (W_\phi^+ h)(b, a) + (W_\phi^- h)(b, a) \text{ (say);} \end{aligned} \quad (11)$$

where, $\mathcal{G} = \frac{1}{a}$ and b is assumed to be a fixed real number.

Setting $h(x+b) = \chi(x)$ and assume that $\chi(x)$ and $\overline{\phi(x)}$ are locally integrable on $(0, \infty)$. Further assume that $\chi(x)$ and $\overline{\phi(x)}$ have asymptotic expansions of the form

$$\overline{\phi(x)} = \sum_{i=0}^{n-1} c_i x^{i-p} + \overline{\phi_b(x)}; \quad \text{as } x \rightarrow 0+, \quad (12)$$

$$\chi(x) = \sum_{i=0}^{n-1} d_i x^{-i-q} + \chi_n(x); \quad \text{as } x \rightarrow \infty. \quad (13)$$

Also assume that

$$\overline{\phi(x)} = O(x^{-\beta}); \quad \text{as } x \rightarrow \infty, \quad \beta \in \mathbb{R}, \quad (14)$$

and

$$\chi(x) = O(x^{-\gamma}); \quad \text{as } x \rightarrow 0+, \quad \gamma \in \mathbb{R}. \quad (15)$$

with the parameters p, q, β and γ satisfy the following condition

$$p + \beta < 1 < q + \gamma; \quad \beta < q \quad \text{and} \quad p < \gamma. \quad (16)$$

Then by using, Theorem 1, we obtain asymptotic expansion of $(W_\phi^+ h)(b, a)$ for large value of a .

Case I: When $n = 1, 2, 3, \dots$ and $m = n + \lfloor p + q \rfloor$ with $p + q \notin \mathbb{Z}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{n-1} b_i M[\bar{\phi}; 1-i-q] \mathcal{G}^{i+q-1/2} + \sum_{i=0}^{m-1} a_i M[\chi; 1+i-p] \mathcal{G}^{i-p+1/2} + O(\mathcal{G}^{n+q-1/2}) \\ &= \sum_{i=0}^{n-1} b_i M[\bar{\phi}; 1-i-q] a^{-i+q+1/2} + \sum_{i=0}^{m-1} a_i M[\chi; 1+i-p] a^{-i+p-1/2} + O(a^{-n+q+1/2}). \end{aligned} \quad (17)$$

Case II: When $n = 1, 2, 3, \dots$ and $m = n + p + q - 1$ with $p + q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{p+q-2} a_i M[\chi; 1+i-p] \mathcal{G}^{i-p+1/2} \\ &+ \sum_{i=0}^{n-1} \mathcal{G}^{i+q-1/2} \left\{ -b_i a_{i+p+q-1} \log \mathcal{G} + \lim_{z \rightarrow 0} \left[b_i M[\bar{\phi}; z+1-i-q] + a_{i+p+q-1} M[\chi; z+i+q] \right] \right\} + O(\mathcal{G}^{m-p+1/2} \log \mathcal{G}) \\ &= \sum_{i=0}^{p+q-2} a_i M[\chi; 1+i-p] a^{-i+p-1/2} \\ &+ \sum_{i=0}^{n-1} a^{-i+q+1/2} \left\{ -b_i a_{i+p+q-1} \log(1/a) + \lim_{z \rightarrow 0} \left[b_i M[\bar{\phi}; z+1-i-q] + a_{i+p+q-1} M[\chi; z+i+q] \right] \right\} + O(a^{-m+p-1/2} \log(1/a)). \end{aligned} \quad (18)$$

Case III: When $m = 1, 2, 3, \dots$ and $n = m + 1 - p - q$ with $1 - p - q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{-p-q} b_i M[\bar{\phi}; 1-i-q] \mathcal{G}^{i+q-1/2} \\ &+ \sum_{i=0}^{m-1} \mathcal{G}^{i-p+1/2} \left\{ -a_i b_{i+1-p-q} \log \mathcal{G} + \lim_{z \rightarrow 0} \left[b_{i+1-p-q} M[\bar{\phi}; z+p-i] + a_i M[\chi; z+i+1-p] \right] \right\} \\ &+ O(\mathcal{G}^{m-p+1/2} \log \mathcal{G}) \\ &= \sum_{i=0}^{-p-q} b_i M[\bar{\phi}; 1-i-q] a^{-i+q+1/2} \\ &+ \sum_{i=0}^{m-1} a^{-i+p-1/2} \left\{ -a_i b_{i+1-p-q} \log(1/a) + \lim_{z \rightarrow 0} \left[b_{i+1-p-q} M[\bar{\phi}; z+p-i] + a_i M[\chi; z+i+1-p] \right] \right\} \\ &+ O(a^{-m+p-1/2} \log(1/a)). \end{aligned} \quad (19)$$

Similarly, we can also obtain the asymptotic expansion of $(W_\phi^- h)(b, a)$ as $a \rightarrow -\infty$.

3. Application

In this section, we apply the previous result and obtain the asymptotic expansions of Shannon wavelet transform, Morlet Wavelet transform and Mexican hat wavelet transform.

3.1. Asymptotic Expansion of the Shannon Wavelet Transform

Let us consider ϕ to be Shannon wavelet and it is given by [1] $\phi(x) = \frac{\sin\left(\frac{\pi x}{2}\right)}{\left(\frac{\pi x}{2}\right)} \cos\left(\frac{3\pi x}{2}\right)$. Since, ϕ is locally integrable on $(0, \infty)$ and has the asymptotic expansion:

$$\overline{\phi(x)} = 1 - \frac{7\pi^2 x^2}{6} + \frac{31\pi^4 x^4}{120} - \frac{217\pi^6 x^6}{5040} + O(x^7) \quad \text{as } x \rightarrow 0+ \quad (20)$$

with

$$\overline{\phi(x)} = O(1); \quad \text{as } x \rightarrow \infty. \quad (21)$$

Consider, $\chi(x)$ is locally integrable on $(0, \infty)$ and satisfies (13) and (15) with parameters

$$1 < q + \gamma; \quad q > 0 \quad \text{and} \quad 0 < \gamma. \quad (22)$$

Now, by using (17), (18) and (19) respectively and by means of formula ([6], p. 321, (41)), then the asymptotic expansions of Shannon wavelet transform are given by

Case I: When $m = 7 + \lfloor q \rfloor$ and $q \notin \mathbb{Z}$, we get

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^6 b_i \left\{ (1 - 2^{i+q}) \pi^{-1+i+q} \cos\left[\frac{1}{2}\pi(-1+i+q)\right] \Gamma[-i-q] \right\} a^{-i-q+1/2} \\ &\quad + \sum_{i=0}^{m-1} a_i M[\chi; 1+i] a^{-i-1/2} + O(a^{-13/2-q}). \end{aligned} \quad (23)$$

Case II: When $m = 6 + q$ and $q \in \mathbb{N}$, we get

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{q-2} a_i M[\chi; 1+i] a^{-i-1/2} + \sum_{i=0}^6 a^{-i-q+1/2} \left\{ -b_i a_{i+q-1} \log(1/a) \right. \\ &\quad \left. + \lim_{z \rightarrow 0} \left[b_i \left\{ (1 - 2^{i+q-z}) \pi^{-1+i+q-z} \cos\left[\frac{1}{2}\pi(-1+i+q-z)\right] \Gamma[-i-q+z] \right\} + a_{i+q-1} M[\chi; z+i+q] \right] \right\} \\ &\quad + O(a^{-m-1/2} \log(1/a)). \end{aligned} \quad (24)$$

Case III: When $m = 6 + q$ and $1 - q \in \mathbb{N}$, we get

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{-q} b_i \left\{ (1 - 2^{i+q}) \pi^{-1+i+q} \cos\left[\frac{1}{2}\pi(-1+i+q)\right] \Gamma[-i-q] \right\} a^{-i-q+1/2} + \sum_{i=0}^{m-1} a^{-i-1/2} \left\{ -a_i b_{i+1-q} \log(1/a) \right. \\ &\quad \left. + \lim_{z \rightarrow 0} \left[b_{i+1-q} \left\{ (1 - 2^{i+q-z}) \pi^{-1+i+q-z} \cos\left[\frac{1}{2}\pi(-1+i+q-z)\right] \Gamma[-i-q+z] \right\} + a_i M[\chi; z+i+1] \right] \right\} \\ &\quad + O(a^{-m-1/2} \log(1/a)). \end{aligned} \quad (25)$$

3.2. Asymptotic Expansion of the Morlet Wavelet Transform

We choose ϕ to be Morlet wavelet and it is given by [1] $\phi(x) = e^{i w_\phi x - \frac{x^2}{2}}$. Since, ϕ is locally integrable on

$(0, \infty)$ and has the asymptotic expansion as

$$\overline{\phi(x)} = 1 - iw_o x + \left(\frac{-1}{2} - \frac{w_o^2}{2} \right) x^2 + \frac{1}{6} i (3w_o + w_o^3) x^3 + O(x^4); \quad \text{as } x \rightarrow 0+, \quad (26)$$

with

$$\overline{\phi(x)} = O(1); \quad \text{as } x \rightarrow \infty.$$

Let $\chi(x)$ is locally integrable on $(0, \infty)$ and satisfy (13) and (15) with parameters (22). Now by using (17), (18) and (19) respectively and by formula ([6], pp. 318, 320, (10,30)), then the asymptotic expansions of Morlet wavelet transform are given by

Case I: When $m = 4 + [q]$ and $q \notin \mathbb{Z}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^3 b_i \left\{ 2^{\frac{1}{2}(-1-i-q)} \left(\Gamma \left[\frac{1}{2}(1-i-q) \right]_1 F_1 \left[\frac{1}{2}(1-i-q), \frac{1}{2}, -\frac{w_o^2}{2} \right] \right. \right. \\ &\quad \left. \left. + i\sqrt{2}\Gamma \left[1 - \frac{i}{2} - \frac{q}{2} \right]_1 F_1 \left[1 - \frac{i}{2} - \frac{q}{2}, \frac{3}{2}, -\frac{w_o^2}{2} \right] w_o \right\} a^{-i-q+\frac{1}{2}} \\ &\quad + \sum_{i=0}^{m-1} a_i M[\chi; 1+i] a^{-i-\frac{1}{2}} + O(a^{-7/2-q}). \end{aligned} \quad (27)$$

Case II: When $m = 3 + q$ and $q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{q-2} a_i M[\chi; 1+i] a^{-i-1/2} \\ &\quad + \sum_{i=0}^3 a^{-i-q+1/2} \left\{ -b_i a_{i+q-1} \log(1/a) + \lim_{z \rightarrow 0} \left[b_i \left\{ 2^{\frac{1}{2}(-1-i-q+z)} \left(\Gamma \left[\frac{1}{2}(1-i-q+z) \right]_1 F_1 \left[\frac{1}{2}(1-i-q+z), \frac{1}{2}, -\frac{w_o^2}{2} \right] \right. \right. \right. \right. \\ &\quad \left. \left. \left. + i\sqrt{2}\Gamma \left[1 - \frac{i}{2} - \frac{q}{2} + \frac{z}{2} \right]_1 F_1 \left[1 - \frac{i}{2} - \frac{q}{2} + \frac{z}{2}, \frac{3}{2}, -\frac{w_o^2}{2} \right] w_o \right\} \right] + a_{i+q-1} M[\chi; z+i+q] \right\} \\ &\quad + O(a^{-m-1/2} \log(1/a)). \end{aligned} \quad (28)$$

Case III: When $m = 3 + q$ and $1 - q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \\ &\quad \sum_{i=0}^{-q} b_i \left\{ 2^{\frac{(-1-i-q)}{2}} \left(\Gamma \left[\frac{1}{2}(1-i-q) \right]_1 F_1 \left[\frac{1}{2}(1-i-q), \frac{1}{2}, -\frac{w_o^2}{2} \right] + i\sqrt{2}\Gamma \left[1 - \frac{i}{2} - \frac{q}{2} \right]_1 F_1 \left[1 - \frac{i}{2} - \frac{q}{2}, \frac{3}{2}, -\frac{w_o^2}{2} \right] w_o \right) \right\} a^{-i-q+\frac{1}{2}} \\ &\quad + \sum_{i=0}^{m-1} a^{-i-1/2} \left\{ -a_i b_{i+1-q} \log(1/a) + \lim_{z \rightarrow 0} \left[b_{i+1-q} \left\{ 2^{\frac{1}{2}(-1-i-q+z)} \left(\Gamma \left[\frac{1}{2}(1-i-q+z) \right]_1 F_1 \left[\frac{1}{2}(1-i-q+z), \frac{1}{2}, -\frac{w_o^2}{2} \right] \right. \right. \right. \right. \\ &\quad \left. \left. \left. + i\sqrt{2}\Gamma \left[1 - \frac{i}{2} - \frac{q}{2} + \frac{z}{2} \right]_1 F_1 \left[1 - \frac{i}{2} - \frac{q}{2} + \frac{z}{2}, \frac{3}{2}, -\frac{w_o^2}{2} \right] w_o \right\} \right] + a_i M[\chi; z+i+1] \right\} + O(a^{-m-1/2} \log(1/a)). \end{aligned} \quad (29)$$

3.3. Asymptotic Expansion of the Mexican Hat Wavelet Transform

We choose ϕ to be Mexican hat wavelet $\phi(x) = (1-x^2)e^{-\frac{x^2}{2}}$ [1]. Since ϕ is locally integrable on $(0, \infty)$

and has the asymptotic expansion:

$$\overline{\phi(x)} = 1 - \frac{3x^2}{2} + \frac{5x^4}{8} - \frac{7x^6}{48} + \frac{3x^8}{128} + O(x^9); \quad \text{as } x \rightarrow 0 + \tag{30}$$

with

$$\overline{\phi(x)} = O(1); \quad \text{as } x \rightarrow \infty$$

As $\chi(x)$ is locally integrable on $(0, \infty)$ and satisfies (13) and (15) with parameters (22). Now by using (17), (18) and (19) respectively, we can obtain the asymptotic expansion of Mexican hat wavelet transform by using formula ([6], p. 313, (13))

Case I: When $m = 9 + [q]$ and $q \notin \mathbb{Z}$, we have

$$(W_\phi^+ h)(b, a) = \sum_{i=0}^8 b_i \left\{ 2^{\frac{1}{2}(-1-i-q)} (i+q) \Gamma \left[\frac{1}{2}(1-i-q) \right] \right\} a^{-i-q+\frac{1}{2}} + \sum_{i=0}^{m-1} a_i M[\chi; 1+i] a^{-i-\frac{1}{2}} + O(a^{-17/2-q}). \tag{31}$$

Case II: When $m = 8 + q$ and $q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{q-2} a_i M[\chi; 1+i] a^{-i-1/2} \\ &+ \sum_{i=0}^8 a^{-i-q+1/2} \left\{ -b_i a_{i+q-1} \log(1/a) + \lim_{z \rightarrow 0} \left[b_i \left\{ 2^{\frac{1}{2}(-1-i-q+z)} (i+q-z) \Gamma \left[\frac{1}{2}(1-i-q+z) \right] \right\} \right] + a_{i+q-1} M[\chi; z+i+q] \right\} \\ &+ O(a^{-m-1/2} \log(1/a)). \end{aligned} \tag{32}$$

Case III: When $m = 8 + q$ and $1 - q \in \mathbb{N}$, we have

$$\begin{aligned} (W_\phi^+ h)(b, a) &= \sum_{i=0}^{-q} b_i \left\{ 2^{\frac{1}{2}(-1-i-q)} (i+q) \Gamma \left[\frac{1}{2}(1-i-q) \right] \right\} a^{-i-q+1/2} \\ &+ \sum_{i=0}^{m-1} a^{-i-1/2} \left\{ -a_i b_{i+1-q} \log(1/a) + \lim_{z \rightarrow 0} \left[b_{i+1-q} \left\{ 2^{\frac{1}{2}(-1-i-q+z)} (i+q-z) \Gamma \left[\frac{1}{2}(1-i-q+z) \right] \right\} + a_i M[\chi; z+i+1] \right] \right\} \\ &+ O(a^{-m-1/2} \log(1/a)). \end{aligned} \tag{33}$$

Acknowledgements

The authors are thankful to Prof. R. S. Pathak, DST Center for Interdisciplinary Mathematical Sciences, Banaras Hindu University, Varanasi-221005, India, for his valuable suggestion for the improvement of the article. We thank the referee for their comments. The research of the first author was supported by U.G.C-BSR start-up grant No. F.30-12/2014 (BSR).

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