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# Necessary and Sufficient Conditions for a Class Positive Local Martingale

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## Abstract

**Let $X$ be a Markov process, which is assumed to be associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$. For $u \in D(\mathcal{E})_e$, the extended Dirichlet space, we give necessary and sufficient conditions for a multiplicative functional to be a positive local martingale.**

## Keywords

**Markov Process, Dirichlet Form, Multiplicative Functional, Positive Local Martingale**

## 1. Introduction

Let $X = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, (X_t)_{t\ge 0}, (P_x)_{x\in E_\Delta}\right)$ be a (non-symmetric) Markov process on a metrizable Lusin space $E$ and $m$ be a $\sigma$-finite positive measure on its Borel $\sigma$-algebra $\mathcal{B}(E)$. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;m)$ associated with Markov process $X$ (we refer the reader to [1] [2] for notations and terminologies of this paper). To simplify notation, we will denote by $u \in D(\mathcal{E})_e$ its $\mathcal{E}$-quasi-continuous $m$-version. If $u \in D(\mathcal{E})_e$, then there exist unique martingale additive functional (MAF in short) $M^{[u]}$ of finite energy and continuous additive functional (CAF in short) $N^{[u]}$ of zero energy such that

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}$$

Let $(N(x,\mathrm{d}y), H_t)$ be a Lévy system for $X$ and $\nu$ be the Revuz measure of the positive continuous additive functional (PCAF in short) $H$. For $t \ge 0$, we define the $[0,\infty]$-valued functional

$$A_t^u = \int_0^t \left(\int_{E_\partial} \left(\mathrm{e}^{u(y)-u(X_s)} - 1 - \left(u(y) - u(X_s)\right)\right) N(X_s, \mathrm{d}y)\right) \mathrm{d}H_s$$

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This paper is concerned with the following multiplicative functionals for $X$:

$$Z_t^u = \mathrm{e}^{M_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t - A_t^u}, \tag{1}$$

where $\langle M^{u,c}\rangle_t$ is the sharp bracket PCAF of the continuous part $M^{u,c}$ of $M^u$.

In [3] under the assumption that $X$ is a diffusion process, then $Z_t^u = \mathrm{e}^{M_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t}$ is a positive local martingale and hence a positive supermartingale. In [4], under the assumption that $u$ is bounded or $\mathrm{e}^u \in D(\mathcal{E})_e$, it is shown that $(Z_t^u, \mathcal{F}_t)_{t\geq 0}$ is a positive local martingale and hence induces another Markov process $Y$, which is called the Girsanov transformed process of $X$ (see [5]). Chen *et al*. in [5] give some necessary and sufficient conditions for $(Z_t^u, \mathcal{F}_t)_{t\geq 0}$ to be a positive supermartingale when the Markov processes are symmetric. It is worthy to point out that the Beurling-Deny formula and Lyons-Zheng decomposition do not apply well to non-symmetric Dirichlet forms setting. For the non-symmetric situations, $u \in D(\mathcal{E})_e$, an interesting and important question is that under what condition is $(Z_t^u, \mathcal{F}_t)_{t\geq 0}$ a positive local martingale?

In this paper, we will try to give a complete answer to this question when the Dirichlet forms are non-symmetric. We present necessary and sufficient conditions for $(Z_t^u, \mathcal{F}_t)_{t\geq 0}$ to be a positive local martingale.

## 2. Main Result

Recall that a positive measure $\mu$ on $(E, \mathcal{B}(E))$ is called smooth with respect to $(\mathcal{E}, D(\mathcal{E}))$ if $\mu(N)=0$ whenever $N \in B(\mathcal{E})$ is $\mathcal{E}$-exceptional and there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ of compact subsets of E such that

$$\mu(F_n) < \infty \text{ for all } k \geq 1$$

Let $J(\mathrm{d}x, \mathrm{d}y) = \frac{1}{2} N(x, \mathrm{d}y)\nu(\mathrm{d}x)$, $k(\mathrm{d}x) = N(x, \partial)\nu(\mathrm{d}x)$, We know from [6] that *J*, *k* are Randon measures. Let $u \in D(\mathcal{E})_e$, $Z_t^u$ be defined as in (1). Denote

$$\mu_u(\mathrm{d}x) = 2\int_{y\in E}\left(\mathrm{e}^{(u(y)-u(x))} - 1 - (u(y)-u(x))\right)J(\mathrm{d}x,\mathrm{d}y) + \left(\mathrm{e}^{-u(x)} - 1 + u(x)\right)k(\mathrm{d}x),$$

$$B_t^u = \sum_{s\leq t}\left[\mathrm{e}^{(u(X_s)-u(X_{s-}))} - 1 - (u(X_s) - u(X_{s-}))\right], \quad t \geq 0.$$

Now we can state the main result of this paper.

**Theorem 1** *The following are equivalent:*

(i) $(Z_t^u, \mathcal{F}_t)_{t\geq 0}$ is a positive $P_x$-local martingale on $[0,\zeta)$ for *q.e.* $x \in E$.

(ii) $(B_t^u)_{t\geq 0}$ is locally $P_x$-integrable on $[0,\zeta)$ for *q.e.* $x \in E$.

(iii) $\mu_u$ is a smooth measure on $(E, \mathcal{B}(E))$.

*Proof.* (iii) $\Rightarrow$ (ii) Suppose that $\mu_u$ is a smooth measure on $(E, \mathcal{B}(E))$ and $\{F_n\}_{n\geq 1}$ is an $\mathcal{E}$-nest such that $\mu_u(F_n) < \infty$ and $I_{F_n}\cdot\mu_u$ is of finite energy integral for $n \geq 1$. Similar to Lemma 2.4 of [4], $h^{t,n} := E_x\left[\left(I_{F_n}\cdot A^u\right)_t\right]$ is quasi-continuous and hence *q.e.* finite. Denote $\tau_n := \inf\left\{t > 0 \middle| X_t \notin F_n\right\}$. Then for $t \geq 0$,

$$\begin{aligned} E_x\left[B_{t\wedge\tau_n}^u\right] &\leq E_x\left\{\sum_{s\leq t} I_{F_n}(X_s)\left[\mathrm{e}^{u(X_s)-u(X_{s-})} - 1 - (u(X_s) - u(X_{s-}))\right]\right\} \\ &= E_x\left\{\int_0^t\int_{E_\partial} I_{F_n}(X_s)\left[\mathrm{e}^{u(y)-u(X_s)} - 1 - (u(y) - u(X_s))\right]N(X_s, \mathrm{d}y)\mathrm{d}H_s\right\} \\ &\leq E_x\left[\left(I_{F_n}\cdot A^u\right)_t\right] < \infty. \end{aligned}$$

Hence by proposition IV 5.30 of [1] $(B_t^u)_{t\geq 0}$ is locally $P_x$-integrable on $[0,\zeta)$ for *q.e.* $x \in E$.

(ii) $\Rightarrow$ (i) Assume that $(B_t^u)_{t\geq 0}$ is locally $P_x$-integrable on $[0,\zeta)$ for *q.e.* $x \in E$. One can check that for *q.e.* $x \in E$ the dual predictable projection of $(B_t^u)_{t\geq 0}$ on $[0,\zeta)$ is $A_t^u$. We set

$$B_t^d := B_t^u - A_t^u,$$
$$M_t := M_t^u + B_t^d.$$

Then $M_t$ is a local martingale on $[0,\zeta)$ and the solution $V_t^u$ of the stochastic differential equation (SDE)

$$V_t^u = 1 + \int_o^t V_{s-}^u \mathrm{d}M_s$$

is a local martingale on $[0,\zeta)$. Moreover, by Doleans-Dade formula (cf. 9.39 of [7]), Note that $\langle M^c \rangle_t = \langle M^{u,c} \rangle_t$, we have that

$$\begin{aligned} V_t^u &= \exp\left\{M_t - \frac{1}{2}\langle M^c \rangle_t\right\}\prod_{s\le t}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s} \\ &= \exp\left\{M_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t - A_t^u + B_t^u + \sum_{s\le t}\left(u(X_s) - u(X_{s-}) + 1 - \mathrm{e}^{u(X_s)-u(X_{s-})}\right)\right\} \\ &= \exp\left\{M_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t - A_t^u\right\} = Z_t^u. \end{aligned}$$

So $(Z_t^u, \mathcal{F}_t)_{t\ge 0}$ is a $P_x$-local martingale.

Let $W_t := \prod_{s\le t}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s}$. Note that $M_s$ is a *càdlàg* process, there are at most countably many points at which $M_s - M_{s-} \ne 0$. Since by Lemma 7.27 of [7] $\sum_{s\le t}(M_s - M_{s-})^2 < \infty$ $P_x$-*a.e.*, there are only finitely many points $s$ at which $|M_s - M_{s-}| > 1/2$, which give a finite non-zero contribution to the product. Using the inequality $|\ln(1+x) - x| \le x^2$ when $|x| \le 1/2$, we get

$$\begin{aligned} W_t &= \prod_{s\le t:|M_s - M_{s-}|\le 1/2}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s} \prod_{s\le t:|M_s - M_{s-}|>1/2}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s} \\ &\ge \prod_{s\le t:|M_s - M_{s-}|\le 1/2}\mathrm{e}^{-(M_s - M_{s-})2} \prod_{s\le t:|M_s - M_{s-}|>1/2}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s} \\ &\ge \mathrm{e}^{-\sum_{s\le t}(M_s - M_{s-})^2} \prod_{s\le t:|M_s - M_{s-}|>1/2}(1+M_s - M_{s-})\mathrm{e}^{M_{s-}-M_s} > 0. \end{aligned}$$

Therefore $(Z_t^u, \mathcal{F}_t)_{t\ge 0}$ is a positive $P_x$-local martingale on $[0,\zeta)$ for *q.e.* $x\in E$.

(i) $\Rightarrow$ (iii) Assume that $(Z_t^u, \mathcal{F}_t)_{t\ge 0}$ is a positive $P_x$-local martingale on $[0,\zeta)$ for *q.e.* $x \in E$, by Lemma 2.2 and Lemma 2.4 of [8],

$$\begin{aligned} L_t &:= \int_0^t \frac{1}{Z_{s-}^u}\mathrm{d}Z_t^u \\ &= \ln Z_t^u - \ln Z_0^u + \frac{1}{2}\int_0^t \frac{1}{\left(Z_{s-}^u\right)^2}\mathrm{d}\langle Z^{u,c}\rangle_s - \sum_{s\le t}\left(\ln\frac{Z_s^u}{Z_{s-}^u} + 1 - \frac{Z_s^u}{Z_{s-}^u}\right) \\ &= M_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t - A_t^u + \frac{1}{2}\int_0^t \frac{1}{\left(Z_{s-}^u\right)^2}\mathrm{d}\langle Z^{u,c}\rangle_s + B_t^u. \end{aligned}$$

is a local martingale on $[0,\zeta)$. We set

$$N_t := L_t - M_t^u$$

then $N_t = B_t^u - A_t^u - \frac{1}{2}\langle M^{u,c}\rangle_t + \frac{1}{2}\int_0^t \frac{1}{\left(Z_{s-}^u\right)^2}\mathrm{d}\langle Z^{u,c}\rangle_s$ is also a local martingale on $[0,\zeta)$. Denote $N_t^d$ is the purely discontinuous part of $N_t$, by Theorem 7.17 of [7], there exist a locally bounded martingale $U_t$ and a local martingale of integrable variation $V_t$ such that $N_t^d = N_0^d + U_t + V_t$. Since $u$ is $E$-quasi-continuous, take an $\mathcal{E}$-nest $\{F_n\}_{n\ge1}$ consisting of compact sets such that $\mathrm{Cap}(E \setminus F_n) \le \frac{1}{3^{n+1}}$ and $\tilde{u}|_{F_n}$ is continuous hence bounded $q.e.$ for each $n \ge 1$. Denote

$$D(\mathcal{E})_{F_k} := \left\{ f \in D(\mathcal{E}) \middle| f = 0 \ m-a.e. \ on \ E \setminus F_k \right\},$$
$$D(\mathcal{E})_{F_k,b} := D(\mathcal{E})_{F_k} \cap \mathcal{B}(E)_b.$$

Take a $\varphi \in L^2(E;m)$, $0 < \varphi < 1$. Set $h := G_1\varphi$, where $(G_\alpha)_{\alpha > o}$ is the family of resolvents associated with $(\mathcal{E}, D(\mathcal{E}))$. Since $\bigcup_{n\ge1} D(\mathcal{E})_{F_n,b}$ is dense in $D(\mathcal{E})$ $w.r.t.$ the $\mathcal{E}_1^{1/2}$-norm, by proposition III. 3.5 and 3.6 of [1], there exists an $\mathcal{E}$-nest $\{F_n'\}_{n\ge1}$ consisting of compact sets and a sequence $\{f_k\}_{k\ge1} \subset \bigcup_{n\ge1} D(\mathcal{E})_{F_n,b}$ such that $\mathrm{Cap}(E \setminus F_n') \le \frac{1}{3^{n+1}}$, $\widetilde{G_1\varphi}\big|_{F_n'} \ge \delta_n$ for some $\delta_n > 0$ and $\widetilde{f_k}$ converges to $\widetilde{G_1\varphi}$ uniformly on $F_n'$ as $k \to \infty$ for each $n \ge 1$. Set $F_n'' = F_n \bigcap F_n'$. So there exists an non-negative $h_n \in \bigcup_{k\ge1} D(\mathcal{E})_{F_k,b}$ and constant $a_n > 0$ such that $h_n \ge a_n$ on $F_n''$. Suppose $h_n \in D(\mathcal{E})_{F_{k_n},b}$, then

$$\begin{aligned}
\left[N^d, M^{h_n}\right]_t &= \sum_{s\le t} \left(N_s^d - N_{s-}^d\right)\left(M_s^{h_n} - M_{s-}^{h_n}\right) \\
&= \sum_{s\le t} \left(U_s^d - U_{s-}^d\right)\left(M_s^{h_n} - M_{s-}^{h_n}\right) + \sum_{s\le t} \left(V_s^d - V_{s-}^d\right)\left(h_n(X_s) - h_n(X_{s-})\right) \\
&\le \left[\sum_{s\le t} \left(U_s - U_{s-}\right)^2 \sum_{s\le t} \left(M_s^{h_n} - M_{s-}^{h_n}\right)^2\right]^{1/2} + 2\|h_n\|_\infty \sum_{s\le t} |V_s - V_{s-}|.
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Recall that a locally bounded martingale $U_t$ is a locally square integrable martingales, $M_t^{h_n}$ is a locally square integrable martingales and $V_t$ is a local martingale of integrable variation. Therefore the quadratic variation $\left[N^d, M^{h_n}\right]_t$ is $P_x$-locally integrable for $q.e. x \in E$, hence there exist a predictable dual projection $\left\langle N^d, M^{h_n}\right\rangle_t$ which is a CAF of finite variation. Since

$$\begin{aligned}
\left[N^d, M^{h_n}\right]_t &= \sum_{s\le t} \left(B_s^u - B_{s-}^u\right)\left(M_s^{h_n} - M_{s-}^{h_n}\right) \\
&= \sum_{s\le t} \left(\mathrm{e}^{u(X_s) - u(X_{s-})} - 1 - \left(u(X_s) - u(X_{s-})\right)\right)\left(h_n(X_s) - h_n(X_{s-})\right).
\end{aligned}$$

the Revuz measure of $\left\langle N^d, M^{h_n}\right\rangle_t$ is

$$\begin{aligned}
\mu_{\langle N^d, M^{h_n}\rangle} = {}& 2\int_{\{y\in E: y\ne x\}} \left(h_n(y) - h_n(x)\right)\left(\mathrm{e}^{(u(y)-u(x))} - 1 - \left(u(y) - u(x)\right)\right) J(\mathrm{d}x, \mathrm{d}y) \\
&+ \left(\mathrm{e}^{-u(x)} - 1 + u(x)\right) k(\mathrm{d}x).
\end{aligned}$$

Let $\{F_k'''\}_{k\ge1}$ be a generalized $\mathcal{E}$-nest associated with $\mu_{\langle N^d, M^{h_n}\rangle}$ such that $\mu_{\langle N^d, M^{h_n}\rangle}(F_k''') < \infty$ for each $k \ge 1$. Denote $D_n := F_n'' \bigcap F_n'''$, then $\mathrm{Cap}(E \setminus D_n) \le \frac{1}{3^n}$ and $\left\{\bigcup_{k=1}^n D_k\right\}_{n\ge1}$ is an $\mathcal{E}$-nest. Hence for any $g \in D(\mathcal{E})_{D_n,b}$, we have $\int_E g(x) \mathrm{d}\mu_{\langle N^d, M^{h_n}\rangle} < \infty$. On the other hand, as $\tilde{u}|_{F_n}$ is bounded, there exists a positive constant $b_n$ such that $\mathrm{e}^{u(y)-u(x)} - 1 - \left(u(y) - u(x)\right)\Big|_{D_n \times F_{k_n}}$ and $\mathrm{e}^{-u(x)} - 1 + u(x)\Big|_{D_n \bigcap F_{k_n}}$ are not larger than $b_n$. Because $J(\mathrm{d}x, \mathrm{d}y)$, $k(\mathrm{d}x)$ are Radon measure and $h_n$, $g$ are bounded,

$$\begin{aligned}
&\int_{E\times E\setminus d} g(x)h_n(x)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)\\
&=\int_{E\times E\setminus d} g(x)h_n(\mathrm{y})\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)\\
&\quad-\int_{E\times E\setminus d} g(x)\left(h_n(y)-h_n(x)\right)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)\\
&=\int_{E\times E\setminus d} g(x)h_n(y)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)\\
&\quad+\frac{1}{2}\int_E h_n(x)g(x)\left(\mathrm{e}^{-u(x)}-1+u(x)\right)k(\mathrm{d}x)-\frac{1}{2}\int_E g(x)\,\mathrm{d}\mu_{\langle N^d,M^{h_n}\rangle}\\
&=\int_{D_n\times F_{k_n}\setminus d} g(x)h_n(y)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)\\
&\quad+\frac{1}{2}\int_{D_n\bigcap F_{k_n}} h_n(x)g(x)\left(\mathrm{e}^{-u(x)}-1+u(x)\right)k(\mathrm{d}x)-\frac{1}{2}\int_E g(x)\,\mathrm{d}\mu_{\langle N^d,M^{h_n}\rangle}<\infty
\end{aligned}$$

As inequality $\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\geq 0$ on $E$ and $h_n\geq a_n$ on $F_n''$, we have for any non-negative $f\in D(\mathcal{E})_{D_n,b}$,

$$\begin{aligned}
&\int_E f(x)\mu_u(\mathrm{d}x)\\
&=2\int_{E\times E\setminus d} f(x)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)+\int_E f(x)\left(\mathrm{e}^{-u(x)}-1+u(x)\right)k(\mathrm{d}x)\\
&\leq 2(a_n)^{-1}\int_{D_n\times E\setminus d} h_n(x)f(x)\left(\mathrm{e}^{u(y)-u(x)}-1-\left(u(y)-u(x)\right)\right)J(\mathrm{d}x,\mathrm{d}y)+\int_{D_n} f(x)\left(\mathrm{e}^{-u(x)}-1+u(x)\right)k(\mathrm{d}x)<\infty.
\end{aligned}$$

For $\left\{\bigcup_{k=1}^n D_k\right\}_{n\geq 1}$ is an $\mathcal{E}$-nest consisting of compact sets, similar to $h_n$, we can construct an $\mathcal{E}$-nest $\left\{\bigcup_{k=1}^n D_k'\right\}$ consisting of compact sets such that $D_n'\subset D_n$ for each $n\geq 1$. And there exists a sequence non-negative $\{h_n'\}_{n\geq 1}\subset\bigcup_{k\geq 1} D(\mathcal{E})_{D_k,b}$ such that $h_n'\geq c_n$ on $D_n'$ for each $n\geq 1$ and some positive $c_n>0$. Since $\mu_u\left(\bigcup_{k=1}^n D_k'\right)\leq\sum_{k=1}^{k=n} c_k^{-1}\int_E\left(h_{k'}(x)\right)\mu_u(\mathrm{d}x)<\infty$, $\mu_u$ is a smooth measure on $\left(E,\mathcal{B}(E)\right)$.

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