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On Extensions of Right Symmetric Rings without Identity

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Abstract

Let us call a ring R (without identity) to be right symmetric if for any triple $a, b, c \in R$ $abc = 0$ then $acb = 0$. Such rings are neither symmetric nor reversible (in general) but are semicommutative. With an idempotent they take care of the sheaf representation as obtained by Lambek. Klein 4-rings and their several generalizations and extensions are proved to be members of such class of rings. An extension obtained is a McCoy ring and its power series ring is also proved to be a McCoy ring.

Keywords

Right (Left) Symmetric Rings, Klein 4-Rings, McCoy Rings

1. Introduction

A ring R is symmetric if for any triple $a_1, a_2, a_3 \in R$, $a_1a_2a_3 = 0$ then for any permutation $\sigma:\{1,2,3\} \to \{1,2,3\}$ $a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} = 0$. J. Lambek in [1] introduced symmetric rings, and got a characterization that a ring R with one is symmetric if and only if R contains a subring which is isomorphic to the rings of sections of a sheaf of prime torsion free symmetric rings. Lambek also noticed that the symmetric property is a weaker notion than that of primeness (see [1: p. 362]). The class of symmetric rings lie between the classes of reduced and reversible rings and they have been extensively studied and generalized in various directions, for instance, some references are [2]-[4], and [5]. Most of the studies on symmetric rings were carried over rings with identity. In this note we assume that rings, in general, are not equipped with the multiplicative identity. Let us say that a ring R is right symmetric if for any triple $a_1, a_2, a_3 \in R$, $a_1a_2a_3 = 0$, then $a_1a_3a_2 = 0$. Left symmetric rings are defined analogously. Some concrete examples are given here to show that right (as well as left) symmetric rings are different than symmetric rings. It is observed that, the Lambek criterion about symmetric rings with one, as

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given in [1], can be extended to right (or left) symmetric rings with idempotents (Proposition 2.7).

A weaker notion of symmetric is reversible which P.M. Cohn defined in [6] as: a ring R is reversible if for any $a_1, a_2 \in R$, $a_1a_2 = 0$ implies $a_2a_1 = 0$. In [7], Anderson and Camillo defined that a ring R (may not be with 1) satisfies ZC_n, if for any $a_i \in R$, where $i = 1, \cdots, n$, the product $\prod_{i=1}^{n} a_i = 0$ implies that the product $\prod_{i=1}^{n} a_{\sigma(i)} = 0$, where $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$, is a permutation. Thus, in their terminology, if a ring satisfies ZC_2, it is reversible, and if it satisfies ZC_3, it is symmetric. They proved that ZC_3 implies ZC_n, $\forall n \geq 3$, but the converse need not be true in general ([7]; Example I-4).

For a ring R with 1_R, clearly, every symmetric ring is reversible, but the converse may not be true, for instance, see ([7]; Example 1-5). In ([8]: Example 7], Mark proved that the group ring $\mathbb{Z}_2(Q_8) := \{x_t : t \in Q_8\}$ where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the group of quaternions, is reversible but not symmetric. For a ring without one, a symmetric ring may not be reversible. For instance, for any ring *A* consider the ring of strictly upper triangular matrices $\mathrm{SUTM}_3[A]$. Then $\forall x, y, z \in \mathrm{SUTM}_3[A]$, $xyz = 0$, so $\mathrm{SUTM}_3[A]$ is symmetric.

On the other hand one sees that if $1 \in A$, then $e_{23}e_{12} = 0$, but $e_{12}e_{23} = e_{13} \neq 0$. Hence $\mathrm{SUTM}_3[A]$ is not reversible. Thus by above fairly simple examples we firmly state that, for rings, in general,

$$\text{symmetry} \not\Leftrightarrow \text{reversibility}$$

In Section 2, after definition, we gave some examples of right symmetric rings which are not symmetric, and developed some interactions with other classes of rings such as von Neuman regular, semicommutative, and Armendariz. In Section 3 we did some extensions of Klein 4-rings and a McCoy ring is constructed in the last section.

2. Right and Left Symmetric Rings

One notices that, in a ring without 1, and with $abc = 0$ implies that $acb = 0$, the commutation only appears on the last two elements. There is no guaranty that $cab = 0$, for the support of this claim we provide below some examples. So, let us define that:

Definitions 2.1. A ring R is right (respt. left) symmetric if for any triple, $a, b, c \in R$, $abc = 0$ implies that $acb = 0$ (respt. $bac = 0$). R is symmetric if R is both, left and right symmetric.

Examples 2.2. (1) *Klein* 4-*rings*. ([8]: Example 1] Consider the so called Klein-4 ring: $V = \{0, a, b, c\}$, which has two generators *a* and *b*, and is a Klein 4-group with respect to addition. Its characteristic is 2 and the relations among its elements are:

$$c = a + b, \quad a^2 = ab = a, \quad b^2 = ba = b$$

Let us consider all possible products of the three non-zero elements of *V*. There are total 3^3 products, among them 15 are zero and 12 are non zero. Consider a typical product xyz of three nonzero elements $x, y, z \in V$. Then $xyz = 0$, if either $y = c$ or $z = c$. This means that $xzy = 0$. So V is right symmetric. If $y \neq c$ and $z \neq c$, then clearly $xyz \neq 0$. For instance $abc = 0 = acb$ but $cab = c \neq 0$. This implies that V is not symmetric. (Erroneously it is mentioned in ([8], Example 1]) that V is symmetric). Obviously, there is no question of reversibility as well, as $ac = 0$ but $ca \neq 0$.

Similarly, the opposite ring, V^{op} is left symmetric and is neither symmetric nor reversible. Both rings are not reduced also, because $c \in V$ is a non-zero nilpotent element.

(2) For any ring R define the $n \times n$ - kth-column (respt. row) matrix ring, denoted by $M_n\mathrm{Col}_k(R)$ (respt. $M_n\mathrm{Row}(R)$, to be a subring, without identity, of the full matrix ring $M_n(R)$ such that it has non-zero elements only in the kth-column (respt. kth-row). In fact $M_n\mathrm{Col}_k(R)$ (respt. $M_n\mathrm{Row}(R)$) is a left (respt. right) ideal of $M_n(R)$. Note that, R is right symmetric if and only if $M_n\mathrm{Col}_k(R)$ is right symmetric. Indeed, if we let R to be right symmetric and $[a_{ik}], [b_{ik}], [c_{ik}] \in M_n\mathrm{Col}_k(R)$, then

$$[a_{ik}][b_{ik}][c_{ik}] = [a_{ik}]b_{kk}c_{kk} = 0 \Rightarrow [a_{ik}]c_{kk}b_{kk} = [a_{ik}][c_{ik}][b_{ik}] = 0$$

The converse is obvious. Analogously, $M_n\mathrm{Row}(R)$ is left symmetric. Note that, if R is symmetric or even a commutative domain or a field, $M_n\mathrm{Col}_k(R)$ may not be symmetric. For instance, in $M_3\mathrm{Col}_2(\mathbb{Z}_2)$ if we let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then one can easily observe that $ABC = ACB = 0$

but $BAC \neq 0$.

Similarly, $M_n\text{Row}(R)$ is left symmetric and is not symmetric.

(3) Let D be any domain. Then the direct sum $V \oplus D$ and $V^{op} \oplus D$ are right and left symmetric rings, respectively, under component wise addition and multiplication. Similarly, $V \oplus V \oplus \cdots$ and $V^{op} \oplus V^{op} \oplus \cdots$ are right and left symmetric rings, respectively.

(4) In [5] Kwak defined left and right α-symmetric rings as follows: Let α be an endomorphism on a ring R. Then R is right (respt. left) α-symmetric, if for any triple, $a, b, c \in R$, $abc = 0 \Rightarrow ac\alpha(b) = 0$ (respt. $abc = 0 \Rightarrow \alpha(b)ac = 0$). Thus right and left symmetric rings are special cases of right and left α-symmetric rings, with $\alpha = \text{Identity map}$. It follows immediately from ([5], Proposition 2.3(2)) that if a reversible ring is left (or right) symmetric, then it is symmetric. Note that in [5] rings are with identity.

There is a symmetry between right and left symmetric rings, because a ring R is right symmetric if and only if its opposite ring is left symmetric. So in the following we will only deal with right symmetric rings, left symmetric rings will appear when needed.

A ring R is said to be semicommutative as defined by Bell in [9], if for any pair $a, b \in R$, $ab = 0$ then for all $r \in R$, $arb = 0$. There are several names of a semicommutative ring in literature. For historical remarks and other details we refer the reader to [10]. All reduced rings are symmetric and symmetric rings are semicommutative. The ring V in Example 2.2. is semicommutative (can be checked easily). A ring R is abelian if every idempotent $e \in R$ is central, duo if every right and left ideals are ideals, and reflexive if for any pair $a, b \in R$, $aRb = 0$, then $bRa = 0$. A ring R is von Neumann regular if $\forall a \in R$, there exists an $x \in R$, such that $a = axa$.

A right symmetric ring in general is non abelian, non duo, non reflexive, and not a von Neumann regular ring. We pose quick counter examples for these claims. The Klein 4-ring $V = \{0, a, b, c\}$ is right symmetric in which a is an idempotent. Because $ac = 0$ and $ca = c$, so a is not central, so V is non abelian. In V, $\{o, a\}$ is a right ideal but $ca = c$ means that $\{0, a\}$ is not an ideal, so V is not right duo. Because $aVc = 0$ but $cVa = \{0, c\} \neq 0$, hence V is not reflexive. Finally, $ccc = cac = cbc = c0c = 0$, so V is not von Neumann regular.

It is defined in [11] that a ring R with an involution $*$ is $*$-reversible, in case for every pair of elements $a, b \in R$, such that $ab = 0$, then $ba^* = 0$.

There are several right symmetric rings without one which are symmetric. For instance, the ring of strictly upper triangular matrices over any ring is without one and is symmetric. Few more cases are given in the following:

Proposition 2.3. (1) *Every symmetric ring is right symmetric and every right symmetric ring with one is symmetric*.

(2) *Every reduced ring is right symmetric* ([1]: (G); [7]: Theorem I-3). *Conversely*, *a right symmetric ring which is not symmetric cannot be reduced*.

(3) *Every right symmetric ring is semicommutative*.

(4) *Every von Neumann regular ring which is right symmetric is symmetric*.

(5) *Every reversible ring which is right symmetric is symmetric*.

(6) *Every ring with involution which is right symmetric is symmetric*.

(7) *Every ring with a reversible involution is right symmetric and hence symmetric*.

(8) (1) - (7) *all hold if we replace right by left*.

Proof: (1) and (5) are obvious.

(2) Let a ring R be reduced. Assume that for some $a, b, c \in R$, $abc = 0$ then $abcb = cbab = cbacba = 0 = cba$ this means that $acb = 0$. Hence R is right symmetric.

Conversely, let R be right symmetric but not symmetric. Assume that $a, b, c \in R$ such that $abc = 0$ and $acb = 0$ but at least one of cab, cba, bac and bca is not equal to zero. Thus if $cab \neq 0$ then $cabcab = 0$. Hence R is not reduced. If $cab = 0$, then $cba = 0$, so $bac \neq 0$ But then $bacbac = 0$.

(3) Let R be a right symmetric ring. Assume that for any pair $a, b \in R$, $ab = 0$. Then for all $r \in R$, $abr = 0$. Hence $arb = 0$, and so R is semicommutative.

(4) Assume that R is von Neumann regular and is right symmetric. Let $a \in R$ be such that $a^2 = 0$. Then for some $x \in R$, $a = axa$. Then $axa = axaxa$ or $ax(a - axa) = 0 = a(a - axa)x = 0$. Thus $a^2xax = a^2x = 0 = axa = a$, we conclude that R is reduced. Then by (2) R is left symmetric, hence symmetric.

(6) Let R be a ring with an involution $*$. This means that $*$ is an anti-automorphism on R of order two. In addition, let R be right symmetric. If $abc = 0$ for some $a, b, c \in R$ then, because R is right symmetric,

$acb=0$. Then $(acb)^*=b^*c^*a^*=0$ or that because R is right symmetric, $b^*a^*c^*=0$. By doubling the involution we get $cab=0$ which implies that $cba=0$. Again, $abc=0$ gives $c^*b^*a^*=c^*a^*b^*=0$, and by the doubling of involution we get $bac=0$ and so the right symmetry gives $bca=0$.

(7) Let R be a ring with an involution $*$ and let R be $*$-reversible. Now assume that $abc=0$ for some $a, b, c \in R$. Then $(bc)a^*=0=a^*c^*b^*$ which gives $bca=0$ or that $a(bc)^*=(bc)^*a^*=c^*b^*a^*=acb=0$. By similar techniques we get the remaining permutations equal to zero. So $*$-reversible rings are right and left symmetric, hence symmetric.

(8) holds by left and right symmetry. □

A quick consequence of Proposition 2.3 (6) is the following.

***Corollary* 2.4.** *Every right symmetric ring which is not symmetric cannot adhere to an involution.*

Examples 2.5. Hence, V and V^{op}, and their generalizations as discussed in Sections 3 & 4 cannot adhere to any involution.

2.6. Some minimalities: (1) V and V^{op} are smallest noncommutative rings (up to isomorphism). These are right and left symmetric, respectively. So the *minimal noncommutative right* (*or left*) *symmetric rings are* V and V^{op}.

(2) Next higher order noncommutative rings are of order eight. So two minimal noncommutative symmetric rings are strictly upper and lower triangular matrix rings $\mathrm{SUTM}_3[\mathbb{Z}_2]$ and $\mathrm{SLTM}_3[\mathbb{Z}_2]$, respectively. Both are without identity and are not reversible (can be checked easily).

(3) ([3]; Example 2.6) A minimal non-commutative symmetric ring with identity is the ring $GF(4)\oplus GF(4)$, in which addition and multiplication are defined by the rules:

$$(a,b)+(c,d)=(a+c,b+d);\quad (a,b)(c,d)=(ac,ad+bc^2)$$

(see details in [3]; Example 2.6). This ring has sixteen elements and is also reversible.

Reappearance of the Lambek Criterion: Lambek proved in [1] that *a ring with one is symmetric if and only if it is isomorphic to the rings of sections of a sheaf of prime - torsion free symmetric rings*. Following is an extension of it.

Proposition 2.7. *A ring* R *with an idempotent is right symmetric if and only if* R *contains a subring which is isomorphic to the rings of sections of a sheaf of prime - torsion free symmetric rings.*

Proof: "Only if", is obvious, because a symmetric ring with 1 is a right symmetric ring with an idempotent. For "if", consider that R is right symmetric. Let $e\in R$ be an idempotent. Then the corner ring eRe being a subring of R is right symmetric and because e is the multiplicative identity, so eRe becomes a symmetric ring. Rest follows from ([1]: Corollary 1]. □

A ring R is called Armendariz as introduced by Rege, S. Chhawchharia in [12] if for any pair of polynomials $f(x)=\sum_{i=0}^{\alpha}a_ix^i$ and $g(x)=\sum_{j=0}^{\beta_1}b_jx^j$ in $R[x]$ such that $f(x)g(x)=0$, then $a_ib_j=0 \quad \forall i=0,1,\cdots,\alpha,$, $\forall j=0,1,\cdots,\beta$. In this section we construct an Armendariz Boolean ring and a polynomial semicommutative ring.

The first part of the following lemma is proved by Nielsen in ([13]; Lemma 1]. The remaining relations are just tautologies.

Lemma 2.8. *Let R be a right symmetric ring. Let* $f(x)=\sum_{i=0}^{\alpha}a_ix^i$ *and* $g(x)=\sum_{j=0}^{\beta}b_jx^j$ *be two polynomials in* $R[x]$ *such that* $f(x)g(x)=0$. *Then the following relations hold*:

(1) $a_ib_0^{i+1}=0, \quad \forall i=0,1,\cdots,\alpha$

(2) $a_0^{j+1}b_j=0, \quad \forall j=0,1,\cdots,\beta$

(3) $a_{\alpha-i}b_{\beta}^{i+1}=0, \quad \forall i=0,1,\cdots,\alpha$

(4) $a_{\alpha}^{j+1}b_{\beta-j}=0, \quad \forall j=0,1,\cdots,\beta$

Theorem 2.9. *Let R be a right symmetric ring. Let* $f(x)=\sum_{i=0}^{\alpha}a_ix^i$ *and* $g(x)=\sum_{j=0}^{\beta}b_jx^j$ *be two polynomials in* $R[x]$ *such that* $f(x)g(x)=0$ *If the coefficients of* $f(x)$ (*or* $g(x)$) *are idempotents, then* $f(x)Rg(x)=0$

Proof: Assume that the coefficients of $f(x)$ are idempotents. If a_0 is idempotent, then by (2) of above lemma

$$a_0 b_j = 0, \quad \forall j = 0,1,\cdots,\beta \Rightarrow \forall r \in R, \quad a_0 b_j r = 0 \Rightarrow a_0 r b_j = 0, \quad \forall j = 0,1,\cdots,\beta$$

Now the coefficients in $fg = 0$ are of the form $\sum_{i=0}^{j} a_i b_{j-i} = 0$, $\forall j = 0,1,\cdots,\beta$. The induction step suggests that: $a_0^j \left(\sum_{i=0}^{j} a_i b_{j-i} \right) = a_0 \left(\sum_{i=0}^{j} a_i b_{j-i} \right) = a_0 b_j = 0$, $\forall j = 0,1,\cdots,\beta$ then remove the term $a_0 b_j$ from this sum and multiply the remaining sum by a_k, where $k = 1,2,\cdots,\alpha$ consecutively, and removing the zero terms until we get the last term $a_\alpha b_j = 0$, $\forall j = 0,1,\cdots,\beta$. In this process the relation (4) of Lemma 2.8 is also involved to delete the unwanted terms. Hence we conclude that $a_i b_j = 0 \Rightarrow a_i r b_j = 0$, $\forall i = 0,1,\cdots,\alpha$, $\forall j = 0,1,\cdots,\beta$ and $\forall r \in R$. If the coefficients of $g(x)$ are idempotents, then (1) and (3) of Lemma 2.8 are involved to prove the desired result. □

Corollary 2.10. *Let* R *be a right symmetric Boolean ring. Then*:

(1) R *is Armendariz.*

(2) *For every pair of polynomials* $f, g \in R[x]$, $fg = 0 \Rightarrow fRg = 0$

(3) *For every pair of polynomials* $f, g \in R[x]$, $fg = 0 \Rightarrow fR[x]g = 0$ *In other words,* $R[x]$ *is semicommutative*.

Proof: (1) and (2) are followed from Theorem 2.9.

(3) Let $f(x) = \sum_{i=0}^{\alpha} a_i x^i$ and $g(x) = \sum_{j=0}^{\beta} b_j x^j$. Then $fg = 0 \Rightarrow a_i b_j = 0$, $\forall i = 0,1,\cdots,\alpha$ $\forall j = 0,1,\cdots,\beta$. Let $h(x) = \sum_{k=0}^{\gamma} c_k x^k \in R[x]$. Then $a_i b_j c_k = 0$, $\forall i = 0,1,\cdots,\alpha$, $\forall j = 0,1,\cdots,\beta$, $k = 0,1,\cdots,\gamma \Rightarrow a_i c_k b_j = 0 \Rightarrow a_i c_k b_j x^l = 0$. Because all terms in the product of polynomials $f(x)$, $h(x)$ and $g(x)$ are of the form $a_i c_k b_j x^l$,we conclude that: $f(x)h(x)g(x) = 0$. □

3. Some Extensions of Klein 4-Rings

Now we pose few more examples of one sided symmetric rings. These rings are extensions of V and V^{op} First result gives a criterion of all rings of order p^2 as symmetric and non symmetric.

Theorem 3.1. *For any prime* p, *a ring* R *of order* p^2 *is symmetric if and only if it is reversible. The non-symmetric ring is either left or right symmetric.*

Proof: It is known that up to isomorphism there are eleven rings of order p^2. These can be classified as commutative and non-commutative rings. The first statement trivially holds for commutative rings. There are nine commutative rings and the only non-commutative rings are:

$$S := \left\{ \langle a,b \rangle \mid pa = pb = 0,\ a^2 = ab = a,\ b^2 = ba = b \right\}$$

and its opposite ring

$$S^{op} := \left\{ \langle a,b \rangle \mid pa = pb = 0,\ a^2 = ba = a,\ b^2 = ab = b \right\}$$

Both rings are of characteristic p and can be verified that these are neither symmetric nor reversible. Note that the non-commutative rings S and S^{op} of order p^2, for all primes p, are right symmetric and left symmetric, respectively. For instance, in case of S, the non-zero elements of S are of the form ma, nb and $ra + sb$, where $1 \le m, n, r, s < p$, so as in the case of V,

$$(ma)(nb)(ra+sb) = mn(r+s)a = 0, \quad \text{provided } r+s = p,$$
$$(ma)(ra+sb)(nb) = mn(r+s)a = 0, \quad \text{provided } r+s = p,$$
$$(mb)(na)(ra+sb) = mn(r+s)b = 0, \quad \text{provided } r+s = p,$$
$$(mb)(ra+sb)(na) = mn(r+s)b = 0, \quad \text{provided } r+s = p.$$

Hence S is right symmetric. But S is not symmetric, because

$$(ra+sb)(ma)(nb)=mn(ra+sb)\neq 0, \quad \forall 1\leq m,n,r,s<p$$

Clearly S is not reversible as well. □

Let $X=\{x_i | i=1,\cdots,n\}$ be a set of symbols and consider the additive group $\mathbb{Z}_2\langle X\rangle$ generated by these symbols. This group has 2^n elements. Define the multiplication on $\mathbb{Z}_2\langle X\rangle$ by the rule: $xy=x \quad \forall x, y\in X$ Then clearly,

$$x(yz)=(xy)z, \quad (x+y)z=xz+yz \quad \text{and} \quad x(y+z)=xy+xz, \quad \forall x,y,z\in X$$

These rules clearly imply that $\mathbb{Z}_2\langle X\rangle$ is an associative ring without 1 and is of characteristic 2. Let us denote this ring by V_{2^n} as its order is 2^n. The Klein-4 ring V as discussed in Example 2.2 above is V_{2^2} and is smallest in the series.

Theorem 3.2. *The ring* V_{2^n} *is right symmetric but not left symmetric. Likewise,* $V_{2^n}^{op}$ *is left symmetric but not right symmetric.*

Proof: Assume that $a, b, c\in V_{2^n}$, such that $abc=0$. If any one of a, b, or c is zero, then we are done. So consider only non-zero elements.

Assume that $a=\sum_{i=1}^{\alpha}x_i$, $b=\sum_{j=1}^{\beta}y_j$ and $c=\sum_{k=1}^{\gamma}z_k$, where $x_i, y_j, z_k\in X$ and $\alpha,\beta,\gamma\in\mathbb{Z}$ such that $1\leq\alpha,\beta,\gamma\leq n$.

Note that for any $x\in X$ and b as above,

$$xb=x\sum_{j=1}^{\beta}y_j=\beta x=\begin{Bmatrix}0 & \text{when } \beta \text{ is even}\\ x & \text{when } \beta \text{ is odd}\end{Bmatrix}.$$

Same will be the consequences if we replace x by a, *i.e.*

$$ab=\sum_{i=1}^{\alpha}x_i\sum_{j=1}^{\beta}y_j=\sum_{i=1}^{\alpha}\beta x_i=\begin{Bmatrix}0 & \text{when } \beta \text{ is even}\\ a & \text{when } \beta \text{ is odd}\end{Bmatrix},$$

and

$$bc=\sum_{j=1}^{\beta}y_j\sum_{k=1}^{\gamma}z_k=\sum_{j=1}^{\beta}\gamma y_j=\begin{Bmatrix}0 & \text{when } \gamma \text{ is even}\\ b & \text{when } \gamma \text{ is odd}\end{Bmatrix}.$$

So $abc=0$ if and only if either β is even or γ is even. This means that $abc=0\Leftrightarrow acb=0$. Hence V_{2^n} is right symmetric.

On the other hand, assume that β is even. Because α and γ are odd, then $abc=0$, but $bac=b\neq 0$. This completes the proof.

The second part can be obtained by symmetry. □

Trivial extension of a ring: Let R be any ring, a trivial extension $T(R,R)$ of R, is a subring of the upper triangular matrix ring over R and is defined as:

$$T(R,R)=\left\{\begin{bmatrix}r & s\\ 0 & r\end{bmatrix}: r,s\in R\right\}.$$

Theorem 3.3. *The trivial extension ring* $T(V,V)$ *is a right symmetric ring where* V *is the Klein* 4-*ring.*

Proof: In short we write (r,s) as an element of $T(V,V)$ but we will follow the rule of matrix multiplication on such ordered pairs. So let (r_1,s_1), (r_2,s_2), $(r_3,s_3)\in T(V,V)$ with $(r_1,s_1)(r_2,s_2)(r_3,s_3)=0$, where r_i, $s_i\in V$. Then

(a) $r_1r_2r_3=0$ and

(b) $r_1r_2s_3+r_1s_2r_3+s_1r_2r_3=0$

We want to prove that $(r_1,s_1)(r_3,s_3)(r_2,s_2)=0$. For this we need to establish that

(c) $r_1r_3r_2=0$ and that

(d) $r_1r_3s_2+r_1s_3r_2+s_1r_3r_2=0$

As in Example 2.2, (a) holds if either $r_2=c$ or $r_3=c$. Assume that $r_3=c$ and $r_2\neq c$, Then (b) holds if $s_3=c$. We substitute $r_3=s_3=c$ in (c) and (d). We see that these are also satisfied.

If $r_2=c$ and $r_3\neq c$, then (b) holds if $s_2=c$. Again we substitute $r_2=s_2=c$ in (c) and (d), we see that these are satisfied. If $r_2 = r_3 = c$, then all (a,b,c,d) are satisfied. Hence we conclude that $T(V,V)$ is right symmetric. □

Remarks 3.4. (1) $T(V,V)$ is not symmetric, as one can easily work out that

$$(a,a)(a,b)(c,c) = (aac, aac + abc + aac) = (0,0),$$

but

$$(c,c)(a,a)(a,b) = (caa, cab + caa + caa) = (c,c) \neq (0,0)$$

(2) It is known that if R is reduced then $T(R,R)$ is symmetric ([8]; Corollary 2.4). Note that V is not a reduced ring.

(3) The ring of 2×2 upper triangular matrices over V, $\mathrm{UTM}_2(V)$ is not right symmetric, because for $a, b, c \in V$,

$$\begin{bmatrix} a & a \\ 0 & b \end{bmatrix}\begin{bmatrix} c & b \\ 0 & b \end{bmatrix}\begin{bmatrix} c & b \\ 0 & c \end{bmatrix} = 0$$

and

$$\begin{bmatrix} a & a \\ 0 & b \end{bmatrix}\begin{bmatrix} c & b \\ 0 & c \end{bmatrix}\begin{bmatrix} c & b \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \neq 0$$

Thus, in general, $M_2[V]$ or $M_n[V]$ are not right symmetric. Hence, being right symmetric is not Morita invariant.

Theorem 3.5. *For a commutative indeterminate* x, *the polynomial ring* $V[x]$ *is right symmetric*.

Proof: Because V is without 1, so $V[x]$ is also without 1 and so $x \notin V[x]$. Now let $f(x) = \sum_{i=0}^{\alpha} a_i x^i$, $g(x) = \sum_{j=0}^{\beta} b_j x^j$, $h(x) = \sum_{k=0}^{\gamma} c_k x^k \in V[x]$, where a_i, b_j, $c_k \in V$ $\forall i = 0,1,\cdots,\alpha$, $j = 0,1,\cdots,\beta$, $k = 0,1,\cdots,\gamma$, and assume that

$$f(x)g(x)h(x) = d(x) = \sum_{t=0}^{\delta} d_t x^t,$$

where

$$d_t = \sum_{t=i+j+k} a_i b_j c_k; \quad t = 0,1,\cdots,\delta = \alpha+\beta+\gamma$$

Also assume that

$$f(x)h(x)g(x) = d'(x) = \sum_{t=0}^{\delta} d'_t x^t$$

where

$$d'_t = \sum_{t=i+j+k} a_i c_k b_j; \quad t = 0,1,\cdots, \ \delta = \alpha+\beta+\gamma.$$

We want to prove that if $d(x) = 0$, then so is $d'(x)$. So assume that $d(x) = 0$. Then $d_t = 0$, $\forall t = 0,1,\cdots,\delta$, and these terms can be expressed as

$$\begin{aligned}
d_0 &= a_0 b_0 c_0, \\
d_1 &= a_0 b_0 c_1 + a_0 b_1 c_0 + a_1 b_0 c_0, \\
d_2 &= a_0 b_0 c_2 + a_0 b_1 c_1 + a_0 b_2 c_0 + a_1 b_0 c_1 + a_1 b_1 c_0 + a_2 b_0 c_0, \\
d_3 &= a_0 b_0 c_3 + a_0 b_1 c_2 + a_0 b_2 c_1 + a_0 b_3 c_0 + a_1 b_1 c_1 + a_1 b_2 c_0 + a_1 b_0 c_2 + a_2 b_1 c_0 + a_2 b_0 c_1 + a_3 b_0 c_0, \\
&\cdots
\end{aligned}$$

where $a_i, b_i, c_i \in V$ For $d'(x) = 0$, we want to establish that $d'_t = 0$, $\forall t = 0,1,\cdots,\delta$, where

$$d_0' = a_0c_0b_0$$
$$d_1' = a_0c_0b_1 + a_0c_1b_0 + a_1c_0b_0,$$
$$d_2' = a_0c_0b_2 + a_0c_1b_1 + a_0c_2b_0 + a_1c_0b_1 + a_1c_1b_0 + a_2c_0b_0,$$
$$d_3' = a_0c_0b_3 + a_0c_1b_2 + a_0c_2b_1 + a_0c_3b_0 + a_1c_1b_1 + a_1c_2b_0 + a_1c_0b_2 + a_2c_1b_0 + a_2c_0b_1 + a_3c_0b_0,$$
$$\cdots$$

(I) We have five options for $d_0 = 0$. These are $a_0 = 0$, $b_0 = 0$, $c_0 = 0$, $b_0 = c$, or $c_0 = c$ Any one choice will give us $d_0' = 0$.

(II) Let $d_1 = 0$. From (I) if we choose $a_0 = 0$, then $a_0b_0c_1 = 0 = a_0b_1c_0$, and so $d_1 = a_1b_0c_0$. For $d_1 = 0$, we again have five choices, $a_1 = 0, b_0 = 0, c_0 = 0, b_0 = c$ or $c_0 = c$, and with the previously chosen $a_0 = 0$, we see that $d_1' = 0$

(III) Let us have $a_0 = a_1 = 0$ as in (I) & (II). Then $d_2 = a_2b_0c_0$. Here again we have five choices for a_2, b_0 and c_0. For every choice we have $d_2 = 0$ which implies $d_2' = 0$ and again we have five options here, each gives $a_1c_0b_0 = 0$. Thus we find that $d_1' = 0$ holds. The choices for $b_0 = 0$ or $c_0 = 0$ will yield same result.

(IV) Now assume that $a_0 = a_1 = a_2 = 0$. This is in continuation of (I), (II), & (III) and the same repetition will give us $d_3 = 0$ and $d_3' = 0$ simultaneously.

The rest are similar.

Definitely, we need to watch the situation for non-zero values, for instance, if we let $a_0, a_1, a_2, \cdots, b_0, b_1, b_2, \cdots \in \{a, b\}$ and $c_0 = c_1 = c_2 = \cdots = c$ then we see that $d_t = d_t' = 0$, $\forall t = 0, 1, \cdots, \delta$. Same situation comes if we let $a_0, a_1, a_2, \cdots, c_0, c_1, c_2, \cdots \in \{a, b\}$ and $b_0 = b_1 = b_2 = \cdots = c$. Hence the required result is obtained. □

4. McCoy Rings without Identity

In [13], Nielsen defined that a ring R is a right McCoy, if $f(x)g(x) = 0$, then there exists an $r \in R$, such that $f(x)r = 0$. Left McCoy and McCoy rings are defined similarly. It is proved in ([13], Theorem 2), that: *every reversible ring is left and right McCoy, hence McCoy*.

In ([13]; Section 3), Nielsen, constructed an example of a right McCoy ring with identity. This example is neither symmetric nor reversible, and there is no question that it is right or left symmetric because it has identity.

In next result we prove that the right symmetric ring V_{2^n} is a right McCoy ring without identity.

Theorem 4.1. *The ring* V_{2^n} *as constructed in Theorem* 2.2. *is a McCoy ring*.

Proof: Again, let $X = \{x_i \mid i = 1, \cdots, n\}$ be a set of symbols and consider the additive group $\mathbb{Z}_2\langle X\rangle$ generated by these symbols and define the multiplication on $\mathbb{Z}_2\langle X\rangle$ by the rule: $xy = x\ \forall x, y \in X$.

Then V_{2^n} is a ring as constructed in Theorem 2.3. The characteristic of this ring is 2. Consider an element of the form $\sum_{k=1}^{\gamma} z_k$, where $z_k \in X$, $k = 1, \cdots, \gamma$, and γ is even and let all z_k be distinct so that $\sum_{k=1}^{\gamma} z_k \neq 0$. Then for any element $t \in V_{2^n}$, $t\sum_{k=1}^{\gamma} z_k = \sum_{k=1}^{\gamma} tz_k = \gamma t = 0$.

Assume that $f(x) = \sum_{i=0}^{\alpha} a_i x^i$ and $g(x) = \sum_{j=0}^{\beta} b_j x^j$ be elements of $V_{2^n}[x]$, with $f(x)g(x) = 0$. Then $f(x)\sum_{k=1}^{\gamma} z_k = \sum_{i=0}^{\alpha} a_i \left(\sum_{k=1}^{\gamma} z_k\right) x^i = 0$. Hence V_{2^n} is right McCoy.

On the other hand, note that $f(x)g(x) = 0$ provided that the coefficients of $g(x)$ are the elements of V_{2^n} of the form $\sum_{k=1}^{\gamma} z_k$, where $z_k \in X$ and γ is even. Hence for any $t \in V_{2^n}$, $tg(x) = 0$ which shows that V_{2^n} is left McCoy. Hence V_{2^n} is McCoy. □

Remarks 4.2. It follows from above that

(i) V_{2^n} is McCoy, right symmetric, and semicommutative, but neither symmetric nor reversible.

(ii) $V_{2^n}^{op}$ is McCoy, left symmetric, and semicommutative, but neither symmetric nor reversible.

Example 4.3. In Section 3 of [14] an example of a McCoy ring is constructed such that its power series ring is not McCoy. Here we prove that the power series ring of Klein 4-ring, which we already have proved that it is

McCoy, is also McCoy. A typical element of $V_{2^2}[[x]][t] = V[[x]][t]$ is of the form $a_x(t) = \sum_{k=0}^{\alpha}\left(\sum_{j=0}^{\infty} a_{kj}x^j\right)t^k$, where a_{kj} is a coefficient in the power series ring $V[[x]]$. Clearly $\forall a_{kj} \in V$, $a_{kj}c = 0$, so $a_x(t)c = 0$. On the other hand, let $a_x(t) \neq 0$ and $b_x(t) \neq 0$ but $a_x(t)b_x(t) = 0$. Then the coefficients b_{hi} in $b_x(t)$ are in the set $\{0, c\}$ and as previously we got the outcome $ab_x(t) = bb_x(t) = 0$. Hence we conclude that $V[[x]]$ *is McCoy*.

We end up at a general statement. The following corollary can be proved by the methods used in Theorem 4.1.

Corollary 4.4. *The power series ring* $V_{2^n}[[x]]$ *is McCoy*.

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