

# Compactness of Composition Operators from the p-Bloch Space to the q-Bloch Space on the Classical Bounded Symmetric Domains

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## Abstract

In this paper, we introduce the weighted Bloch spaces  $\beta^p(\mathfrak{R}_I(m, n))$  on the first type of classical bounded symmetric domains  $\mathfrak{R}_I(m, n)$ , and prove the equivalence of the norms  $\|f\|_{1,p}$  and  $\|f\|_{2,p+(m+n)/2}$ . Furthermore, we study the compactness of composition operator  $C_\phi$  from  $\beta^p(\mathfrak{R}_I(m, n))$  to  $\beta^q(\mathfrak{R}_I(m, n))$ , and obtain a sufficient and necessary condition for  $C_\phi : \beta^p(\mathfrak{R}_I(m, n)) \rightarrow \beta^q(\mathfrak{R}_I(m, n))$  to be compact.

## Keywords

Bloch Space, Classical Bounded Symmetric Domains, Composition Operators, Compactness, Bergman Metric

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## 1. Introduction

Let  $\Omega$  be a bounded homogeneous domain in  $\mathbb{C}^n$ . The class of all holomorphic functions on  $\Omega$  will be denoted by  $H(\Omega)$ . For  $\phi$  a holomorphic self-map of  $\Omega$  and  $f \in H(\Omega)$ , the composition  $f \circ \phi$  is denoted by  $C_\phi f$ , and  $C_\phi$  is called the composition operator with symbol  $\phi$ .

The composition operators as well as related operators known as the weighted composition operators between

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the weighted Bloch spaces were investigated in [1] [2] in the case of the unit disk, and in [3]-[7] for the case of the unit ball. The study of the weighted composition operators from the Bloch space to the Hardy space  $H^\infty$  was carried out in [8] [9] for the unit ball. Characterizations of the boundedness and the compactness of the composition operators and the weighted ones between the Bloch spaces were given in [10]-[12] for the polydisc case, and in [13]-[18] for the case of the bounded symmetric domains. Furthermore, we will give some results about the composition operators for the case of the weighted Bloch space on the bounded symmetric domains.

In 1930s all irreducible bounded symmetric domains were divided into six types by E. Cartan. The first four types of irreducible domains are called the classical bounded symmetric domains, the other two types, called exceptional domains, consist of one domain each (a 16 and 27 dimensional domain).

The first three types of classical bounded symmetric domains can be expressed as follows [19]:

$$\mathfrak{R}_I(m, n) = \{Z : Z \text{ is an } m \times n \text{ complex matrix, } I_m - Z\bar{Z}^T > 0\},$$

where  $m \leq n$  and  $I_m$  is the  $m \times m$  identity matrix,  $Z^T$  is the transpose of  $Z$ ;

$$\mathfrak{R}_{II}(p) = \{Z : Z \text{ is a } p \times p \text{ symmetric matrix, } Z = Z^T, I_p - Z\bar{Z} > 0\}$$

$$\mathfrak{R}_{III}(q) = \{Z : Z \text{ is a } q \times q \text{ antisymmetric matrix, } Z = -Z^T, I_q + Z\bar{Z} > 0\}.$$

Let  $A = (a_{ij})_{m \times n}$  and  $B = (b_{kl})_{s \times t}$ . The Kronecker product  $A \times B$  of  $A$  and  $B$  is defined as the  $ms \times nt$  matrix  $C = (c_{ijkl})$  such that the element at the  $ik$ -th row and  $jl$ -th column  $c_{ijkl} = a_{ij}b_{kl}$  [19]. Then the Bergman metric of  $\mathfrak{R}_I(m, n)$  is as follows (see [19]):

$$H_Z(u, u) = (m + n)u(I - Z\bar{Z}^T)^{-1} \times (I - \bar{Z}^T Z)^{-1} \bar{u}^T, \tag{1.1}$$

where  $u = (u_{11}, \dots, u_{1n}, \dots, u_{m1}, \dots, u_{mn})$  is a complex vector,  $\bar{u}^T$  is the conjugate transpose of  $u$ , and  $Z = (z_{ij})_{m \times n} \in \mathfrak{R}_I(m, n)$ .

Following Timoney's approach (see [18]), a holomorphic function  $f$  is in the Bloch space  $\beta(\mathfrak{R}_I(m, n))$ , if

$$\|f\|_\beta = \sup_{Z \in \mathfrak{R}_I(m, n)} Q_f(Z) < \infty.$$

Now we define a holomorphic function  $f$  to be in the p-Bloch space  $\beta^p(\mathfrak{R}_I(m, n))$ , if

$$\|f\|_{\beta^p} = \sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - Z\bar{Z}^T)^p Q_f(Z) < \infty, \tag{1.2}$$

where

$$Q_f(Z) = \sup \left\{ \frac{|\nabla f(Z)u|}{H_Z^{1/2}(u, u)} : u \in \mathbb{C}^{mn} - \{0\} \right\},$$

$$\nabla f(Z) = \left( \frac{\partial f}{\partial z_{11}}(Z), \dots, \frac{\partial f}{\partial z_{1n}}(Z), \dots, \frac{\partial f}{\partial z_{m1}}(Z), \dots, \frac{\partial f}{\partial z_{mn}}(Z) \right).$$

We can prove that  $\beta^p(\mathfrak{R}_I(m, n))$  is a Banach space with norm  $\|f\|_{\beta^p} = |f(0)| + \|f\|_{\beta^p}$  which is similar with the case on  $\beta^\alpha(B^n)$ .

Let  $\phi = (\phi_{ij})_{m \times n}$  be a holomorphic self-map of  $\mathfrak{R}_I(m, n)$ . We are concerned here with the question of when  $C_\phi : \beta^p(\mathfrak{R}_I(m, n)) \rightarrow \beta^q(\mathfrak{R}_I(m, n))$  will be a compact operator.

Let  $\text{diag}(d_1, \dots, d_n)$  denote a diagonal matrix with diagonal elements  $d_1, \dots, d_n$ . In this work, we shall denote by  $C$  a positive constant, not necessarily the same on each occurrence.

In Section 2, we prove the equivalence of the norms defined in this paper and in [20].

In Section 3, we state several auxiliary results most of which will be used in the proofs of the main results.

Finally, in Section 4, we establish the main result of the paper. We give a sufficient and necessary condition for the composition operator  $C_\phi$  from the p-Bloch space  $\beta^p(\mathfrak{R}_I(m, n))$  to the q-Bloch space  $\beta^q(\mathfrak{R}_I(m, n))$  to be compact, where  $p \geq 0$  and  $q \geq 0$ . Specifically, we prove the following result:

**Theorem 1.1.** Let  $\phi$  be a holomorphic self-map of  $\mathfrak{R}_I(m, n)$ . Then  $C_\phi : \beta^p(\mathfrak{R}_I(m, n)) \rightarrow \beta^q(\mathfrak{R}_I(m, n))$  is compact if and only if, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$\frac{\det(I - Z\bar{Z}^T)^q}{\det(I - \phi(Z)\overline{\phi(Z)}^T)^p} \frac{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)}{H_Z^{1/2}(u, u)} < \varepsilon, \tag{1.3}$$

for all  $u \in \mathbb{C}^{m+n} - \{0\}$  whenever  $\text{dist}(\phi(Z), \partial\mathfrak{R}_I) < \delta$ ,  $Z \in \mathfrak{R}_I(m, n)$ .

The compactness of the composition operators for the weighted Bloch space on the bounded symmetric domains of  $\mathfrak{R}_{II}(p)$ ,  $\mathfrak{R}_{III}(q)$  is similar with the case of  $\mathfrak{R}_I(m, n)$ ; we omit the details.

## 2. The Equivalence of the Norms

Denote [20]  $\|f\|_{2, \beta^p} = \sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - Z\bar{Z}^T)^p |\nabla f(Z)|$  and  $\|f\|_{2, p} = |f(0)| + \|f\|_{2, \beta^p}$ .

**Lemma 2.1.** (Bloomfield-Watson) [21] Let  $A = (a_{ij}) \geq 0$  be an  $m \times m$  Hermitian matrix. Then

$$\det(\bar{B}^T A B) \leq \det A \tag{2.1}$$

where  $B$  is any  $m \times n$  matrix and satisfies  $B\bar{B}^T = I$ .

**Theorem 2.1.**  $\|f\|_{1, p}$  and  $\|f\|_{2, p+(m+n)/2}$  are equivalent.

*Proof.* The metric matrix of  $\mathfrak{R}_I(m, n)$  is

$$T(Z, Z) = (m+n)(I_m - Z\bar{Z}^T)^{-1} \times (I_n - \bar{Z}^T Z)^{-1}.$$

For any  $Z \in \mathfrak{R}_I(m, n)$ , let  $\varphi_Z \in \text{Aut}(\mathfrak{R}_I(m, n))$  with  $\varphi_Z(Z) = 0$ . Then

$$T(Z, Z) = ((J\varphi_Z)(Z))^T T(0, 0) \overline{(J\varphi_Z)(Z)} = (m+n)((J\varphi_Z)(Z))^T \overline{(J\varphi_Z)(Z)}$$

Denote  $\varphi_Z^{-1} = \psi_Z$  then  $\psi_Z(0) = Z$ ,  $J\psi_Z = (J\varphi_Z)^{-1}$  and

$$T^{-1}(Z, Z) = \frac{1}{m+n} \overline{(J\varphi_Z)(Z)}^{-1} ((J\varphi_Z)(Z))^T = \frac{1}{m+n} \overline{(J\psi_Z)(0)} (J\psi_Z)(0)^T.$$

Thus

$$\overline{\nabla f(Z)} T^{-1}(Z, Z) (\nabla f(Z))^T = \frac{1}{m+n} \overline{\nabla f(Z) (J\psi_Z)(0)} (\nabla f(Z) (J\psi_Z)(0))^T = \frac{1}{m+n} |\nabla(f \circ \psi_Z)(0)|^2.$$

Hence

$$\begin{aligned} \det(I - Z\bar{Z}^T)^p Q_{f \circ \psi_Z}(0) &= \det(I - Z\bar{Z}^T)^p \sup \left\{ \frac{|\nabla(f \circ \psi_Z)(0)u|}{H_0^{1/2}(u, u)} : u \in \mathbb{C}^{m+n} - \{0\}, |u| = 1 \right\} \\ &= \frac{1}{\sqrt{m+n}} \det(I - Z\bar{Z}^T)^p \sup \{ |\nabla(f \circ \psi_Z)(0)u| : u \in \mathbb{C}^{m+n} - \{0\}, |u| = 1 \} \\ &= \det(I - Z\bar{Z}^T)^p \left\{ \overline{\nabla f(Z)} T^{-1}(Z, Z) (\nabla f(Z))^T \right\}^{1/2}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|f\|_{\beta^p} &= \sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - Z\bar{Z}^T)^p Q_f(Z) \\ &= \sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - Z\bar{Z}^T)^p \left\{ \overline{\nabla f(Z)} T^{-1}(Z, Z) (\nabla f(Z))^T \right\}^{1/2}. \end{aligned}$$

Since

$$\begin{aligned} \overline{\nabla f(Z)} T^{-1}(Z, Z) (\nabla f(Z))^T &\leq |\nabla f(Z)|^2 \det T^{-1}(Z, Z) \\ &= (m+n)^{-1} |\nabla f(Z)|^2 \det(I - Z\bar{Z}^T)^{m+n}. \end{aligned}$$

Thus

$$\|f\|_{\beta^p} \leq \frac{1}{\sqrt{m+n}} \|f\|_{2,\beta^{p+\frac{m+n}{2}}}. \tag{2.2}$$

For

$$\begin{aligned} \det(I - Z\bar{Z}^T)^p Q_f(Z) &= \det(I - Z\bar{Z}^T)^p \sup \left\{ \frac{|\nabla f(Z)u|}{\{uT(Z,Z)\bar{u}^T\}^{1/2}} : u \in \mathbb{C}^{mn} - \{0\}, |u|=1 \right\} \\ &\geq \det(I - Z\bar{Z}^T)^p \{ \det T^{-1}(Z,Z) \}^{1/2} |\nabla f(Z)| \\ &= \frac{1}{\sqrt{m+n}} \det(I - Z\bar{Z}^T)^{p+\frac{m+n}{2}} |\nabla f(Z)|, \end{aligned}$$

then we have

$$\|f\|_{\beta^p} \geq \frac{1}{\sqrt{m+n}} \|f\|_{2,\beta^{p+\frac{m+n}{2}}}. \tag{2.3}$$

Combining (2.2) and (2.3),

$$\|f\|_{\beta^p} = \frac{1}{\sqrt{m+n}} \|f\|_{2,\beta^{p+\frac{m+n}{2}}}.$$

Next,

$$\|f\|_{1,p} = |f(0)| + \|f\|_{\beta^p} \leq |f(0)| + \|f\|_{2,\beta^{p+\frac{m+n}{2}}} = \|f\|_{2,p+\frac{m+n}{2}};$$

and

$$\|f\|_{2,p+\frac{m+n}{2}} = |f(0)| + \|f\|_{2,\beta^{p+\frac{m+n}{2}}} \leq \sqrt{m+n} (|f(0)| + \|f\|_{\beta^p}) = \sqrt{m+n} \|f\|_{1,p}.$$

Therefore, the proof is completed. □

### 3. Some Lemmas

Here we state several auxiliary results most of which will be used in the proof of the main result.

**Lemma 3.1.** [18] *Let  $\mathcal{D} \subset \mathbb{C}^N$  be a bounded homogeneous domain. Then there exists a constant  $C$ , depending only on  $\mathcal{D}$ , such that*

$$H_{\phi(z)}(J\phi(z)u, J\phi(z)u) \leq CH_z(u, u) \tag{3.1}$$

for each  $z \in \mathcal{D}$  whenever  $\phi$  holomorphically maps  $\mathcal{D}$  into itself. Here  $H_z(u, u)$  denotes the Bergman metric

on  $\mathcal{D}$ ,  $J\phi(z) = \left( \frac{\partial \phi_l(z)}{\partial z_k} \right)_{1 \leq l, k \leq N}$  denotes the Jacobian matrix of  $\phi$ .

**Lemma 3.2.** *Let  $\phi$  be a holomorphic self-map of  $\mathfrak{R}_1(m, n)$  and  $K$  a compact subset of  $\mathfrak{R}_1(m, n)$ . Then there exists a constant  $C > 0$  such that*

$$\frac{\det(I - Z\bar{Z}^T)^q}{\det(I - \phi(Z)\overline{\phi(Z)}^T)^p} \frac{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)}{H_Z^{1/2}(u, u)} \leq C \tag{3.2}$$

for all  $u \in \mathbb{C}^{mn} - \{0\}$  whenever  $\phi(Z) \in K$ .

*Proof.* For  $\delta \in (0, 1)$ , let  $E_W^\delta := \{W \in \mathfrak{R}_1(m, n) : \text{dist}(W, \partial\mathfrak{R}_1) \geq \delta\}$ .

For any compact  $K \subset \mathfrak{R}_1(m, n)$ , there exists a constant  $\delta \in (0, 1)$  such that  $K \subset E_W^\delta$ . Then there exists  $M \in (0, 1)$  such that  $\det(I - \phi(Z)\overline{\phi(Z)}^T) > M$ , whenever  $\phi(Z) \in K$ .

Thus

$$\frac{\det(I - ZZ^T)^q}{\det(I - \phi(Z)\overline{\phi(Z)}^T)^p} < \frac{1}{M^p}. \tag{3.3}$$

Combining Lemma 3.1 with (3.3) shows that (3.2) holds. □

**Lemma 3.3.** (Hadamard) [21] *Let  $A = (a_{ij}) \geq 0$  be an  $n \times n$  Hermitian matrix. Then*

$$\det A \leq \prod_{i=1}^n a_{ii} \tag{3.4}$$

and equality holds if and only if  $A$  is a diagonal matrix.

**Lemma 3.4.** *Let  $Z = (z_{kl})_{m \times n} \in \mathfrak{R}_I(m, n)$ . Then*

$$\det(I_m - ZZ^T) \leq \prod_{i=1}^m (1 - |z_{ii}|^2). \tag{3.5}$$

*Proof.* For any  $Z \in \mathfrak{R}_I(m, n)$ , we have  $I_m - ZZ^T = \left( \delta_{st} - \sum_{j=1}^n z_{sj} \bar{z}_{tj} \right)_{1 \leq s, t \leq m} > 0$ .

Thus we have  $0 < 1 - \sum_{j=1}^n |z_{ij}|^2 < 1, i = 1, 2, \dots, m$ .

It follows from Lemma 3.3 that  $\det(I_m - ZZ^T) \leq \prod_{i=1}^m \left( 1 - \sum_{j=1}^n |z_{ij}|^2 \right) \leq \prod_{i=1}^m (1 - |z_{ii}|^2)$ . □

**Lemma 3.5.** *Let  $\mathfrak{R}_I(m, n)$  be a classical bounded symmetric domain, and  $T(z, z)$  denote its metric matrix. Then a holomorphic function  $f$  on  $\mathfrak{R}_I(m, n)$  is in  $\beta^p(\mathfrak{R}_I(m, n))$  if and only if*

$$\sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - ZZ^T)^p \left\{ \nabla f(Z) T^{-1}(Z, Z) \overline{\nabla f(Z)}^T \right\}^{1/2} < \infty. \tag{3.6}$$

If (3.6) holds, then

$$\|f\|_{\beta^p} \leq \sup_{Z \in \mathfrak{R}_I(m, n)} \det(I - ZZ^T)^p \left\{ \nabla f(Z) T^{-1}(Z, Z) \overline{\nabla f(Z)}^T \right\}^{1/2}. \tag{3.7}$$

*Proof.* We can get the conclusion by the process of the proof on Theorem 2.1. □

**Lemma 3.6.** [18] *Let*

$$\begin{aligned} P &= U(\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m), 0)V \in \mathfrak{R}_I, \\ Q &= U \text{diag} \left( \frac{1}{\sqrt{1-\lambda_1^2}}, \frac{1}{\sqrt{1-\lambda_2^2}}, \dots, \frac{1}{\sqrt{1-\lambda_m^2}} \right) \bar{U}^T, \\ R &= \bar{V}^T \text{diag} \left( \frac{1}{\sqrt{1-\lambda_1^2}}, \frac{1}{\sqrt{1-\lambda_2^2}}, \dots, \frac{1}{\sqrt{1-\lambda_m^2}}, 1, \dots, 1 \right) V, \end{aligned}$$

where  $U_{m \times m}$  and  $V_{n \times n}$  are unitary matrices and  $0 \leq \lambda_m \leq \dots \leq \lambda_2 \leq \lambda_1 < 1$ .

Denote  $\Phi_p(Z) = Q(P - Z)(I_n - \bar{P}^T Z)^{-1} R^{-1}, Z \in \mathfrak{R}_I(m, n)$ . Then

- (1)  $\Phi_p(Z) \in \text{Aut}(\mathfrak{R}_I(m, n))$ ;
- (2)  $(\Phi_p)^{-1} = \Phi_p$ ;
- (3)  $\Phi_p(0) = P$  and  $\Phi_p(P) = 0$ ;
- (4)  $d\Phi_p(Z)|_{Z=P} = -QdZR$  and  $d\Phi_p(Z)|_{Z=0} = -Q^{-1}dZR^{-1}$  for  $Z \in \mathfrak{R}_I(m, n)$ ;

$$(5) \quad \Phi_p(Z) = Q^{-1} (I_m - Z\bar{P}^T)^{-1} (P - Z)R \quad \text{for } Z \in \mathfrak{R}_I(m, n);$$

$$(6) \quad (I_m - Z\bar{P}^T)Q \left( I_m - \Phi_p(Z) \overline{\Phi_p(Z)^T} \right) \bar{Q}^T (I_m - P\bar{Z}^T) = I_m - Z\bar{Z}^T \quad \text{for } Z \in \mathfrak{R}_I(m, n).$$

**Lemma 3.7.**  $C_\phi : \beta^p(\mathfrak{R}_I(m, n)) \rightarrow \beta^q(\mathfrak{R}_I(m, n))$  is compact if and only if  $\|C_\phi f_k\|_{\beta^q} \rightarrow 0$  as  $k \rightarrow \infty$  for any bounded sequence  $\{f_k\}$  in  $\beta^p(\mathfrak{R}_I(m, n))$  that converges to 0 uniformly on compact subsets of  $\mathfrak{R}_I(m, n)$ .

*Proof.* The proof is trial by using the normal methods.  $\square$

#### 4. Proof of Theorem 1.1

*Proof.* Let  $\{f_k\}$  be a bounded sequence in  $\beta^p(\mathfrak{R}_I(m, n))$  with  $\|f_k\|_{\beta^p} \leq C$ , and  $f_k \rightarrow 0$  uniformly on compact subsets of  $\mathfrak{R}_I(m, n)$ .

Suppose (1.3) holds. Then for any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that

$$\frac{\det(I - Z\bar{Z}^T)^q}{\det(I - \phi(Z)\overline{\phi(Z)^T})^p} \frac{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)}{H_Z^{1/2}(u, u)} < \frac{\varepsilon}{C} \quad (4.1)$$

for all  $u \in \mathbb{C}^{mn} - \{0\}$  whenever  $\text{dist}(\phi(Z), \partial\mathfrak{R}_I(m, n)) < \delta$  and  $Z \in \mathfrak{R}_I(m, n)$ .

By the chain rule, we have  $\nabla(f_k \circ \phi)(Z) = \nabla(f_k)(\phi(Z))J\phi(Z)$ .

If  $u \in \mathbb{C}^{mn} - \{0\}$  and  $J\phi(Z)u = 0$ , then we get  $\nabla(f_k \circ \phi)(Z)u = 0$ . If  $u \in \mathbb{C}^{mn} - \{0\}$  and  $J\phi(Z)u \neq 0$ , then

$$\frac{\nabla(f_k \circ \phi)(Z)u}{H_Z^{1/2}(u, u)} = \frac{\nabla(f_k)(\phi(Z))J\phi(Z)u}{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)} \frac{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)}{H_Z^{1/2}(u, u)}. \quad (4.2)$$

It follows from (4.1) and (4.2) that

$$\begin{aligned} \det(I - Z\bar{Z}^T)^q Q_{C_\phi f_k}(Z) &= \sup \left\{ \det(I - Z\bar{Z}^T)^q \frac{|\nabla(f_k \circ \phi)(Z)u|}{H_Z^{1/2}(u, u)}, u \in \mathbb{C}^{mn} - \{0\}, J\phi(Z)u \neq 0 \right\} \\ &\leq \|f_k\|_{\beta^p} \sup \left\{ \frac{\det(I - Z\bar{Z}^T)^q}{\det(I - \phi(Z)\overline{\phi(Z)^T})^p} \frac{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)}{H_Z^{1/2}(u, u)}, u \in \mathbb{C}^{mn} - \{0\} \right\} \end{aligned} \quad (4.3)$$

$$< C \frac{\varepsilon}{C} = \varepsilon, \quad (4.4)$$

whenever  $\text{dist}(\phi(Z), \partial\mathfrak{R}_I(m, n)) < \delta$  and  $Z \in \mathfrak{R}_I(m, n)$ .

On the other hand, there exists a constant  $m > 0$  such that

$$\inf \{ H_W^{1/2}(u, u) : |u| = 1, \text{dist}(W, \partial\mathfrak{R}_I(m, n)) \geq \delta \} = m.$$

So if  $\text{dist}(W, \partial\mathfrak{R}_I(m, n)) \geq \delta$ , then

$$\det(I - W\bar{W}^T)^p \frac{|\nabla f_k(W)u|}{H_W^{1/2}(u, u)} \leq \det(I - W\bar{W}^T)^p \frac{|\nabla f_k(W)|}{H_W^{1/2}(|u|/|u|, |u|/|u|)} \leq \det(I - W\bar{W}^T)^p \frac{|\nabla f_k(W)|}{m}.$$

We assume that  $\{f_k\}$  converges to 0 uniformly on compact subsets of  $\mathfrak{R}_I(m, n)$ . By Weierstrass Theorem, it is easy to see that  $\{\nabla f_k\}$  converges to 0 uniformly on compact subsets of  $\mathfrak{R}_I(m, n)$ . Thus, for given  $\varepsilon > 0$ , there exists  $k$  large enough such that

$$\frac{\det(I - \phi(Z)\overline{\phi(Z)^T})^p |\nabla(f_k)(\phi(Z))J\phi(Z)u|}{H_{\phi(Z)}^{1/2}(J\phi(Z)u, J\phi(Z)u)} < \frac{\varepsilon}{C} \quad (4.5)$$

for any  $u \in \mathbb{C}^{mn} - \{0\}$ ,  $J\phi(Z)u \neq 0$  whenever  $\text{dist}(\phi(Z), \partial\mathfrak{R}_l(m, n)) \geq \delta$  and  $Z \in \mathfrak{R}_l(m, n)$ . Then by inequalities (4.3) and (4.5) and Lemma 3.2, it follows that, for  $k$  large enough,

$$\|C_\phi f_k\|_{\beta^q} < \varepsilon \tag{4.6}$$

whenever  $\text{dist}(\phi(Z), \partial\mathfrak{R}_l(m, n)) \geq \delta$  and  $Z \in \mathfrak{R}_l(m, n)$ .

Combining (4.4) and (4.6) shows that  $\|C_\phi f_k\|_{\beta^q} < \varepsilon$  as  $k$  large enough. So

$C_\phi : \beta^p(\mathfrak{R}_l(m, n)) \rightarrow \beta^q(\mathfrak{R}_l(m, n))$  is compact.

For the converse, arguing by contradiction, suppose  $C_\phi : \beta^p(\mathfrak{R}_l(m, n)) \rightarrow \beta^q(\mathfrak{R}_l(m, n))$  is compact and the condition (1.3) fails. Then there exist an  $\varepsilon_0 > 0$ , a sequence  $\{Z^j\}$  in  $\mathfrak{R}_l(m, n)$  with  $\phi(Z^j) \rightarrow \partial\mathfrak{R}_l(m, n)$  as  $j \rightarrow \infty$  and a sequence  $\{u^j\}$  in  $\mathbb{C}^{mn} - \{0\}$ , such that

$$\frac{\det(I - Z^j \bar{Z}^{jT})^q}{\det(I - \phi(\bar{Z}^j)\overline{\phi(\bar{Z}^j)^T})^p} \frac{H_{\phi(\bar{Z}^j)}^{1/2}(J\phi(\bar{Z}^j)u^j, J\phi(\bar{Z}^j)u^j)}{H_{Z^j}^{1/2}(u^j, u^j)} \geq \varepsilon_0 \tag{4.7}$$

for all  $j = 1, 2, \dots$ .

Now we will construct a sequence of functions  $\{f_j\}$  satisfying the following three conditions :

- (I)  $\{f_j\}$  is a bounded sequence in  $\beta^p(\mathfrak{R}_l(m, n))$ ;
- (II)  $\{f_j\}$  tends to 0 uniformly on any compact subset of  $\mathfrak{R}_l(m, n)$ ;
- (III)  $\|C_\phi f_j\|_{\beta^q} \rightarrow 0$  as  $j \rightarrow \infty$ .

The existence of this sequence will contradict the compactness of  $C_\phi$ .

We will construct the sequence of functions  $\{f_j\}$  according to the following four parts: A - D.

Part A: Suppose that  $\phi(Z^j) = r_j E_{11}$ ,  $j = 1, 2, \dots$

where  $E_{kl}$  is the  $m \times n$  matrix whose element at the  $k$ th row and  $l$ th column is 1 and the other elements are 0. Since  $\phi$  maps  $\mathfrak{R}_l(m, n)$  into itself,  $0 < r_j < 1$  and  $r_j \rightarrow 1$  as  $j \rightarrow \infty$ .

Denote  $J\phi(Z^j)u^j$  by  $w^j = (w_{11}^j, \dots, w_{1n}^j, w_{21}^j, \dots, w_{2n}^j, \dots, w_{m1}^j, \dots, w_{mn}^j)$  Using formula (1.1), we have

$$\begin{aligned} H_{\phi(Z^j)}(w^j, w^j) &= (m+n)w^j \text{diag}\left(\left(1-r_j^2\right)^{-1}, 1, \dots, 1\right) \times \text{diag}\left(\left(1-r_j^2\right)^{-1}, 1, \dots, 1\right) \overline{w^j}^T \\ &= (m+n) \left[ \frac{|w_{11}^j|^2}{(1-r_j^2)^2} + \frac{1}{1-r_j^2} \left( \sum_{l=2}^n |w_{1l}^j|^2 + \sum_{k=2}^m |w_{k1}^j|^2 \right) + \sum_{2 \leq k \leq m, 2 \leq l \leq n} |w_{kl}^j|^2 \right]. \end{aligned}$$

Denote

$$A_j = \frac{|w_{11}^j|^2}{(1-r_j^2)^2}, \quad B_j = \frac{1}{1-r_j^2} \left( \sum_{l=2}^n |w_{1l}^j|^2 + \sum_{k=2}^m |w_{k1}^j|^2 \right), \quad C_j = \sum_{2 \leq k \leq m, 2 \leq l \leq n} |w_{kl}^j|^2,$$

then

$$H_{\phi(Z^j)}(w^j, w^j) = (m+n)(A_j + B_j + C_j). \tag{4.8}$$

We construct the sequence of functions  $\{f_j\}$  according to the following three different cases.

Case 1. If for some  $j$ ,

$$\max(B_j, C_j) \leq A_j \tag{4.9}$$

then set

$$f_j(Z) = \frac{1}{p} \left( \frac{1}{(1-z_{11})^p} - \frac{1}{(1-e^{-a(1-r_j)}z_{11})^p} \right), \tag{4.10}$$

where  $a$  is any positive number.

Case 2. If for some  $j$ ,

$$\max(A_j, C_j) \leq B_j \tag{4.11}$$

then set

$$f_j(Z) = \left( \sum_{l=2}^n e^{-i\theta_{1l}^j} z_{1l} + \sum_{k=2}^m e^{-i\theta_{k1}^j} z_{k1} \right) \left( \frac{1}{(1-z_{11})^{p+1/2}} - \frac{1}{(1-e^{-a(1-r_j)} z_{11})^{p+1/2}} \right) \tag{4.12}$$

where  $\theta_{kl}^j = \arg w_{kl}^j$ , if for some  $l$ ,  $w_{kl}^j \neq 0$  or for some  $k$ ,  $w_{kl}^j = 0$ , replace the corresponding term  $e^{-i\theta_{kl}^j} z_{kl}$  by 0 (the same below).

Case 3. If for some  $j$ ,

$$\max(A_j, B_j) \leq C_j \tag{4.13}$$

then set

$$f_j(Z) = \left( \sum_{2 \leq k \leq m, 2 \leq l \leq n} e^{-i\theta_{kl}^j} z_{kl} \right) \left( \frac{1}{(1-z_{11})^p} - \frac{1}{(1-e^{-a(1-r_j)} z_{11})^p} \right). \tag{4.14}$$

Next, we will prove that the sequences of functions  $\{f_j(Z)\}$  defined by (4.10), (4.12) and (4.14) all satisfy the conditions (I), (II) and (III).

To begin with, we will prove the sequence of functions  $\{f_j(Z)\}$  defined by (4.10) satisfies the three conditions. We can get that

$$\begin{aligned} & \det(I - Z\bar{Z}^T)^p \left\{ \nabla f_j(Z) T^{-1}(Z, Z) \overline{\nabla f_j(Z)}^T \right\}^{1/2} \\ &= (m+n)^{-1/2} \det(I - Z\bar{Z}^T)^p \left\{ \left( 1 - \sum_{l=1}^n |z_{1l}|^2 \right) \left( 1 - \sum_{k=1}^m |z_{k1}|^2 \right) \right\}^{1/2} \left| \frac{1}{(1-z_{11})^{p+1}} - \frac{e^{-a(1-r_j)}}{(1-e^{-a(1-r_j)} z_{11})^{p+1}} \right| \\ &\leq (m+n)^{-1/2} (1-|z_{11}|^2)^p (1-|z_{11}|^2) \frac{2}{(1-|z_{11}|)^{p+1}} \\ &\leq 2^{p+2} (m+n)^{-1/2}. \end{aligned}$$

It follows from Lemma 3.5 that  $\|f_j\|_{\beta^p} \leq C 2^{p+2} (m+n)^{-1/2}$ .

This proves that the sequence of functions  $\{f_j(Z)\}$  defined by (4.10) satisfies condition (I).

Let  $E$  be any compact subset of  $\mathfrak{R}_I(m, n)$ . Then there exists a  $\rho \in (0, 1)$  such that

$$|z_{11}| \leq \rho \tag{4.15}$$

for any  $Z = (z_{kl})_{m \times n} \in E$ . By (4.10), we have

$$|f_j(Z)| = \left| \frac{1}{p} \frac{1}{(1-e^{-a(1-r_j)} z_{11})^p} \left\{ \left( \frac{1-e^{-a(1-r_j)} z_{11}}{1-z_{11}} \right)^p - 1 \right\} \right| \leq \frac{1}{p} \frac{1}{(1-\rho)^p} \left| \left( \frac{1-e^{-a(1-r_j)} z_{11}}{1-z_{11}} \right)^p - 1 \right|.$$

Since

$$\left| \frac{1-e^{-a(1-r_j)} z_{11}}{1-z_{11}} - 1 \right| = \left| \frac{z_{11}}{1-z_{11}} \right| \left| 1-e^{-a(1-r_j)} \right| \leq \frac{1}{1-\rho} (1-e^{-a(1-r_j)}).$$



But  $1 - e^{-a(1-r_j)} \rightarrow 0$  as  $j \rightarrow \infty$ . Thus,  $\left(\frac{1 - e^{-a(1-r_j)} z_{11}}{1 - z_{11}}\right)^p - 1$  converges to 0 uniformly on  $E$ . Therefore,  $\{f_j\}$  converges to 0 uniformly on  $E$  as  $j \rightarrow \infty$ . Thus, the sequence of functions  $\{f_j(Z)\}$  defined by (4.10) satisfies the condition (II).

Now (4.8) and (4.9) mean that

$$H_{\phi(Z^j)}(w^j, w^j) = (m, n)(A_j + B_j + C_j) \leq 3(m, n)A_j. \tag{4.16}$$

Combining (4.7) and (4.16), we have

$$\begin{aligned} \|C_{\phi f_j}\|_{\beta^q} &\geq \det(I - Z^j \bar{Z}^{jT})^q \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{Z^j}^{1/2}(u^j, u^j)} \\ &\geq \varepsilon_0(1 - r_j^2)^p \frac{|\nabla(f_j)(r_j E_{11})w^j|}{H_{r_j E_{11}}^{1/2}(w^j, w^j)} \geq \frac{\varepsilon_0(1 - r_j^2)^p}{\sqrt{3(m+n)}} \frac{\left|\frac{\partial f_j}{\partial z_{11}}(r_j E_{11})w_{11}^j\right|}{|w_{11}^j|/(1 - r_j^2)} \\ &\geq \frac{\varepsilon_0}{\sqrt{3(m+n)}} \left[1 - \frac{(1 - r_j)^{p+1} e^{-a(1-r_j)}}{(1 - e^{-a(1-r_j)})r_j^{p+1}}\right]. \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \left[1 - \frac{(1 - r_j)^{p+1} e^{-a(1-r_j)}}{(1 - e^{-a(1-r_j)})r_j^{p+1}}\right] = 1 - \left(\frac{1}{a+1}\right)^{p+1} \neq 0.$$

This proves that  $\|C_{\phi f_j}\|_{\beta^q} \rightarrow 0$  as  $j \rightarrow \infty$ , which means that the sequence of functions  $\{f_j(Z)\}$  defined by (4.10) satisfies condition (III).

We can prove that the sequence of functions  $\{f_j(Z)\}$  defined by (4.12) or (4.14) satisfies the conditions (I) - (III) by using the analogous method as above.

Part B: Now we assume that

$$\phi(Z^j) = r_j^{(1)} E_{11} + r_j^{(2)} E_{22}, \quad j = 1, 2, \dots$$

It is clear that  $1 > r_j^{(1)} \geq r_j^{(2)} \geq 0$  and for  $\phi(Z^j) \rightarrow \partial \mathfrak{R}_l(m, n)$  we can assume that  $r_j^{(1)} \rightarrow 1$  and  $r_j^{(2)} \rightarrow \lambda_0$  as  $j \rightarrow \infty$ , where  $\lambda_0 \leq 1$ .

If  $\lambda_0 = 1$ , we can use the same methods as in Part A to construct a sequence of functions  $\{f_j(Z)\}$  satisfying conditions (I)-(III).

Using formula (1.1), we have

$$\begin{aligned} H_{\phi(Z^j)}(w^j, w^j) &= (m+n)w^j \text{diag}\left(\left(1 - r_j^{(1)2}\right)^{-1}, \left(1 - r_j^{(2)2}\right)^{-1}, \dots, 1\right) \times \text{diag}\left(\left(1 - r_j^{(1)2}\right)^{-1}, \left(1 - r_j^{(2)2}\right)^{-1}, \dots, 1\right) \overline{w^j}^T \\ &= (m+n) \left[ \frac{|w_{11}^j|^2}{\left(1 - r_j^{(1)2}\right)^2} + \frac{|w_{12}^j|^2 + |w_{21}^j|^2}{\left(1 - r_j^{(1)2}\right)\left(1 - r_j^{(2)2}\right)} + \frac{|w_{22}^j|^2}{\left(1 - r_j^{(2)2}\right)^2} + \frac{1}{\left(1 - r_j^{(1)2}\right)} \left(\sum_{l=3}^n |w_{1l}^j|^2 + \sum_{k=3}^m |w_{k1}^j|^2\right) \right. \\ &\quad \left. + \frac{1}{\left(1 - r_j^{(2)2}\right)} \left(\sum_{l=3}^n |w_{2l}^j|^2 + \sum_{k=3}^m |w_{k2}^j|^2\right) + \sum_{3 \leq k \leq m, 3 \leq l \leq n} |w_{kl}^j|^2 \right]. \end{aligned}$$

Denote

$$A_j = \frac{|w_{11}^j|^2}{(1-r_j^{(1)})^2}, \quad B_j = \frac{|w_{12}^j|^2 + |w_{21}^j|^2}{(1-r_j^{(1)}) (1-r_j^{(2)})}, \quad C_j = \frac{|w_{22}^j|^2}{(1-r_j^{(2)})^2},$$

$$D_j = \frac{1}{(1-r_j^{(1)})^2} \left( \sum_{l=3}^n |w_{1l}^j|^2 + \sum_{k=3}^m |w_{k1}^j|^2 \right), \quad E_j = \frac{1}{(1-r_j^{(2)})^2} \left( \sum_{l=3}^n |w_{2l}^j|^2 + \sum_{k=3}^m |w_{k2}^j|^2 \right), \quad F_j = \sum_{3 \leq k \leq m, 3 \leq l \leq n} |w_{kl}^j|^2.$$

Then,

$$H_{\phi(z^j)}(w^j, w^j) = (m+n)(A_j + B_j + C_j + D_j + E_j + F_j) \tag{4.17}$$

We construct the sequence of functions  $\{f_j\}$  according to the following six different cases.

Case 1. If for some  $j$ ,

$$\max(B_j, C_j, D_j, E_j, F_j) \leq A_j$$

then set

$$f_j(Z) = \frac{1}{p} \left( \frac{1}{(1-z_{11})^p} - \frac{1}{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right)^p} \right) \tag{4.18}$$

Case 2. If for some  $j$ ,

$$\max(A_j, C_j, D_j, E_j, F_j) \leq B_j,$$

then set

$$f_j(Z) = \left( e^{-i\theta_{12}^j} z_{12} + e^{-i\theta_{21}^j} z_{21} \right) \left( \frac{1}{\left[ (1-z_{11})(1-z_{22}) \right]^{p+1/2}} - \frac{1}{\left[ \left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right) \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right) \right]^{p+1/2}} \right). \tag{4.19}$$

Case 3. If for some  $j$ ,

$$\max(A_j, B_j, D_j, E_j, F_j) \leq C_j$$

then set

$$f_j(Z) = \frac{1}{p} \left( \frac{1}{(1-z_{22})^p} - \frac{1}{\left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)^p} \right). \tag{4.20}$$

Case 4. If for some  $j$ ,

$$\max(A_j, B_j, C_j, E_j, F_j) \leq D_j$$

then set

$$f_j(Z) = \left( \sum_{l=3}^n e^{-i\theta_{1l}^j} z_{1l} + \sum_{k=3}^m e^{-i\theta_{k1}^j} z_{k1} \right) \left( \frac{1}{(1-z_{11})^{p+1/2}} - \frac{1}{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right)^{p+1/2}} \right). \tag{4.21}$$

Case 5. If for some  $j$ ,

$$\max(A_j, B_j, C_j, D_j, F_j) \leq E_j,$$

then set

$$f_j(Z) = \left( \sum_{l=3}^n e^{-i\theta_{2l}^j} z_{2l} + \sum_{k=3}^m e^{-i\theta_{k2}^j} z_{k2} \right) \left( \frac{1}{(1-z_{22})^{p+1/2}} - \frac{1}{\left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)^{p+1/2}} \right). \tag{4.22}$$

Case 6. If for some  $j$ ,

$$\max(A_j, B_j, C_j, D_j, E_j) \leq F_j$$

then set

$$f_j(Z) = \left( \sum_{3 \leq k \leq m, 3 \leq l \leq n} e^{-i\theta_{kl}^j} z_{kl} \right) \left( \frac{1}{(1-z_{11})^p} - \frac{1}{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right)^p} \right). \tag{4.23}$$

By using the same methods as in Part A, we can prove the sequences of functions  $\{f_j(Z)\}$  defined by (4.18)-(4.23) satisfying conditions (I) - (III).

Now, as an example, we will prove that the sequence of functions  $\{f_j(Z)\}$  defined by (4.19) satisfying the conditions (I) - (III).

For any  $Z \in \mathfrak{R}_j(m, n)$ , we have

$$|z_{12}|^2 < 1 - |z_{ii}|^2, \quad |z_{21}|^2 < 1 - |z_{ii}|^2, \quad i = 1, 2. \tag{4.24}$$

Thus

$$\begin{aligned} |\nabla f_j(Z)| &= \left\{ \sum_{1 \leq k \leq m, 1 \leq l \leq n} \left| \frac{\partial f_j}{\partial z_{kl}}(Z) \right|^2 \right\}^{1/2} \\ &\leq \left| \left( p + \frac{1}{2} \right) \left( e^{-i\theta_{12}^j} z_{12} + e^{-i\theta_{21}^j} z_{21} \right) \left( \frac{1}{(1-z_{11})^{p+\frac{3}{2}} (1-z_{22})^{p+\frac{1}{2}}} - \frac{e^{-a(1-r_j^{(1)})}}{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right)^{p+\frac{3}{2}} \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)^{p+\frac{1}{2}}} \right) \right| \\ &\quad + \left| \left( p + \frac{1}{2} \right) \left( e^{-i\theta_{12}^j} z_{12} + e^{-i\theta_{21}^j} z_{21} \right) \left( \frac{1}{(1-z_{11})^{p+\frac{1}{2}} (1-z_{22})^{p+\frac{3}{2}}} - \frac{e^{-a(1-r_j^{(2)})}}{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right)^{p+\frac{1}{2}} \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)^{p+\frac{3}{2}}} \right) \right| \\ &\quad + 2 \left| \frac{1}{\left[ (1-z_{11})(1-z_{22}) \right]^{p+1/2}} - \frac{1}{\left[ \left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right) \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right) \right]^{p+1/2}} \right| \\ &< \left( p + \frac{1}{2} \right) (|z_{12}| + |z_{21}|) \left( \frac{2}{(1-|z_{11}|)^{p+\frac{3}{2}} (1-|z_{22}|)^{p+\frac{1}{2}}} + \frac{2}{(1-|z_{11}|)^{p+\frac{1}{2}} (1-|z_{22}|)^{p+\frac{3}{2}}} \right) + \frac{4}{(1-|z_{11}|)^{p+\frac{1}{2}} (1-|z_{22}|)^{p+\frac{1}{2}}} \\ &\leq 4\sqrt{2} \left( p + \frac{1}{2} \right) \left( \frac{1}{(1-|z_{11}|)^{p+1} (1-|z_{22}|)^{p+\frac{1}{2}}} + \frac{1}{(1-|z_{11}|)^{p+\frac{1}{2}} (1-|z_{22}|)^{p+1}} \right) + \frac{4}{(1-|z_{11}|)^{p+\frac{1}{2}} (1-|z_{22}|)^{p+\frac{1}{2}}}. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
 & \det(I - Z\bar{Z}^T)^p \left\{ \nabla f_j(Z) T^{-1}(Z, Z) \overline{\nabla f_j(Z)}^T \right\}^{1/2} \\
 & \leq \det(I - Z\bar{Z}^T)^p |\nabla f_j(Z)| \left\{ \det T^{-1}(Z, Z) \right\}^{1/2} \\
 & \leq \frac{1}{\sqrt{m+n}} (1 - |z_{11}|^2)^{p + \frac{m+n}{2}} (1 - |z_{22}|^2)^{p + \frac{m+n}{2}} \\
 & \quad \times \left( \frac{4\sqrt{2} \left(p + \frac{1}{2}\right)}{(1 - |z_{11}|)^{p+1} (1 - |z_{22}|)^{p+\frac{1}{2}}} + \frac{4\sqrt{2} \left(p + \frac{1}{2}\right)}{(1 - |z_{11}|)^{p+\frac{1}{2}} (1 - |z_{22}|)^{p+1}} + \frac{4}{(1 - |z_{11}|)^{p+\frac{1}{2}} (1 - |z_{22}|)^{p+\frac{1}{2}}} \right) \\
 & \leq \frac{2^{2p+m+n+2}}{\sqrt{m+n}} \left( 2\sqrt{2} \left(p + \frac{1}{2}\right) + 1 \right).
 \end{aligned} \tag{4.25}$$

It follows from Lemma 3.5 and (4.25) that  $\|f_j\|_{\beta^p} \leq C$ . This proves that the sequence of functions  $\{f_j(Z)\}$  defined by (4.19) satisfy the condition (I).

Let  $E$  be any compact subset of  $\mathfrak{R}_l(m, n)$ . Since there exists a  $\rho \in (0, 1)$  such that  $1 - |z_{ii}| \geq 1 - \rho > 0$ ,  $i = 1, 2$ . Thus

$$|f_j(Z)| \leq \frac{2}{(1 - \rho)^{2p+1}} \left\{ \left[ \frac{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right) \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)}{(1 - z_{11})(1 - z_{22})} \right]^{p+1/2} - 1 \right\}.$$

Since

$$\left| \frac{1 - e^{-a(1-r_j^{(i)})} z_{ii}}{1 - z_{ii}} - 1 \right| = \left| \frac{z_{ii}}{1 - z_{ii}} \right| \left| 1 - e^{-a(1-r_j^{(i)})} \right| \leq \frac{1}{1 - \rho} \left( 1 - e^{-a(1-r_j^{(i)})} \right), \quad i = 1, 2.$$

So  $1 - e^{-a(1-r_j^{(i)})} \rightarrow 0$  as  $j \rightarrow \infty$ . Thus,  $\left[ \frac{\left(1 - e^{-a(1-r_j^{(1)})} z_{11}\right) \left(1 - e^{-a(1-r_j^{(2)})} z_{22}\right)}{(1 - z_{11})(1 - z_{22})} \right]^{p+1/2} - 1$  converges to 0 uni-

formly on  $E$ . Therefore, the sequence of  $\{f_j\}$  converges to 0 uniformly on  $E$  as  $j \rightarrow \infty$ . Thus, the sequence of functions  $\{f_j(Z)\}$  defined by (4.19) satisfies the condition (II).

For case 2,

$$H_{\phi(Z^j)}(w^j, w^j) \leq 6(m+n) B_j. \tag{4.26}$$

Combining (4.7) and (4.26), we have

$$\begin{aligned}
 \|C_\phi f_j\|_{\beta^q} & \geq \det(I - Z^j \bar{Z}^j)^q \frac{|\nabla(f_j)(\phi(Z^j)) J\phi(Z^j) u^j|}{H_{Z^j}^{1/2}(u^j, u^j)} \geq \varepsilon_0 (1 - r_j^{(1)^2})^p (1 - r_j^{(2)^2})^p \frac{|\nabla(f_j)(r_j^{(1)} E_{11} + r_j^{(2)} E_{22}) w^j|}{H_{r_j^{(1)} E_{11} + r_j^{(2)} E_{22}}^{1/2}(w^j, w^j)} \\
 & \geq \frac{\varepsilon_0}{\sqrt{6(m+n)}} \left[ 1 - \left( \frac{(1 - r_j^{(1)})(1 - r_j^{(2)})}{\left(1 - e^{-a(1-r_j^{(1)})} r_j^{(1)}\right) \left(1 - e^{-a(1-r_j^{(2)})} r_j^{(2)}\right)} \right)^{p+1/2} \right].
 \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \left[ 1 - \left( \frac{(1-r_j^{(1)})(1-r_j^{(2)})}{\left(1 - e^{-a(1-r_j^{(1)})} r_j^{(1)}\right) \left(1 - e^{-a(1-r_j^{(2)})} r_j^{(2)}\right)} \right)^{p+1/2} \right] = 1 - \left( \frac{1}{a+1} \right)^{2p+1} \neq 0.$$

This proves that  $\|C_\phi f_j\|_{\beta^q} \rightarrow 0$  as  $j \rightarrow \infty$ , which means that the sequence of functions  $\{f_j(Z)\}$  defined by (4.19) satisfies condition (III).

If  $\lambda_0 < 1$ , then by Lemma 3.6, there exist  $\Phi_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}}$  and  $\Phi_{r_j^{(1)}E_{11}}$  in  $\mathfrak{R}_I(m, n)$  such that

$$\Phi_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}}(r_j^{(1)}E_{11} + r_j^{(2)}E_{22}) = 0 \quad \text{and} \quad \Phi_{r_j^{(1)}E_{11}}(r_j^{(1)}E_{11}) = 0 \quad (j = 1, 2, \dots).$$

If we denote  $\Psi^{(j)}(Z) = \left(\Phi_{r_j^{(1)}E_{11}}\right)^{-1} \circ \Phi_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}}$ , then  $\Psi^{(j)} \in \text{Aut}(\mathfrak{R}_I(m, n))$  and

$$\Psi^{(j)}(\phi(Z^j)) = \Psi^{(j)}(r_j^{(1)}E_{11} + r_j^{(2)}E_{22}) = r_j^{(1)}E_{11} = r_j E_{11}, \quad \text{where } r_j = r_j^{(1)}.$$

Denote  $g_j = f_j \circ \Psi^{(j)}$ , where the sequence of functions  $\{f_j\}$  is the sequence obtained in Part A. We have

$$H_{\phi(Z^j)}(w^j, w^j) = H_{\Psi^{(j)} \circ \phi(Z^j)}(J\Psi^{(j)}(\phi(Z^j))w^j, J\Psi^{(j)}(\phi(Z^j))w^j) = H_{r_j E_{11}}(v^j, v^j), \quad (4.27)$$

where  $w^j = J\phi(Z^j)u^j$  and  $v^j = J\Psi^{(j)}(\phi(Z^j))w^j$ . Now (4.27) implies that

$$\frac{|\nabla(g_j)(\phi(Z^j))w^j|}{H_{\phi(Z^j)}^{1/2}(w^j, w^j)} = \frac{|\nabla(f_j)(r_j E_{11})v^j|}{H_{r_j E_{11}}^{1/2}(v^j, v^j)}$$

and

$$\begin{aligned} \|C_\phi g_j\|_{\beta^q} &\geq \det(I - ZZ^T)^q \frac{|\nabla(g_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{Z^j}^{1/2}(u^j, u^j)} \\ &\geq \varepsilon_0 \det\left(I - \phi(Z^j)\overline{\phi(Z^j)}^T\right)^p \frac{|\nabla(g_j)(\phi(Z^j))w^j|}{H_{\phi(Z^j)}^{1/2}(w^j, w^j)} \\ &= \varepsilon_0 \left(1 - r_j^{(2)^2}\right)^p \left(1 - r_j^{(1)^2}\right)^p \frac{|\nabla(f_j)(r_j E_{11})v^j|}{H_{r_j E_{11}}^{1/2}(v^j, v^j)}. \end{aligned}$$

It is clear that  $\lim_{j \rightarrow \infty} (1 - r_j^{(2)^2}) = 1 - \lambda_0^2 \neq 0$ , and combining the discussion in Part A, we can get that

$\|C_\phi g_j\|_{\beta^q} \rightarrow 0$  as  $j \rightarrow \infty$ ; that means the sequence of functions  $\{g_j\}$  satisfies condition (III).

We prove that the sequence of functions  $\{g_j\}$  is a bounded sequence in  $\beta^p(\mathfrak{R}_I(m, n))$ .

Since  $\Psi^j(Z) \in \text{Aut}(\mathfrak{R}_I(m, n))$ ,

$$\det(I - ZZ^T)^p Q_{g_j}(Z) = \det(I - ZZ^T)^p Q_{f_j}(\Psi^{(j)}(Z)).$$

So  $\|g_j\|_{\beta^p} = \|f_j\|_{\beta^p}$  is bounded.

Next we prove that  $\{g_j\}$  converges to 0 uniformly on any compact subset  $E$  of  $\mathfrak{R}_I(m, n)$ . Let  $\Psi^{(j)}(Z) = (\Psi_{lk}^{(j)}(Z))_{1 \leq l \leq m, 1 \leq k \leq n}$ , then by the definition of  $\Psi^{(j)}$  and Lemma 3.6, we can get a calculation directly that

$$\Psi_{11}^{(j)}(Z) = z_{11} + r_j^{(2)} \frac{z_{12}z_{21}}{1 - r_j^{(2)}z_{22}}.$$

It is clear that  $\Psi_{11}^{(j)}(Z)$  converges uniformly to  $\Psi_{11}(Z) = z_{11} + \lambda_0 \frac{z_{12}z_{21}}{1 - \lambda_0 z_{22}}$  in  $\mathfrak{R}_I(m, n)$ .

Since  $\lambda_0 < 1$  and  $\lambda_0 E_{11} + \lambda_0 E_{22} \in \mathfrak{R}_I(m, n)$ , there similarly exist  $\Psi(Z)$  in  $Aut(\mathfrak{R}_I(m, n))$  such that  $\Psi(\lambda_0 E_{11} + \lambda_0 E_{22}) = \lambda_0 E_{11}$ , and the first component of  $\Psi(Z)$  is  $\Psi_{11}(Z)$ . It is clear that  $\Psi_{11}(Z)$  is holomorphic on  $\mathfrak{R}_I(m, n)$ . Let  $M_1 = \sup_{Z \in E} |\Psi_{11}(Z)| = \Psi_{11}(Z_0)$  for  $Z_0 \in E$ . For  $\Psi(Z) \in Aut(\mathfrak{R}_I(m, n))$ , we know

$M_1 = \Psi_{11}(Z_0) < 1$ . We may choose  $M_0 > 0$  such that  $M_1 < M_0 < 1$ . Thus, for  $j$  large enough,  $|\Psi_{11}^{(j)}(Z_0)| < M_0$  and from this it follows that

$$1 - |\Psi_{11}^{(j)}(Z)| > 1 - M_0 > 0$$

by the definition of  $\{f_j(Z)\}$ ,  $g_j(Z) = f_j \circ \Psi^{(j)}(Z)$  converges to 0 uniformly on  $E$ .

Hence  $\{g_j(Z)\}$  satisfies conditions (I)–(III), and this contradicts the compactness of  $C_\phi$ .

Part C: Assume that

$$\phi(Z^j) = \sum_{k=1}^m r_j^{(k)} E_{kk}, \quad j = 1, 2, \dots$$

where  $1 > r_j^{(1)} \geq r_j^{(2)} \geq \dots \geq r_j^{(m)} \geq 0$ . For  $\phi(Z^j) \rightarrow \partial \mathfrak{R}_I(m, n)$  we may assume that  $r_j^{(1)} \rightarrow 1, r_j^{(2)} \rightarrow \lambda_2, \dots, r_j^{(m)} \rightarrow \lambda_m$  as  $j \rightarrow \infty$ , where  $\lambda_k \leq 1, k = 2, 3, \dots, m$ .

Just as in Part B, we can use the same methods to prove the conclusion. And for  $\lambda_k = 1, k = 2, 3, \dots, m$ , we may only show the sequence of functions  $\{f_j(Z)\}$  which satisfy the conditions (I) - (III) here.

Using formula (1.1), we have

$$\begin{aligned} H_{\phi(Z^j)}(w^j, w^j) &= (m+n) w^j \text{diag} \left( (1 - r_j^{(1)^2})^{-1}, (1 - r_j^{(2)^2})^{-1}, \dots, (1 - r_j^{(m)^2})^{-1} \right) \\ &\quad \times \text{diag} \left( (1 - r_j^{(1)^2})^{-1}, (1 - r_j^{(2)^2})^{-1}, \dots, (1 - r_j^{(m)^2})^{-1}, 1, \dots, 1 \right) w^{jT} \\ &= (m+n) \left[ \sum_{k=1}^m \frac{|w_{kk}^j|^2}{(1 - r_j^{(k)^2})^2} + \sum_{1 \leq k < l \leq m} \frac{|w_{kl}^j|^2 + |w_{lk}^j|^2}{(1 - r_j^{(k)^2})(1 - r_j^{(l)^2})} + \sum_{k=1}^m \frac{\sum_{l=m+1}^n |w_{kl}^j|^2}{1 - r_j^{(k)^2}} \right]. \end{aligned}$$

Denote

$$\begin{aligned} A_j^{(k)} &= \frac{|w_{kk}^j|^2}{(1 - r_j^{(k)^2})^2}, \quad k = 1, \dots, m, \quad B_j^{(kl)} = \frac{|w_{kl}^j|^2 + |w_{lk}^j|^2}{(1 - r_j^{(k)^2})(1 - r_j^{(l)^2})}, \quad 1 \leq k < l \leq m, \\ C_j^{(k)} &= \frac{\sum_{l=m+1}^n |w_{kl}^j|^2}{1 - r_j^{(k)^2}}, \quad k = 1, \dots, m. \end{aligned}$$

then,

$$H_{\phi(Z^j)}(w^j, w^j) = (m+n) \left( \sum_{k=1}^m A_j^{(k)} + \sum_{1 \leq k < l \leq m} B_j^{(kl)} + \sum_{k=1}^m C_j^{(k)} \right). \tag{4.28}$$

We construct the sequence of functions  $\{f_j\}$  according to the following three different cases.

Case 1. If for some  $j$ ,

$$\max(A_j^{(i)}, B_j^{(st)}, C_j^{(i)}) \leq A_j^{(k)}, \quad i, k = 1, \dots, m, \quad 1 \leq s < t \leq m,$$

then set

$$f_j^{(k)}(Z) = \frac{1}{p} \left( \frac{1}{(1-z_{kk})^p} - \frac{1}{\left(1 - e^{-a(1-r_j^{(k)})} z_{kk}\right)^p} \right), \quad k = 1, \dots, m. \tag{4.29}$$

Case 2. If for some  $j$ ,

$$\max(A_j^{(i)}, B_j^{(st)}, C_j^{(i)}) \leq B_j^{(kl)}, \quad i = 1, \dots, m, \quad 1 \leq s < t \leq m, \quad 1 \leq k < l \leq m,$$

then set

$$f_j^{(kl)}(Z) = \left( e^{-i\theta_{kl}^j} z_{kl} + e^{-i\theta_{lk}^j} z_{lk} \right) \left( \frac{1}{\left[ (1-z_{kk})(1-z_{ll}) \right]^{p+1/2}} - \frac{1}{\left[ \left(1 - e^{-a(1-r_j^{(k)})} z_{kk}\right) \left(1 - e^{-a(1-r_j^{(l)})} z_{ll}\right) \right]^{p+1/2}} \right), \tag{4.30}$$

$1 \leq k < l \leq m.$

Case 3. If for some  $j$ ,

$$\max(A_j^{(i)}, B_j^{(st)}, C_j^{(i)}) \leq C_j^{(k)}, \quad i, k = 1, \dots, m, \quad 1 \leq s < t \leq m,$$

then set

$$f_j^{(k)}(Z) = \left( \sum_{l=m+1}^n e^{-i\theta_{kl}^j} z_{kl} \right) \left( \frac{1}{(1-z_{kk})^{p+1/2}} - \frac{1}{\left(1 - e^{-a(1-r_j^{(k)})} z_{kk}\right)^{p+1/2}} \right), \quad k = 1, \dots, m. \tag{4.31}$$

Using the same methods as in Part A and Part B, we can prove the sequences of functions  $\{f_j(Z)\}$  defined by (4.29)-(4.31) satisfying conditions (I) - (III).

Part D: In the general situation. For  $\phi(Z^j) \in \mathfrak{R}_l(m, n)$ , there exist an  $m \times m$  unitary matrix  $P_j$  and an  $n \times n$  unitary matrix  $Q_j$  such that

$$P_j(\phi(Z^j))Q_j = \sum_{k=1}^m r_j^{(k)} E_{kk}.$$

We may assume that  $P_j \rightarrow P$  and  $Q_j \rightarrow Q$  as  $j \rightarrow \infty$ . Let  $P_j = (p_j^{kl})$  and  $P = (p^{kl})$ ;  $P_j \rightarrow P$  means that  $p_j^{kl} \rightarrow p^{kl}$  as  $j \rightarrow \infty$  for any  $1 \leq k, l \leq m$ . Let  $\Psi^j(Z) = P_j Z Q_j$  and  $\Psi(Z) = P Z Q$  for  $Z \in \mathfrak{R}_l(m, n)$ . Of course,  $P$  is an  $m \times m$  unitary matrix,  $Q$  is an  $n \times n$  unitary matrix, and  $\{\Psi^{(j)}(Z)\}$  converges uniformly to  $\Psi(Z)$  on  $\mathfrak{R}_l(m, n)$ .

Let  $g_j(Z) = f_j(\Psi^j(Z))$ ,  $j = 1, 2, \dots$  where the sequence of  $\{f_j\}$  are the functions obtained in Part C. From the same discussion as that in Part B, we know that  $\{g_j(Z)\}$  satisfies conditions (I) and (III). For the compact subset  $E \subset \mathfrak{R}_l(m, n)$ ,  $\Psi(E)$  is also a compact subset of  $\mathfrak{R}_l(m, n)$ , so we can choose an open subset  $D_1$  of  $\mathfrak{R}_l(m, n)$  such that  $\Psi(E) \subset D_1 \subset \overline{D_1} \subset \mathfrak{R}_l(m, n)$ . Since  $\{\Psi^{(j)}(Z)\}$  converges uniformly to  $\Psi(Z)$  on  $\mathfrak{R}_l(m, n)$ , it follows that  $\Psi^j(E) \subset D_1$  as  $j \rightarrow \infty$ . Since  $\{f_j(Z)\}$  tends to 0 uniformly on  $\overline{D_1}$ , we know  $\{g_j(Z)\}$  tends to 0 uniformly on  $E$ . Thus,  $\{g_j\}$  satisfies condition (II). □

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