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# Existence Theory for Single Positive Solution to Fourth-Order Boundary Value Problems

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## Abstract

**By fixed point theorem of a mixed monotone operator, we study boundary value problems to nonlinear singular fourth-order differential equations, and provide sufficient conditions for the existence and uniqueness of positive solution. The nonlinear term in the differential equation may be singular.**

## Keywords

**Mixed Monotone Operator, Singular, Existence, Uniqueness**

## 1. Introduction

Fourth-order differential equations play an important role in various fields of science and engineering. With the help of boundary value conditions, we can describe the natural phenomena and mathematical model more accurately. Therefore, the fourth-order differential equations have received much attention and the theory and application have been greatly developed (see [1]-[4] and their references). Most of the results told us that the equations had at least single and multiple positive solutions. In papers [1]-[3], the authors obtained some newest results for the singular fourth-order boundary value problems. But there is no result on the uniqueness of solution in them.

In this paper, we consider the following singular fourth-order boundary value problem:

$$\begin{cases} \left[p(t)x'''(t)\right]' + q(t)x''(t) = \lambda f\left(t, x(t)\right), \quad 0 < t < 1,\ \lambda > 0, \\ \alpha_1 x(0) - \beta_1 x'(0) = 0, \\ \gamma_1 x(1) + \delta_1 x'(1) = 0, \\ \alpha_2 x''(0) - \beta_2 x'''(0) = 0, \\ \gamma_2 x''(1) + \delta_2 x'''(1) = 0. \end{cases} \tag{1.1}$$

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Throughout this paper, we always suppose that

$(S_1)$ $p(t)\in C^1([0,1],R)$, $p(t)>0$, $q(t)\in C([0,1],R)$, $q(t)\le 0$, $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i\ge 0$, $(i=1,2)$, and $\beta_i\gamma_i+\alpha_i\gamma_i+\alpha_i\delta_i>0$ $(i=1,2)$. $f\in C((0,1)\times(0,+\infty),(0,+\infty))$.

Moreover, $f(t,u)$ may be singular at $t=0$, $t=1$, or $x=0$.

Equation (1.1) is often referred to as the deformation for an elastic beam under a variety of boundary conditions. A brief discussion of the physical interpretation under some boundary conditions associated with the linear beam equation can be found in Zill and Cullen [5]. In this article, we consider the existence and uniqueness of positive solutions for fourth-order singular boundary value problems by using mixed monotone method.

## 2. Preliminary

Let $P$ be a normal cone of a Banach space $E$, and $e\in P$ with $\|e\|\le 1$, $e\ne\theta$. Define

$$Q_e=\left\{x\in P\,\middle|\,x\ne\theta,\ \text{there exist constants } m, M>0 \text{ such that } me\le x\le Me\right\}.$$

Now we give a definition(see [7]).

**Definition 2.1.** Assume $A:Q_e\times Q_e\to Q_e$. $A$ is said to be mixed monotone if $A(x,y)$ is nondecreasing in $x$ and nonincreasing in $y$, *i.e.* if $x_1\le x_2\ (x_1,x_2\in Q_e)$ implies $A(x_1,y)\le A(x_2,y)$ for any $y\in Q_e$, and $y_1\le y_2\ (y_1,y_2\in Q_e)$ implies $A(x,y_1)\ge A(x,y_2)$ for any $x\in Q_e$. $x^*\in Q_e$ is said to be a fixed point of $A$ if $A(x^*,x^*)=x^*$.

**Theorem 2.1.** Suppose that $A:Q_e\times Q_e\to Q_e$ is a mixed monotone operator and $\exists$ a constant $\alpha$, $0\le\alpha<1$, such that

$$A\left(tx,\frac{1}{t}y\right)\ge t^{\alpha}A(x,y)\quad \forall x,y\in Q_e,\ 0<t<1. \tag{2.1}$$

Then $A$ has a unique fixed point $x^*\in Q_e$. Moreover, for any $(x_0,y_0)\in Q_e\times Q_e$

$$x_n=A(x_{n-1},y_{n-1}),\quad y_n=A(y_{n-1},x_{n-1}),\quad n=1,2,\cdots,$$

satisfy

$$x_n\to x^*,\quad y_n\to x^*$$

where

$$\|x_n-x^*\|=o\left(1-r^{\alpha^n}\right),\quad \|y_n-y^*\|=o\left(1-r^{\alpha^n}\right),$$

$0<r<1$, $r$ is a constant from $(x_0,y_0)$.

**Theorem 2.2.** (See [7]): Suppose that $A:Q_e\times Q_e\to Q_e$ is a mixed monotone operator and $\exists$ a constant $\alpha\in(0,1)$ such that (2.1) holds. If $x_\lambda^*$ is a unique solution of equation

$$A(x,x)=\lambda x,\quad \lambda>0,$$

in $Q_e$, then $\|x_\lambda^*-x_{\lambda_0}^*\|\to 0$, $\lambda\to\lambda_0$. If $0<\alpha<\frac{1}{2}$, then $0<\lambda_1<\lambda_2$ implies $x_{\lambda_1}^*\ge x_{\lambda_2}^*$, $x_{\lambda_1}^*\ne x_{\lambda_2}^*$, and

$$\lim_{\lambda\to+\infty}\|x_\lambda^*\|=0,\quad \lim_{\lambda\to 0^+}\|x_\lambda^*\|=+\infty.$$

## 3. Uniqueness Positive Solution of Problem (1.1)

This section discusses the problem

$$\begin{cases}\left[p(t)x'''(t)\right]'+q(t)x''(t)=\lambda f(t,x(t)), & 0<t<1,\ \lambda>0,\\ \alpha_1x(0)-\beta_1x'(0)=0,\\ \gamma_1x(1)+\delta_1x'(1)=0,\\ \alpha_2x''(0)-\beta_2x'''(0)=0,\\ \gamma_2x''(1)+\delta_2x'''(1)=0.\end{cases}$$

Throughout this section, we assume that

$$f(t,x)=z(t)\big(g(x)+h(x)\big),\quad t\in(0,1), \tag{3.1}$$

where

$$\begin{aligned}&g:[0,+\infty)\to[0,+\infty)\text{ is continuous and nondecreasing;}\\&h:(0,+\infty)\to(0,+\infty)\text{ is continuous and nonincreasing.}\end{aligned} \tag{3.2}$$

Let $Q=I\times I$ and $Q_1=\{(t,s)\in Q\,|\,0\le t\le s\le 1\}$, $Q_2=\{(t,s)\in Q\,|\,0\le s\le t\le 1\}$. We denote the Green's functions for the following boundary value problems

$$\begin{cases}-x''(t)=0, & 0<t<1,\\ \alpha_1 x(0)-\beta_1 x'(0)=0,\\ \gamma_1 x(1)+\delta_1 x'(1)=0,\end{cases}$$

and

$$\begin{cases}-\big(p(t)x'(t)\big)'-q(t)x(t)=0, & 0<t<1,\\ \alpha_2 x(0)-\beta_2 x'(0)=0,\\ \alpha_2 x(1)+\beta_2 x'(1)=0,\end{cases}$$

by $H(t,s)$ and $G(t,s)$, respectively. It is well known that $H(t,s)$ and $G(t,s)$ can be written by

$$H(x,y):=\frac{1}{\rho_1}\begin{cases}(\beta_1+\alpha_1 t)\big(\delta_1+\gamma_1(1-s)\big), & (t,s)\in Q_1,\\ (\beta_1+\alpha_1 s)\big(\delta_1+\gamma_1(1-t)\big), & (t,s)\in Q_2,\end{cases}$$

where $\rho_1=\beta_1\gamma_1+\alpha_1\gamma_1+\alpha_1\delta_1>0$ and

$$G(t,s):=\frac{1}{\omega}\begin{cases}m(t)n(s), & (t,s)\in Q_1,\\ m(s)n(t), & (t,s)\in Q_2.\end{cases}$$

**Lemma 3.1.** Suppose that $(S_1)$ holds, then the Green's function $G(t,s)$, possesses the following properties:

1) $m(t)\in C^2(I,R)$ is increasing and $m(t)>0$, $x\in(0,1]$.

2) $n(t)\in C^2(I,R)$ is decreasing and $n(t)>0$, $x\in[0,1)$.

3) $(Lm)(t)\equiv 0$, $m(0)=\beta_2$, $m'(0)=\alpha_2$.

4) $(Ln)(t)\equiv 0$, $n(1)=\delta_2$, $n'(1)=-\gamma_2$.

5) $\omega$ is a positive constant. Moreover, $p(t)\big(m'(t)n(t)-m(t)n'(t)\big)\equiv\omega$.

6) $G(t,s)$ is continuous and symmetrical over $Q$.

7) $G(t,s)$ has continuously partial derivative over $Q_1$, $Q_2$.

8) For each fixed $s\in I$, $G(t,s)$ satisfies $LG(t,s)=0$ for $s\ne t$, $t\in I$. Moreover, $R_1(G)=R_2(G)=0$ for $s\in(0,1)$.

9) $G_t'$ has discontinuous point of the first kind at $t=s$ and

$$G_t'(s+0,s)-G_t'(s-0,s)=-\frac{1}{p(s)},\quad s\in(0,1).$$

Following from Lemma 3.1, it is easy to see that

(a) $H(t,s)\le H(t,t)$, $H(s,s)\le H(s,s)$,

$$H(t,s)\ge\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}H(t,t)H(s,s),\quad\text{for }(t,s)\in[0,1]\times[0,1].$$

(b) $G(t,s)\le G(t,t)=\dfrac{1}{\omega}m(t)n(t)$, $G(t,s)\le G(s,s)$,

$$G(t,s)\ge G(t,t)G(s,s)\frac{w}{m(1)n(0)},\quad\text{for }(t,s)\in[0,1]\times[0,1].$$

Suppose that $x$ is a positive solution of (1.1). Then

$$x(t)=\lambda\int_0^1\int_0^1 H(t,\tau)G(\tau,s)f(s,x(s))\mathrm{d}s\mathrm{d}\tau \qquad 0\le t\le 1. \tag{3.3}$$

By using (3.3) and (a), we see that for every positive solution $x$ one has

$$\|x\|\le\lambda\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)f(s,x(s))\mathrm{d}s\mathrm{d}\tau,$$

$$x(t)\ge\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}H(t,t)\lambda\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)f(s,x(s))\mathrm{d}s\mathrm{d}\tau\ge\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}\|x\|$$

where $\|x\|=\sup\{|x(t)|;0\le t\le 1\}$. Let

$$K(t,s)=\int_0^1 H(t,\tau)G(\tau,s)\mathrm{d}\tau.$$

Thus by (3.3) one has

$$x(t)=\lambda\int_0^1 K(t,s)f(s,x(s))\mathrm{d}s\mathrm{d}\tau,\quad 0\le t\le 1$$

by (a) one has

$$\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}H(t,t)\int_0^1 H(\tau,\tau)G(\tau,s)\mathrm{d}\tau\le K(t,s)\le H(t,t)\int_0^1 G(\tau,s)\mathrm{d}\tau. \tag{3.4}$$

Let $P=\{x\in C[0,1]\,\|\,x(t)\ge 0,\forall t\in[0,1]\}$. Obviously, $P$ is a normal cone of Banach space $C[0,1]$.

**Theorem 3.1.** Suppose that there exists $\alpha\in(0,1)$ such that

$$\begin{aligned} g(tx)&\ge t^\alpha g(x)\\ h(t^{-1}x)&\ge t^\alpha h(x)\end{aligned} \tag{3.5}$$

for any $t\in(0,1)$ and $x>0$, and $z\in C((0,1),(0,\infty))$ satisfies

$$\int_0^1 H^{-\alpha}(s,s)z(s)\mathrm{d}s<+\infty. \tag{3.6}$$

Then (1.1) has a unique positive solution $x_\lambda^*(t)$. And moreover, $0<\lambda_1<\lambda_2$ implies $x_{\lambda_1}^*\le x_{\lambda_2}^*$, $x_{\lambda_1}^*\ne x_{\lambda_2}^*$. If $\alpha\in\left(0,\frac{1}{2}\right)$, then

$$\lim_{\lambda\to0^+}\|x_\lambda^*\|=0,\quad \lim_{\lambda\to+\infty}\|x_\lambda^*\|=+\infty.$$

**Proof.** Since (3.5) holds, let $t^{-1}x=y$, one has

$$h(y)\ge t^\alpha h(ty)$$

then

$$h(ty)\le\frac{1}{t^\alpha}h(y),\ \forall t\in(0,1),\ y>0. \tag{3.7}$$

Let $y=1$. The above inequality is

$$h(t)\le\frac{1}{t^\alpha}h(1),\ \forall t\in(0,1). \tag{3.8}$$

From (3.5), (3.7) and (3.8), one has

$$\begin{aligned} &h(t^{-1}x)\ge t^\alpha h(x),\quad h\left(\frac{1}{t}\right)\ge t^\alpha h(1),\\ &h(tx)\le\frac{1}{t^\alpha}h(x),\quad h(t)\le\frac{1}{t^\alpha}h(1),\ t\in(0,1),\quad x>0.\end{aligned} \tag{3.9}$$

Similarly, from (3.5), one has

$$g(tx) \ge t^\alpha g(x), \quad g(t) \ge t^\alpha g(1), \quad t \in (0,1), \quad x > 0. \tag{3.10}$$

Let $t = \frac{1}{x}$, $x > 1$, one has

$$g(x) \le x^\alpha g(1), \quad x \ge 1. \tag{3.11}$$

Let $e(t) = \frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)} H(t,t)$. It is clear that $\|e\| < 1$, and now let

$$Q_e = \left\{ x \in C[0,1] \middle| \frac{1}{M} e(t) \le x(t) \le Me(t),\ t \in [0,1] \right\} \tag{3.12}$$

where $M > 1$ is chosen such that

$$M > \max\left\{ \left\{ \lambda g(1) \left( \frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)} \right)^{-1} \int_0^1\int_0^1 G(\tau,s) z(s) \mathrm{d}s\mathrm{d}\tau \right.\right.$$

$$\left. + \lambda h(1) \left( \frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)} \right)^{-\alpha-1} \int_0^1\int_0^1 G(\tau,s) z(s) H^{-\alpha}(s,s) \mathrm{d}s\mathrm{d}\tau \right\}^{\frac{1}{1-\alpha}}$$

$$\left\{ \lambda g(1) \left( \frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)} \right)^{\alpha} \int_0^1\int_0^1 H(\tau,\tau) G(\tau,s) z(s) H^{\alpha}(s,s) \mathrm{d}s\mathrm{d}\tau \right.$$

$$\left.\left. + \lambda h(1) \int_0^1\int_0^1 H(\tau,\tau) G(\tau,s) z(s) \mathrm{d}\tau\mathrm{d}s \right\}^{-\frac{1}{1-\alpha}} \right\}.$$

For any $x, y \in Q_e$, we define

$$A_\lambda(x,y)(t) = \lambda \int_0^1 K(t,s) z(s) \left[ g(x(s)) + h(y(s)) \right] \mathrm{d}s \quad \forall t \in [0,1]. \tag{3.13}$$

First we show that $A_\lambda : Q_e \times Q_e \to Q_e$. Let $x, y \in Q_e$, from (3.10) and (3.11) we have

$$g(x(t)) \le g(Me(t)) \le g(M) \le M^\alpha g(1)$$

and from (3.9) we have

$$h(y(t)) \le h\left( \frac{1}{M} e(t) \right) \le M^\alpha e^{-\alpha}(t) h(1). \tag{3.14}$$

Then from (3.4) and (3.13) we have

$$A_\lambda(x,y)(t) = \lambda \int_0^1 H(t,\tau) G(\tau,s) z(s) \left[ g(x(s)) + h(y(s)) \right] \mathrm{d}s\mathrm{d}\tau$$

$$\le \lambda H(t,t) \left\{ \int_0^1\int_0^1 G(\tau,s) z(s) g(x(s)) \mathrm{d}s\mathrm{d}\tau + \int_0^1\int_0^1 G(\tau,s) z(s) h(y(s)) \mathrm{d}s\mathrm{d}\tau \right\}$$

$$\le \lambda H(t,t) \left\{ M^\alpha g(1) \int_0^1\int_0^1 G(\tau,s) z(s) \mathrm{d}s\mathrm{d}\tau + M^\alpha h(1) \int_0^1\int_0^1 G(\tau,s) z(s) e^{-\alpha}(s) \mathrm{d}s\mathrm{d}\tau \right\}$$

$$= \lambda H(t,t) \left\{ M^\alpha g(1) \int_0^1\int_0^1 G(\tau,s) z(s) \mathrm{d}s\mathrm{d}\tau + M^\alpha h(1) \left( \frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)} \right)^{-\alpha} \int_0^1\int_0^1 G(\tau,s) z(s) H^{-\alpha}(s,s) \mathrm{d}s\mathrm{d}\tau \right\}$$

$$\le e(t) M \qquad t \in [0,1].$$

On the other hand, for any $x, y \in Q_e$, from (3.9) and (3.10), we have

$$g(x(t)) \ge g\left(\frac{1}{M}e(t)\right) \ge M^{-\alpha}e^{\alpha}(t)g(1),$$
$$h(y(t)) \ge h(Me(t)) \ge h(M) \ge M^{-\alpha}h(1). \tag{3.15}$$

Thus, from (3.15), we have

$$A_\lambda(x,y)(t) = \lambda\int_0^1 H(\tau,\tau)G(\tau,s)z(s)\left[g(x(s))+h(y(s))\right]\mathrm{d}s$$
$$\ge \lambda\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}H(t,t)\left\{\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)z(s)g(x(s))\mathrm{d}\tau\mathrm{d}s+\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)z(s)h(y(s))\mathrm{d}\tau\mathrm{d}s\right\}$$
$$\ge \lambda e(t)\left\{M^{-\alpha}g(1)\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)z(s)e^{\alpha}(s)\mathrm{d}\tau\mathrm{d}s + M^{-\alpha}h(1)\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)z(s)\mathrm{d}\tau\mathrm{d}s\right\}$$
$$= \lambda e(t)\left\{M^{-\alpha}g(1)\left(\frac{\rho_1}{(\alpha_1+\beta_1)(\delta_1+\gamma_1)}\right)^{\alpha}\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)H^{\alpha}(s,s)z(s)\mathrm{d}\tau\mathrm{d}s\right.$$
$$\left.+M^{-\alpha}h(1)\int_0^1\int_0^1 H(\tau,\tau)G(\tau,s)z(s)\mathrm{d}\tau\mathrm{d}s\right\}$$
$$\ge e(t)\frac{1}{M}.$$

So, $A_\lambda$ is well defined and $A_\lambda(Q_e\times Q_e)\subset Q_e$.
Next, for any $l\in(0,1)$, one has

$$A_\lambda(lx,l^{-1}y)(t) = \lambda\int_0^1 K(t,s)z(s)\left[g(lx(s))+h(l^{-1}y(s))\right]\mathrm{d}s$$
$$\ge \lambda\int_0^1 K(t,s)z(s)\left[l^{\alpha}g(x(s))+l^{\alpha}h(y(s))\right]\mathrm{d}s$$
$$= l^{\alpha}A_\lambda(x,y)(t),\quad t\in[0,1].$$

So the conditions of Theorems 2.1 and 2.2 hold. Therefore there exists a unique $x_\lambda^*\in Q_e$ such that $A_\lambda(x^*,x^*)=x_\lambda^*$. It is easy to check that $x_\lambda^*$ is a unique positive solution of (1.1) for given $\lambda>0$. Moreover, Theorem 2.2 means that if $0<\lambda_1<\lambda_2$ then $x_{\lambda_1}^*(t)\le x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t)\ne x_{\lambda_2}^*(t)$, and if $\alpha\in\left(0,\frac{1}{2}\right)$, then

$$\lim_{\lambda\to0^+}\left\|x_\lambda^*\right\|=0,\quad \lim_{\lambda\to+\infty}\left\|x_\lambda^*\right\|=+\infty.$$

This completes the proof.

**Example.** Consider the following singular fourth-order boundary value problem:

$$\begin{cases}\left[p(t)x'''(t)\right]'+q(t)x''(t)=\lambda\left(\mu x^{a}+x^{-b}\right), & 0<t<1,\\ \alpha_1x(0)-\beta_1x'(0)=0,\\ \gamma_1x(1)+\delta_1x'(1)=0,\\ \alpha_2x''(0)-\beta_2x'''(0)=0,\\ \gamma_2x''(1)+\delta_2x'''(1)=0,\end{cases}$$

where $\lambda,\ a,\ b>0,\ \mu\ge0,\ \max\{a,b\}<1$, satisfies $\int_0^1 H^{-\alpha}(s,s)z(s)\mathrm{d}s<+\infty$.
Let

$$\alpha=\max\{a,b\},\quad g(x)=\mu x^{a},\quad h(x)=x^{-b},\quad z(t)=1.$$

Thus $0<\alpha<1$ and for any $t\in(0,1)$ $x>0$, $y>0$,

$$g(tx)=t^{a}g(x)\ge t^{\alpha}g(x),\quad h(t^{-1}x)=t^{b}h(x)\ge t^{\alpha}h(x).$$

Now Theorem 3.1 guarantees that the above equation has a positive solution.

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## References

[1] Jiang, D.Q., Liu, H. and Xu, X. (2005) Nonresonant Singular Fourth-Order Boundary Value Problems. *Applied Mathematics Letters*, **18**, 69-75. http://dx.doi.org/10.1016/j.aml.2003.05.016

[2] Jiang, D.Q. (2000) Multiple Positive Solutions to Singular Boundary Value Problems for Superlinear Higher Order ODEs. *Computers Mathematics with Applications*, **40**, 249-259. http://dx.doi.org/10.1016/S0898-1221(00)00158-9

[3] Liu, Y. (2003) Structure of a Class of Singular Boundary Value Problem with Superlinear Effect. *Journal of Mathematical Analysis and Applications*, **284**, 64-75. http://dx.doi.org/10.1016/S0022-247X(03)00214-2

[4] Kong, L. and Wang, J. (2001) The Green's Function for ($k$, $n - k$) Conjugate Boundary Value Problems and Its Applications. *Journal of Mathematical Analysis and Applications*, **255**, 404-422. http://dx.doi.org/10.1006/jmaa.2000.7158

[5] Zill, D.G. and Cullen, M.R. (2001) Differential Equations with Boundary-Value Problems. 5th Edition, Brooks Cole, Belmont.

[6] Liu, Y. (2004) Multiple Positive Solutions of Nonlinear Singular Boundary Value Problem for Fourth Order Equations. *Applied Mathematics Letters*, **17**, 747-757. http://dx.doi.org/10.1016/j.aml.2004.06.001

[7] Guo, D. (2000) The Order Methods in Nonlinear Analysis. Shandong Technology and Science Press, Jinan.