

Dual Quermassintegral Differences for Intersection Body

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Abstract

In this paper, we introduce the concept of dual quermassintegral differences. Further, we give the dual Brunn-Minkowski inequality and dual Minkowski inequality for dual quermassintegral differences for mixed intersection bodies.

Keywords

Intersection Body, Dual Brunn-Minkowski Inequality, Dual Minkowski Inequality

1. Introduction

The projection body was introduced in 1934 by Minkowski [1]. The research on the projection body has attracted much attention. The intersection operator and the class of intersection bodies were introduced in 1988 by Lutwak [2], who found a close connection between those bodies and famous Busemann-Petty problem (See [3]-[6]).

In [2], Lutwak presented the mysterious duality between projection and intersection bodies.

For convex bodies K and L , let ΠK and $\Pi_1(K, L)$ denote the projection body of K and mixed projection body of K and L , respectively. In [7], Lutwak established the following Brunn-Minkowski inequality for projection body and Minkowski inequality for mixed projection body:

Theorem A. *Let K and L be convex bodies in \mathbb{R}^n . Then*

$$V(\Pi(K+L))^{\frac{1}{n(n-1)}} \geq V(\Pi K)^{\frac{1}{n(n-1)}} + V(\Pi L)^{\frac{1}{n(n-1)}}, \quad (1.1)$$

with equality if and only if K and L are homothetic.

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Theorem B. Let K and L be convex bodies in \mathbb{R}^n . Then

$$V(\Pi_1(K, L))^{n-1} \geq V(\Pi K)^{n-2} V(\Pi L), \tag{1.2}$$

with equality if and only if K and L are homothetic.

In [8], Theorem A and Theorem B were extended to volume differences:

Theorem C. Suppose that K, L , and D_1 are convex bodies in \mathbb{R}^n , and $D_1 \subset K, D_2 \subset L, D_2$ is a homothetic copy of D_1 . Then

$$\begin{aligned} & \left[V(\Pi(K+L)) - V(\Pi(D_1+D_2)) \right]^{\frac{1}{n(n-1)}} \\ & \geq \left[V(\Pi K) - V(\Pi D_1) \right]^{\frac{1}{n(n-1)}} + \left[V(\Pi L) - V(\Pi D_2) \right]^{\frac{1}{n(n-1)}}, \end{aligned} \tag{1.3}$$

with equality if and only if K and L are homothetic and $(V(K), V(D_1)) = \mu(V(L), V(D_2))$, where μ is a constant.

Theorem D. Suppose that K, L , and D_1 are convex bodies in \mathbb{R}^n , and $D_1 \subset K, D_2 \subset L, D_2$ is a homothetic copy of D_1 . Then

$$\left[V(\Pi_1(K, L)) - V(\Pi_1(D_1, D_2)) \right]^{n-1} \geq \left[V(\Pi K) - V(\Pi D_1) \right]^{n-2} \left[V(\Pi L) - V(\Pi D_2) \right], \tag{1.4}$$

with equality if and only if K and L are homothetic.

For star bodies K and L , let IK and $I_1(K, L)$ denote the intersection body of K and mixed intersection body of K and L , respectively. In [9], Zhao *et al.* established the following dual Brunn-Minkowski inequality for intersection body and dual Minkowski inequality for mixed intersection body:

Theorem E. Let K and L be star bodies in \mathbb{R}^n . Then

$$V(I(K \tilde{+} L))^{\frac{1}{n(n-1)}} \leq V(IK)^{\frac{1}{n(n-1)}} + V(IL)^{\frac{1}{n(n-1)}}, \tag{1.5}$$

with equality if and only if L is a dilatate of K .

Theorem F. Let K and L be star bodies in \mathbb{R}^n . Then

$$V(I_1(K, L))^{n-1} \leq V(IK)^{n-2} V(IL), \tag{1.6}$$

with equality if and only if L is a dilatate of K .

In this paper, we shall prove the dual forms of inequalities (1.3) and (1.4) for mixed intersection body. In this work new contributions that illustrate this mysterious duality will be presented. Our main results can be stated as follows:

Theorem 1.1. Let K, L and D_1 are star bodies in \mathbb{R}^n and $K \subseteq D_1, L \subseteq D_2, D_2$ is a dilatation of D_1 . Then

$$\begin{aligned} & \left[V(I(D_1 \tilde{+} D_2)) - V(I(K \tilde{+} L)) \right]^{\frac{1}{n(n-1)}} \\ & \geq \left[V(ID_1) - V(IK) \right]^{\frac{1}{n(n-1)}} + \left[V(ID_2) - V(IL) \right]^{\frac{1}{n(n-1)}}, \end{aligned} \tag{1.7}$$

with equality if and only if L is a dilatate of K and $(V(IK), V(ID_1)) = \lambda(V(IL), V(ID_2))$, where λ is a constant.

Theorem 1.2. Let K, L and D_1 are star bodies in \mathbb{R}^n and $K \subseteq D_1, L \subseteq D_2, D_2$ is a dilatation of D_1 . Then

$$\left[V(I_1(D_1, D_2)) - V(I_1(K, L)) \right]^{n-1} \geq \left[V(ID_1) - V(IK) \right]^{n-2} \left[V(ID_2) - V(IL) \right], \tag{1.8}$$

with equality if and only if L is a dilatate of K .

Please see the next section for related definitions and notations.

2. Definitions and Notations

In this section, we will recall some basic results for dual quermassintegrals of star bodies. The reader is referred to Gardner [10], Lutwak [2] [11] and Thompson [12] for the Brunn-Minkowski theory with its dual theory.

As usual, let B_n denote the unit ball in Euclidean n -space, \mathbb{R}^n . While its boundary is S^{n-1} and its volume is denoted by k_n . For a compact subset K of \mathbb{R}^n , with $o \in K$, star-shaped with respect to o , the radial function $\rho(K, \cdot): S^{n-1} \rightarrow \mathbb{R}$, is defined by

$$\rho(K, u) = \rho_K(u) = \max \{ \lambda : \lambda u \in K \}. \quad (2.1)$$

If $\rho(K, \cdot)$ is continuous and positive, K will be called a star body.

Two star bodies K, L are said to be dilatate (of each other) if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$.

The radial sum of two star bodies K_1, K_2 is defined as the star body K satisfying $\rho_K = \rho_{K_1} + \rho_{K_2}$. This operation will be denoted by $\tilde{+}$, i.e., $K = K_1 \tilde{+} K_2$.

For star bodies K_1, \dots, K_n , the dual mixed volume $\tilde{V}(K_1, \dots, K_n)$ is defined by (see e.g. [11])

$$\tilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) du. \quad (2.2)$$

If $K_1 = \dots = K_{n-i} = K$, and $K_{n-i+1} = \dots = K_n = B_n$, then the dual mixed volume $\tilde{V}(K, n-i; B_n, i)$ is called dual i -quermassintegral of K , and denoted by $\tilde{W}_i(K)$ and allow $i \in \mathbb{R}$. It is easily seen that $\tilde{W}_0(K) = V(K)$.

Let M and K be star bodies in \mathbb{R}^n . If $K \subseteq M$, then the dual i -quermassintegral difference function of M and K , $D\tilde{w}_i(M, K)$ ($i \in \mathbb{R}; i \neq n$), can be defined by

$$D\tilde{w}_i(M, K) = \tilde{W}_i(M) - \tilde{W}_i(K). \quad (2.3)$$

If $i = 0$ in (2.3), then we get the volume difference of star bodies M and K :

$$Dv(M, K) = V(M) - V(K).$$

(See [13] for the concept of the volume difference of two compact domains).

The intersection body of a star body K , IK , is the centrally symmetric body whose radial function on S^{n-1} is given by [2]

$$\rho(IK, u) = v(K \cap u^\perp), \quad (2.4)$$

where v is $(n-1)$ -dimensional volume.

Let K_1, \dots, K_{n-1} be star bodies in \mathbb{R}^n . The mixed intersection body $I(K_1, \dots, K_{n-1})$ of star bodies K_1, \dots, K_{n-1} is defined by

$$\rho_{I(K_1, \dots, K_{n-1})}(u) = \tilde{v}(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp) = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho_{K_1}(v) \cdots \rho_{K_{n-1}}(v) d\lambda_{n-2}(v), \quad (2.5)$$

where \tilde{v} is $(n-1)$ -dimensional dual mixed volume.

If $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = L$, then $I(K_1, \dots, K_{n-1})$ will be denoted as $I_i(K, L)$. If $K = B_n$, then $I_i(B_n, L)$ is called the intersection body of order i of L ; it will often be written as $I_i L$. Specially, $I_{n-1} L = IL$. This term was introduced by Zhang [14].

3. Inequalities for Dual Quermassintegral Differences

In this section, we will establish two inequalities for dual quermassintegral differences of star bodies, which are generalizations of Theorem 1.1 and Theorem 1.2 presented in introduction.

Theorem 3.1. Suppose that K, L and D_1 are star bodies in \mathbb{R}^n , D_2 is a dilatate of D_1 . If $0 \leq i \leq n-1$, $K \subseteq D_1, L \subseteq D_2$, then

$$D\tilde{w}_i(I(D_1 \tilde{+} D_2), I(K \tilde{+} L))^{\frac{1}{(n-i)(n-1)}} \geq D\tilde{w}_i(ID_1, IK)^{\frac{1}{(n-i)(n-1)}} + D\tilde{w}_i(ID_2, IL)^{\frac{1}{(n-i)(n-1)}},$$

with equality if and only if L is a dilatate of K and $(\tilde{W}_i(IK), \tilde{W}_i(ID_1)) = \mu(\tilde{W}_i(IL), \tilde{W}_i(ID_2))$, where μ is a constant.

Obviously, the case $i = 0$ in Theorem 3.1 is just Theorem 1.1. Furthermore, taking D_1 and D_2 to be two closed balls with radii r_1 and r_2 in Theorem 1.1, we infer

Corollary 3.2. *Let $R(K)$ and $R(L)$ be the circumradii of star bodies K and L . If $r_1 \geq R(K)$, $r_2 \geq R(L)$, then*

$$\left[(r_1 + r_2)^{n(n-1)} k_{n-1}^n k_n - I(K+L) \right]^{\frac{1}{n(n-1)}} \geq \left[r_1^{n(n-1)} k_{n-1}^n k_n - V(IK) \right]^{\frac{1}{n(n-1)}} + \left[r_2^{n(n-1)} k_{n-1}^n k_n - V(IL) \right]^{\frac{1}{n(n-1)}},$$

with equality if and only if L is a dilatate of K and $V(IK)/V(IL) = (r_1/r_2)^{n(n-1)}$.

Theorem 3.3. *Suppose that K, L and D_1 are star bodies in \mathbb{R}^n , D_2 is a dilatate of D_1 . If $K \subseteq D_1, L \subseteq D_2$, and $0 \leq i, j < n-1$. Then*

$$D\tilde{w}_i(I_j(D_1, D_2), I_j(K, L))^{n-1} \geq D\tilde{w}_i(ID_1, IK)^{n-j-1} D\tilde{w}_i(ID_2, IL)^j,$$

with equality if and only if L is a dilatate of K and $(\tilde{W}_i(IK), \tilde{W}_i(ID_1)) = \mu(\tilde{W}_i(IL), \tilde{W}_i(ID_2))$, where μ is a constant.

Obviously, the case $i = 0, j = 1$ in Theorem 3.3 is just Theorem 1.2.

We will require some additional notations and two analytic inequalities to prove Theorem 3.1 and Theorem 3.3. Firstly, we define a function $\Phi_p(x)$ by

$$\Phi_p(x) = \left(x_1^p - \sum_{i=2}^n x_i^p \right)^{\frac{1}{p}}, x \in \mathbb{R}_p,$$

where $\mathbb{R}_p = \left\{ x = (x_1, \dots, x_n) \mid x_i \geq 0, x_1^p \geq \sum_{i=2}^n x_i^p \right\}$ for $p > 0$. Note that Φ_p^p is an indefinite form of its variables.

Lemma 3.4. *If $p > 1$, $x, y \in \mathbb{R}_p$, then $x + y \in \mathbb{R}_p$, and*

$$\Phi_p(x + y) \geq \Phi_p(x) + \Phi_p(y), \tag{3.1}$$

with equality holds if and only if the coordinates of x, y are proportional.

A proof of Lemma 3.4 can be found in [15]. The inequality (3.1) was first proved by Bellman [16] and is known as Bellman's inequality.

Lemma 3.5. *If $a, b, c, d > 0$, $0 < \alpha < 1, 0 < \beta < 1$ and $\alpha + \beta = 1$. Let $a > b$ and $c > d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \geq (a-b)^\alpha (c-d)^\beta,$$

with equality if and only if $a/b = c/d$.

Proof. Consider the following function

$$f(x) = x^\alpha c^\beta - (x-b)^\alpha (c-d)^\beta, x > 0.$$

Let

$$f'(x) = \alpha c^\beta x^{\alpha-1} - \alpha (c-d)^\beta (x-b)^{\alpha-1} = 0.$$

We get $x = bc/d$.

On the other hand, if $x \in \left(0, \frac{bc}{d} \right)$, then $f'(x) < 0$; if $x \in \left(\frac{bc}{d}, +\infty \right)$, then $f'(x) > 0$, and it follows that

$$\min_{x>0} f(x) = f\left(\frac{bc}{d}\right) = b^\alpha d^\beta.$$

This completes the proof.

Lemma 3.6. [15] *Let K, L be star bodies in \mathbb{R}^n . If $0 \leq i < n, 1 \leq j < n-1$, then*

$$\tilde{W}_i(I(K \tilde{+} L))^{\frac{1}{(n-i)(n-1)}} \leq \tilde{W}_i(IK)^{\frac{1}{(n-i)(n-1)}} + \tilde{W}_i(IL)^{\frac{1}{(n-i)(n-1)}}, \quad (3.2)$$

and

$$\tilde{W}_i(I_j(K, L))^{n-1} \leq \tilde{W}_i(IK)^{n-j-1} \tilde{W}_i(IL)^j, \quad (3.3)$$

with equality if and only if L is a dilatate of K .

Proof of Theorem 3.1. For star bodies K, L , applying inequality (3.2), we have

$$\tilde{W}_i(I(K \tilde{+} L))^{\frac{1}{(n-i)(n-1)}} \leq \tilde{W}_i(IK)^{\frac{1}{(n-i)(n-1)}} + \tilde{W}_i(IL)^{\frac{1}{(n-i)(n-1)}}, \quad (3.4)$$

with equality if and only if L is a dilatate of K .

$$\tilde{W}_i(I(D_1 \tilde{+} D_2))^{\frac{1}{(n-i)(n-1)}} = \tilde{W}_i(ID_1)^{\frac{1}{(n-i)(n-1)}} + \tilde{W}_i(ID_2)^{\frac{1}{(n-i)(n-1)}}. \quad (3.5)$$

Since $K \subseteq D_1, L \subseteq D_2$, we get

$$\tilde{W}_i(IK) \leq \tilde{W}_i(ID_1), \quad \tilde{W}_i(IL) \leq \tilde{W}_i(ID_2)$$

From (3.4) and (3.5), we obtain that

$$\begin{aligned} & D\tilde{W}_i(I(D_1 \tilde{+} D_2), I(K \tilde{+} L)) \\ & \geq \left(\tilde{W}_i(ID_1)^{\frac{1}{(n-i)(n-1)}} + \tilde{W}_i(ID_2)^{\frac{1}{(n-i)(n-1)}} \right)^{(n-i)(n-1)} \\ & \quad - \left(\tilde{W}_i(IK)^{\frac{1}{(n-i)(n-1)}} + \tilde{W}_i(IL)^{\frac{1}{(n-i)(n-1)}} \right)^{(n-i)(n-1)}. \end{aligned}$$

Then by Lemma 3.4, we get

$$\begin{aligned} & D\tilde{W}_i(I(D_1 \tilde{+} D_2), I(K \tilde{+} L))^{\frac{1}{(n-i)(n-1)}} \\ & \geq \left(\tilde{W}_i(ID_1) - \tilde{W}_i(IK) \right)^{\frac{1}{(n-i)(n-1)}} + \left(\tilde{W}_i(ID_2) - \tilde{W}_i(IL) \right)^{\frac{1}{(n-i)(n-1)}}. \end{aligned} \quad (3.6)$$

Note that the equality holds in (3.4) if and only if L is a dilatate of K . By Lemma 3.4 we know that the equality holds in (3.6) if and only if L is a dilatate of K . and $(\tilde{W}_i(IK), \tilde{W}_i(ID_1))$ is proportional to $(\tilde{W}_i(IL), \tilde{W}_i(ID_2))$.

This completes the proof.

Proof of Theorem 3.3. Applying inequality (3.3), we have

$$\tilde{W}_i(I_j(K, L))^{n-1} \leq \tilde{W}_i(IK)^{n-j-1} \tilde{W}_i(IL)^j,$$

with equality if and only if L is a dilatate of K .

$$\tilde{W}_i(I_j(D_1, D_2))^{n-1} = \tilde{W}_i(ID_1)^{n-j-1} \tilde{W}_i(ID_2)^j.$$

Hence, by Lemma 3.5, we obtain that

$$\begin{aligned} & D\tilde{W}_i(I_j(D_1, D_2), I_j(K, L)) \\ & \geq \tilde{W}_i(ID_1)^{\frac{n-j-1}{n-1}} \tilde{W}_i(ID_2)^{\frac{j}{n-1}} - \tilde{W}_i(IK)^{\frac{n-j-1}{n-1}} \tilde{W}_i(IL)^{\frac{j}{n-1}} \\ & \geq \left(\tilde{W}_i(ID_1) - \tilde{W}_i(IK) \right)^{\frac{n-j-1}{n-1}} \left(\tilde{W}_i(ID_2) - \tilde{W}_i(IL) \right)^{\frac{j}{n-1}}. \end{aligned}$$

The proof is complete.

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