Advances in Pure Mathematics, 2014, 4, 580-584
Published Online November 2014 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2014.411066

Approximation Theorems for Exponentially Bounded α-Times Integrated Cosine Function

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Received 26 September 2014; revised 25 October 2014; accepted 31 October 2014

Abstract

In this paper, based on the theories of α-times Integrated Cosine Function, we discuss the approximation theorem for α-times Integrated Cosine Function and conclude the approximation theorem of exponentially bounded α-times Integrated Cosine Function by the approximation theorem of n-times integrated semigroups. If the semigroups are equicontinuous at each point $t \in [0,\infty]$, we give different methods to prove the theorem.

Keywords

α-Times Integrated Cosine Function, Exponentially Bounded, Approximation

1. Introduction

Integrated semigroups were introduced by Arent [1] [2] and Davies and Pang [3] in 1987. The approximation theorem is one of the fundamental theorems in the theory of operater semigroups. There have been many results on approximation [4]-[7]. Cao [8] obtained the approximation theorem for m-times Integrated Cosine Function, $m \in N$. In this paper, we refine the theory by introducing α-times Integrated Cosine Function for positive real numbers α. Moreover, if the semigroups are equicontinuous at each point $t \in [0,\infty]$, we give different methods to prove the theorem.

Throughout this paper, we will denote by X—a Banach space with norm $\|\bullet\|$, by $B(X)$—the Banach space of all bounded linear operators from X to X; A is a linear operator in X, by

$$D(A), R(A), \rho(A), R(\lambda, A)$$

respectively the domain, the range, the resolvent set, and the resolvent of A.

2. Preliminaries

Definition 2.1. Let $\alpha \in R^{+}$, then a strongly continuous family $\{S(t)\}_{t\geq 0}$ in $B(X)$ is called an α-times Integrated Cosine Function, if the following hold:

1) $S(0)=0$;

2) For any $x\in X$, and $\forall s,t\geq 0$,

$$2S(s)S(t)=\frac{1}{\Gamma(\alpha)}\Big\{(-1)^{\alpha}\int_0^{|t-s|}(|t-s|-r)^{\alpha-1}S(r)x\mathrm{d}r$$
$$+\left(\int_0^{t+s}-\int_0^{t}-\int_0^{s}\right)(t+s-r)^{\alpha-1}S(r)x\mathrm{d}r$$
$$+\int_0^{t}(t-s+r)^{\alpha-1}S(r)x\mathrm{d}r+\int_0^{s}(t-s+r)^{\alpha-1}S(r)x\mathrm{d}r\Big\}.$$

Definition 2.2. A is a linear operator in X, $\alpha\in R^{+}$, A is called the generator of an α-times Integrated Cosine Function if there are nonnegative numbers ω, M and a mapping $S:[0,\infty)\to B(X)$ such that

1) $\{S(t)\}_{t\geq 0}$ is strongly continuous and $\left\|\int_0^t S(s)\mathrm{d}s\right\|\leq M\mathrm{e}^{wt}$ for all $t\geq 0$;

2) (ω,∞) is contained in the resolvent set of A;

3) $R(\lambda^2,A)=\lambda^{\alpha-1}\int_0^{\infty}\mathrm{e}^{-\lambda t}S(t)\mathrm{d}t$ for $\lambda>\omega$.

Lemma 2.3. [9] For each $n\in N$ let $f_n\in L^1_{loc}([0,\infty),X)$, with

$$\left\|\int_0^t f_n(s)\mathrm{d}s\right\|\leq M\mathrm{e}^{\omega t},\quad t\geq 0$$

and let

$$F_n(\lambda)=\int_0^{\infty}\mathrm{e}^{-\lambda t}f_n(t)\mathrm{d}t,\qquad \lambda>\omega$$

Assume that

$$\lim_{n\to\infty}F_n(\lambda)\quad\text{exists for }\lambda>\omega,$$

and that for a fixed $t_0\in(0,\infty)$, $\sup_{n\in N}\|f_n(t_0)\|<\infty$, and

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\left(f_n(t_0+s)-f_n(t_0)\right)\mathrm{d}s=0$$

with uniform concergence for $n\in N$. Then $\lim_{m\to\infty}f_n(t_0)$ exists.

Lemma 2.4. [10] If A is a linear operator in X, $\alpha\geq 0$. The following assertions are equivalent:

1) There exist constant $\omega, M\geq 0$, such that $(\omega^2,\infty)\subset\rho(A)$, and

$$\left\|(\lambda-\omega)^{k+1}\left(\lambda^{1-\alpha}R(\lambda^2,A)\right)^{(k)}\right\|\leq Mk!.$$

for $\lambda>\omega$, $k\in N_0=N\cup\{0\}$.

2) $\forall\beta\in(\alpha,\alpha+1]$, A generate a β-times Integrated Cosine Function $\{S_\beta(t)\}_{t\geq 0}$, and exist constant k such that $\alpha+1$-times Integrated Cosine Function $\{S_{\alpha+1}(t)\}_{t\geq 0}$ hold

$$\lim_{h\to 0}\sup\frac{1}{h}\left\|S_{\alpha+1}(t+h)-S_{\alpha+1}(t)\right\|\leq k\mathrm{e}^{\omega t}\quad(t\geq 0,h\geq 0).$$

3. Main Results

Theorem 3.1. If A_n generates a α-times Integrated Cosine Function $\{S_n(t)\}_{t\geq 0}$, and there is $M,\omega\in R^{+}$

such that $\left\|S_n(t)\right\| \le Me^{\omega t}$, then the following statements are equivalent:

1) $\lim_{n\to\infty} R\left(\lambda^2, A_n\right)x = R\left(\lambda^2, A_0\right)x$, $\forall x \in X$, for some $\lambda_0 > \omega$, and $\left\{S_n(t)\right\}_{t\ge 0}$ is equicontinuous at each point $t \in [0,\infty]$;

2) $\lim_{n\to\infty} R\left(\lambda^2, A_n\right)x = R\left(\lambda^2, A_0\right)x$, $\forall x \in X$, $\lambda > \omega$, and $\left\{S_n(t)\right\}_{t\ge 0}$ is equicontinuous at each point $t \in [0,\infty]$;

3) $\lim_{n\to\infty} S_n(t)x = S_0(t)x$, $\forall x \in X$ uniformly on compacts of $t \ge 0$.

Proof: 1) $\Rightarrow$ 2) Consider the set

$$\Omega = \left\{\lambda : \lim_{n\to\infty} R\left(\lambda^2, A_n\right)x = R\left(\lambda^2, A_0\right)x, \forall x \in X, \lambda > \omega\right\},$$

which is nonempty by assumption.

Let $\mu \in \Omega$, then

$$\lambda^2 - A_n = \mu^2 - A_n + \lambda^2 - \mu^2 = \left[I - \left(\mu^2 - \lambda^2\right)R\left(\mu^2, A_n\right)\right]\left(\mu^2 - A_n\right)$$

when $\left|\mu^2 - \lambda^2\right| < \dfrac{1}{\left\|R\left(\mu^2 - A_n\right)\right\|}$

$$R\left(\lambda^2, A_n\right) = R\left(\mu^2 - A_n\right)\left[I - \left(\mu^2 - \lambda^2\right)R\left(\mu^2, A_n\right)\right]^{-1} = \sum_{k=0}^{\infty}\left(\mu^2 - \lambda^2\right)^k R\left(\mu^2, A_n\right)^{k+1}$$

Obviously $R\left(\lambda^2, A_n\right)$ converges as $n \to \infty$. Therefore, the set Ω is open.

On the other hand, taking an accumulation point λ of Ω with $\lambda > \omega$, we can find $\mu \in \Omega$, such that $\left|\mu^2 - \lambda^2\right| < \dfrac{1}{\left\|R\left(\mu^2 - A_n\right)\right\|}$. By the above considerations, λ must belong to Ω, *i.e.*, Ω is relatively closed in $S = \{\lambda : \lambda > \omega\}$, which leads to the conclusion.

$$2) \Rightarrow 3) \text{ Let } F_n(\lambda) = \lambda^{-\alpha+1}R\left(\lambda^2, A_n\right) = \int_0^\infty e^{-\lambda t} S_n(t)\,dt,$$

for

$$\begin{aligned}&\lim_{n\to\infty} R\left(\lambda^2, A_n\right)x = R\left(\lambda^2, A_0\right)x\\ &\lim_{n\to\infty} S_n(t)x\end{aligned},$$

and $\left\{S_n(t)\right\}_{t\ge 0}$ is equicontinuous at each point $t \in [0,\infty]$; using Lemma 2.2, it is easy to know that $\lim_{n\to\infty} S_n(t)x$ exists. We now fix $b > 0$, then for each $\varepsilon > 0$, $\exists K \in N$; when $|t - s| \le \dfrac{b}{K}, t, s \in [0,b]$, we have

$$\left\|S_m(t) - S_m(s)\right\| < \frac{\varepsilon}{3} \tag{1}$$

Pick $t_i = \dfrac{i}{k}b \in [0,b], i = 1,2,3,\cdots,K$, then $\exists N_0 \in N$ such that

$$\left\|S_n(t_i) - S_l(t_i)\right\| < \frac{\varepsilon}{3},\ n,l \ge N_0, i = 1,2,3,\cdots,k. \tag{2}$$

From (1) (2), we have $\left\|S_n(t) - S_l(t)\right\| < \dfrac{\varepsilon}{3}$, $n,l \ge N_0$, $t \in [0,b]$.

It shows that 3) is right.

3) $\Rightarrow$ 2) fix $t_0 \in [0,\infty)$, for each $\varepsilon > 0$, $\exists N_0 \in N$, when $n \ge N_0$.

We have

$$\left\|S_n(s)-S_{N_0}(s)\right\|<\frac{\varepsilon}{3},\quad s\in[0,t+1].$$

For $S_n(t)$ is continuous on $[0,t+1]$, then $\exists\delta_0>0$, $|s-t|<\delta_0$, when $s\in[0,t+1]$
We have

$$\left\|S_n(s)-S_n(t)\right\|<\frac{\varepsilon}{3},\quad n=1,2,3,\cdots,N_0$$

Therefore, if $n\geq N_0$, $s\in[0,t+1]$, then

$$\left\|S_n(s)-S_n(t)\right\|\leq\left\|S_n(s)-S_{N_0}(S)\right\|+\left\|S_{N_0}(s)-S_{N_0}(t)\right\|+\left\|S_{N_0}(t)-S_n(t)\right\|<\varepsilon$$

In conclusion $\{S_n(t),n\in N\}$ is equicontinuous at t.

By using the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}F_n(\lambda)=\lambda^{-\alpha+1}R(\lambda^2,A_n)=\int_0^\infty \mathrm{e}^{-\lambda t}S_n(t)\mathrm{d}t=\int_0^\infty \mathrm{e}^{-\lambda t}S_0(t)\mathrm{d}t$$

So 2) is right.
2) $\Rightarrow$ 1) the proof is obvious.
The proof is completed.

Corollary 3.2. If A_n is the generator of α-times Integrated Cosine Function $\{S_n(t)\}_{t\geq 0}$ satisfying:

$$\left\|S_n(t+h)-S_n(t)\right\|\leq M\mathrm{e}^{\omega(t+h)}h^{\gamma},\ n\in N,t,h\geq 0,\gamma\in(0,1] \tag{3}$$

Then (1)-(3) are equivalent:

1) $\lim_{n\to\infty}R(\lambda^2,A_n)x=R(\lambda^2,A_0)x$, $\forall x\in X$, for some $\lambda_0>\omega$.

2) $\lim_{n\to\infty}R(\lambda^2,A_n)x=R(\lambda^2,A_0)x$, $\forall x\in X$, $\lambda>\omega$.

3) $\lim_{n\to\infty}S_n(t)x=S_0(t)x$, $\forall x\in X$, uniformly on compacts of $t\geq 0$.

Theorem 3.3. If A_n is the generator of α-times Integrated Cosine Function $\{S_n(t)\}_{t\geq 0}$, and there is $M,\omega\in R^+$ such that $\left\|S_n(t)\right\|\leq M\mathrm{e}^{\omega t}$, $\forall x\in X$, $\lambda>\omega$, $\{S_n(t)\}_{t\geq 0}$ is equicontinuous at each point $t\in[0,\infty]$. $\lim_{n\to\infty}R(\lambda^2,A_n)x=R(\lambda^2)x$ exist, for some $\lambda_0>\omega$, $\ker R(\lambda_0^2)=\{0\}$, then there is a linear operator A—generator of α-times Integrated Cosine Function $S(t)$, such that $\lim_{n\to\infty}S_n(t)x=S(t)x$, $\forall x\in X$, and uniformly on compacts of $t\geq 0$.

Proof: By $\lim_{n\to\infty}R(\lambda^2,A_n)x=R(\lambda^2)x$, from the resolvent identity, we have

$$R(\lambda^2,A_n)-R(\mu^2,A_n)=(\mu^2-\lambda^2)R(\lambda^2,A_n)R(\mu^2,A_n)$$

then $R(\lambda^2)-R(\mu^2)=(\mu^2-\lambda^2)R(\lambda^2)R(\mu^2)$, $\lambda,\mu>\omega$ hence $\ker R(\lambda^2)$ and $RangR(\lambda^2)$ independent λ. Since $\ker R(\lambda_0^2)=\{0\}$, then there is a linear operator A, $D(A)=RangR(\lambda^2)$, $R(\lambda^2)x=(\lambda^2 I-A)^{-1}x$.

By Definition 2.2, we know that

$$\lambda^{1-\alpha}R(\lambda^2,A_n)x=\int_0^\infty \mathrm{e}^{-\lambda t}S_n(t)x\mathrm{d}t,\ \forall x\in X,\lambda>\omega, \tag{4}$$

for $\lim_{n\to\infty}R(\lambda^2,A_n)x=R(\lambda^2)x$ exist, by the proof of the Theorem 3.1, we obtain that

$$\lim_{n\to\infty}S_n(t)x=S(t)x\quad\text{exist,}$$

$$\text{hence} \quad \lambda^{1-\alpha} R\left(\lambda^2, A\right)x = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) x \mathrm{d}t, \quad \forall x \in X, \quad \lambda > \omega .$$

then A generates a α -times Integrated Cosine Function $\{S(t)\}_{t\geq 0}$, such that $\lim_{n\to\infty} S_n(t)x = S(t)x$, $\forall x \in X$, and uniformly on compacts of $t \geq 0$.

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