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Weierstrass' Elliptic Function Solution to the Autonomous Limit of the String Equation of Type (2,5)*

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Abstract

In this article, we study the string equation of type (2,5), which is derived from 2D gravity theory or the string theory. We consider the equation as a 4th order analogue of the first Painlevé equation, take the autonomous limit, and solve it concretely by use of the Weierstrass' elliptic function.

Keywords

Painlevé Hierarchy, String Equation, Elliptic Function

1. Introduction

1.1. The String Equation of Type (2,5)

Put $D = \mathrm{d}/\mathrm{d}z$. Consider the commutator equation of ordinary differential operators

$$[Q,P]=1,\quad Q:=\Sigma_{k=2}^{q} w_k D^{q-k},\quad P:=\Sigma_{k=2}^{p} v_k D^{p-k}.$$

We call it the string equation (or Douglas equation) of type (q,p), which appears in the string theory or the theory of quantum gravity in 2D [1]-[9]. In the followings, we set $q=2$, $p=2g+1$.

In the case where $q=2$, $p=3$, the string equation is written as an ODE satisfied by the potential w of Sturm-Liouville operator $Q=D^2+w$, and then, by a fractional linear transformation, it is reduced to the first Painlevé equation [10] [11]

$$w''=6w^2+z, \tag{PI}$$

*Dedicated to Professor Masafumi Yoshino on the occasion of his 60th birthday.

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which is equivalent to the Hamiltonian system:

$$\mathrm{d}w/\mathrm{d}z = \partial H/\partial v,\quad \mathrm{d}v/\mathrm{d}z = -\partial H/\partial w,\quad H = \frac{1}{2}v^2 - 2w^3 - zw.$$

In the case where $q = 2$, $p = 5$, $[Q, P] = 1$ yields

$$C_0 = w^{(4)} + 5w'^2 + 10(w' + C_1)(w'' + 3w^2) - 20w^3 + 16C_2 w + 16z,$$

where C_0, C_1, C_2 are integral constants. By the fractional linear transformation $z \mapsto \alpha z + \beta$, $w \mapsto \gamma w + \delta$,

$$\alpha^7 = -\frac{1}{3},\quad \gamma = 6\alpha^5,\quad \delta = -C_1,\quad 16\beta = C_0 - 20C_1^3 + 16C_1C_2$$

and putting $a = \alpha^4\left(8C_2 - 15C_1^2\right)/4$, the string equation is reduced to

$$w^{(4)} = 20w''w + 10w'^2 - 40w^3 - 8aw - \frac{8}{3}z, \tag{S}$$

We also call it the string equation of type (2,5). Note that (S) coincides the 4th order equation of *the first Painlevé hierarchy* [12]-[15]

$$d_n[w] + 4z = 0, \tag{$_{2n}$PI}$$

for $n \in \mathbb{N}$, where $d_n[w]$ is an expression of a given meromorphic function w defined by $d_0[w] = -4w$ and $Dd_{n+1}[w] = \left(D^3 - 8wD - 4w'\right)d_n[w]$.

1.2. Degenerated Garnier System

Equation (S) is also obtained as follows. Consider a 2D degenerated Garnier system [16] [17]:

$$\partial q_i/\partial t_j = \partial H_j/\partial p_i,\quad \partial p_i/\partial t_j = -\partial H_j/\partial q_i,\quad (i, j \in \{1,2\}), \tag{dG$_{9/2}$}$$

$$H_1 = \frac{1}{3}\left(q_2^2 - q_1 - \frac{1}{3}t_1\right)p_1^2 + \frac{2}{3}q_2p_1p_2 + \frac{1}{3}p_2^2 + 3\left(q_1 + \frac{1}{3}t_1\right)q_2\left(q_2^2 - 2q_1 + \frac{1}{3}t_1\right) - t_2q_1,$$

$$H_2 = \frac{1}{3}q_2p_1^2 + \frac{2}{3}p_1p_2 + 3q_2^4 - 9q_1q_2^2 + 3q_1^2 - t_1q_1 - t_2q_2.$$

which is a 2D analogue of (PI) in the theory of isomonodromic deformations. If we fix one of the independent variables $t_1 \equiv a\,(= \text{const.})$, we get a Hamiltonian system with only one independent variable $t_2 \equiv z$ as follows:

$$\partial q_i/\partial z = \partial H/\partial p_i,\quad \partial p_i/\partial z = -\partial H/\partial q_i,\quad (i \in \{1,2\}),$$

$$H\,(= H_2) = \frac{1}{3}q_2p_1^2 + \frac{2}{3}p_1p_2 + 3q_2^4 - 9q_1q_2^2 + 3q_1^2 - aq_1 - zq_2.$$

From the above system, eliminating q_1, p_1, p_2 and putting $w = q_2$, we obtain (S). So, Equation (S) is 4th order analogue of (PI) in the double sences.

It is already known by Shimomura [18] that every solution to (S) is meromorphic on $\mathbb{C}$, and that every pole of every solution is double one with its residue 0.

1.3. Autonomous Limit of the First Painlevé Equation

The first Painlevé equation (PI) has the autonomous limit [11]. Replacing (w, v, z, H) by $\left(\varepsilon^{-2}w, \varepsilon^{-3}v, \varepsilon z + \varepsilon^{-4}b, \varepsilon^{-6}H\right)$ with a constant $b \in \mathbb{C}$, and taking limit $\varepsilon \to 0$, we obtain $w'' = 6w^2 + b$ which is solved by the Weierstrass' elliptic function [10] [11]. The relation between the fundamental 2-form before and after the replacement is

$$\mathrm{d}w \wedge \mathrm{d}v - \mathrm{d}H \wedge \mathrm{d}z \mapsto \varepsilon^{-5}\left(\mathrm{d}w \wedge \mathrm{d}v - \mathrm{d}H \wedge \mathrm{d}z\right).$$

1.4. Results

It is quite natural to think that:

Conjecture. Each equation of the first Painlevé hierarchy has the autonomous limit, and which is satisfied by the Weierstrass' elliptic function.

For $n = 2$, the statement is valid, *i.e.*

Theorem A. Replacing (w, z, a) by $(\varepsilon^{-2}w, \varepsilon z + \varepsilon^{-6}b, \varepsilon^{-4}a)$, or replacing $(q_1, q_2, p_1, p_2, z, H)$ by $(\varepsilon^{-4}q_1, \varepsilon^{-2}q_2, \varepsilon^{-3}p_1, \varepsilon^{-5}p_2, \varepsilon z + \varepsilon^{-6}b, \varepsilon^{-8}H)$ with a constant $b \in \mathbb{C}$, and taking limit $\varepsilon \to 0$, we obtain the autonomous limit of the 4th order equation of the first Painlevé hierarchy (S). Moreover, the relation between the fundamental 2-form before and after the replacement is

$$\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 - \mathrm{d}H \wedge \mathrm{d}z \mapsto \varepsilon^{-7}\left(\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 - \mathrm{d}H \wedge \mathrm{d}z\right).$$

It is easy to show the above. The autonomous limit is given by

$$w^{(4)} = 20w''w + 10w'^2 - 40w^3 - 8aw - \frac{8}{3}b, \tag{A}$$

Theorem B. The autonomous limit Equation (A) has a solution concretely described by the Weierstrass' elliptic function as

$$w(z) = 4\wp(z)/a_1,$$

where $a_1 = \left(8 \pm 4\sqrt{-2}\right)/3$.

Remark. Modulus of the elliptic function is determined by the constants a and b. g_2 and g_3 in the elliptic function theory are as follows:

$$g_2 = -4a_1a/(40 - 9a_1), \quad g_3 = -a_1^2b/6(10 - 3a_1).$$

The next section is devoted to give the proof of Theorem B.

2. Proof of Theorem B

Put $\varphi = -[\text{l.h.s. of } (\mathrm{A})] + [\text{r.h.s. of } (\mathrm{A})]$, *i.e.*

$$\varphi := w^{(4)} + 20w''w + 10w'^2 - 40w^3 - 8aw - \frac{8}{3}b = 0. \tag{1}$$

Multiplying both sides of $\varphi = 0$ by w', and integrating it, we obtain a first integral of (A)

$$\int \varphi w' \mathrm{d}z := -w'w''' + \frac{1}{2}w''^2 + 10w'^2w - 10w^4 - 4aw^2 - \frac{8}{3}bw = c : \text{const.} \tag{2}$$

In order to find the elliptic function solution, let w satisfy the relation:

$$w'^2 = a_0w^4 + a_1w^3 + a_2w^2 + a_3w + a_4 =: A(w). \tag{3}$$

Substituting (3), $w'' = \frac{1}{2}A_w(w)$ and $w''' = \frac{1}{2}A_{ww}(w)w'$ into (2), we have

$$\begin{aligned}\int \varphi w' \mathrm{d}z &= -\frac{1}{2}A_{ww}(w)A(w) + \frac{1}{8}A_w(w)^2 + 10wA(w) - 10w^4 - 4aw^2 - \frac{8}{3}bw = c \\ &= \left[-4a_0^2\right]w^6 + \left[-6a_0a_1 + 10a_0\right]w^5 + \left[-5a_0a_2 - \frac{15}{8}a_1{}^2 + 10a_1 - 10\right]w^4 \\ &\quad + \left[-5a_0a_3 - \frac{5}{2}a_1a_2 + 10a_2\right]w^3 + \left[-6a_0a_4 - \frac{9}{4}a_1a_3 - \frac{1}{2}a_2^2 + 10a_3 - 4a\right]w^2 \\ &\quad + \left[-3a_1a_4 - \frac{1}{2}a_2a_3 + 10a_4 - \frac{8}{3}b\right]w + \left[-a_2a_4 + \frac{1}{8}a_3^2\right]\cdot 1.\end{aligned}$$

So, if we take

$$a_0 = a_2 = 0,\ a_1 = \frac{1}{3}\left(8 \pm 4\sqrt{-2}\right),\ a_3 = 16a/(40 - 9a_1),\ a_4 = 8b/3(10 - 3a_1), \text{ and } c = \frac{1}{8}a_3^2,$$

then solutions of (3) satisfy (2). Now, in order to reduce $w'^2 = a_1w^3 + a_3w + a_4$ to $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, we use the scale transformation $w = \chi\wp$, $\chi \in \mathbb{C} \setminus \{0\}$. Immediately we obtain $\chi = 4/a_1$, and also $g_2 = -a_3/\chi$, $g_3 = -a_4/\chi$. □

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