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On the Stability of Solutions of Nonlinear Functional Differential Equation of the Fifth-Order

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Abstract

The main purpose of this paper is to investigate global asymptotic stability of the zero solution of the fifth-order nonlinear delay differential equation on the following form

$$x^{(5)}(t)+\psi\left(x(t-r),\dot{x}(t-r),\ddot{x}(t-r),\dddot{x}(t-r),x^{(4)}(t-r)\right)x^{(4)}(t)$$
$$+f\left(\dddot{x}(t-r)\right)+\phi\left(\ddot{x}(t-r)\right)+g\left(\dot{x}(t)\right)+h\left(x(t)\right)=0.$$

By constructing a Lyapunov functional, sufficient conditions for the stability of the zero solution of this equation are established.

Keywords

Global Asymptotic Stability, Lyapunov Functional, Fifth-Order Delay Differential Equations

1. Introduction

As is well-known, the area of differential equations is an old but durable subject, that remains alive and useful to a wide variety of engineers, scientists and mathematicians. Now the subject of differential equations represents a huge body of knowledge including many subfields and a vast array of applications in many disciplines. It should be noted that principles of differential equations are largely related to the qualitative theory of ordinary differential equations. Qualitative theory refers to the study of behaviour of solutions, for example, the investigation of

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stability, instability, boundedness of solutions and etc., without determining explicit formulas for the solutions. In particular one can refer that many authors have dealt with delay differential equations and its problems, and many excellent results have been obtained on the behaviour of solutions for various higher-order: second-, third-, fourth-, and fifth-order nonlinear differential equations with delay, for example, [1]-[27], and references quoted therein, which contain the differential equations without delay or with delay. In many of these references, the authors dealt with the problems by using Lyapunov's second method [28]. By considering Lyapunov functionals we obtained the conditions which ensured the stability of the problem. It is worth-mentioning that construction of these Lyapunov functionals remains a general problem. We know that a similar problem exists for ordinary differential equations for higher-order [12]. Clearly, it is even more difficult to construct Lyapunov functionals for delay differential equations of higher-order. Up to this moment the investigations concerning the stability of solutions of nonlinear equations of fifth-order with delay have not been fully developed.

In particular in 2010 Tunç [29] obtained sufficient conditions, which ensure the stability of the zero solution of a nonlinear delay differential equation of fifth-order

$$\begin{aligned}&x^{(5)}(t)+\psi\left(x(t-r),\dot{x}(t-r),\ddot{x}(t-r),\dddot{x}(t-r),x^{(4)}(t-r)\right)x^{(4)}(t)\\&+f\left(\ddot{x}(t-r),\dddot{x}(t-r)\right)+\alpha_3\ddot{x}(t)+\alpha_4\dot{x}(t)+\alpha_5 x(t)=0,\end{aligned}$$

where ψ and f are continuous functions; α_3, α_4 and α_5 are positive constants, r is a bounded delay and positive constant; the derivatives $f_z(z,w)$, $f_w(z,w)$ exist and are continuous for all z, w and $f(z,0)=0$.

Later in 2011 Abou-El-Ela, Sadek and Mahmoud [30] obtained the sufficient conditions for the uniform stability of the zero solution of a nonlinear fifth-order delay differential equation of the following form

$$x^{(5)}(t)+\phi(\dddot{x}(t))x^{4}(t)+\psi(\dddot{x}(t))+f(\ddot{x}(t))+g(\dot{x}(t))+h(x(t-r))=0,$$

where r is a positive constant; $\phi(\dddot{x}),\psi(\dddot{x}), f(\ddot{x}), g(\dot{x})$ and $h(x)$ are continuous functions and $\psi(0)=f(0)=g(0)=h(0)=0.$

In the present paper, we are concerned with the stability of the zero solution of the fifth-order nonlinear delay differential equation on the form

$$\begin{aligned}&x^{(5)}(t)+\psi\left(x(t-r),\dot{x}(t-r),\ddot{x}(t-r),\dddot{x}(t-r),x^{(4)}(t-r)\right)x^{(4)}(t)\\&+f\left(\dddot{x}(t-r)\right)+\phi\left(\ddot{x}(t-r)\right)+g\left(\dot{x}(t)\right)+h\left(x(t)\right)=0,\end{aligned}\tag{1.1}$$

or its equivalent system form

$$\begin{aligned}&\dot{x}=y,\qquad \dot{y}=z,\qquad \dot{z}=w,\qquad \dot{w}=u,\\&\dot{u}=-\psi\left(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)\right)u-f(w)\\&\quad-\phi(z)-g(y)-h(x)+\int_{t-r}^{t}f'(w(s))u(s)\mathrm{d}s+\int_{t-r}^{t}\phi'(z(s))w(s)\mathrm{d}s,\end{aligned}\tag{1.2}$$

where ψ, f, ϕ, g and h are continuous functions for the arguments displayed explicitly in (1.1) with $f(0)=\phi(0)=g(0)=h(0)=0,$ r is a bounded delay and positive constant; the derivatives $f'(w)$ and $\phi'(z)$ exist and are continuous for all w,z.

2. Preliminaries and Stability Results

In order to reach the main result of this paper, we will give some basic information to the stability criteria for the general autonomous delay differential system. We consider

$$\dot{x}=f(x_t),\ x_t=x(t+\theta),\ -r\le\theta\le 0,\ t\ge 0,\tag{2.1}$$

where $f:C_H\to\mathbb{R}^n$ is a continuous mapping, $f(0)=0, C_H:=\left\{\varphi\in C\left([-r,0],\mathbb{R}^n\right):\|\varphi\|\le H\right\}$ and for $H_1<H$, there exists $L(H_1)>0,$ with $|f(\varphi)|\le L(H_1)$ when $\|\varphi\|\le H_1.$

The following are the classical theorems on uniform stability and global asymptotic stability for the solution of (2.1).

Theorem 2.1. [31]. Let $V(\varphi):\mathcal{C}_H\to\mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition $V(0)=0$ and the functions $W_i(r),(i=1,2)$ are wedges such that

i) $W_1\left(\left|\varphi(0)\right|\right) \le V(\varphi) \le W_2\left(\left\|\varphi\right\|\right)$ and

ii) $\dot{V}_{(2,1)}(\varphi) \le 0$.

Then the zero solution of (2.1) is uniformly stable.

Theorem 2.2. [32]. Suppose $f(0)=0$, let V be a continuous functional defined on $C_H = C$ with $V(0)=0$, and let $u(s)$ be non-negative and continuous function for $0 \le s < \infty$, $u(s) \to \infty$ as $s \to \infty$ such that for all $\varphi \in C$

i) $u\left(\left|\varphi(0)\right|\right) \le V(\varphi)$, $V(\varphi) \ge 0$ and

ii) $\dot{V}_{(2,1)}(\varphi) < 0$ for $\varphi \ne 0$.

Then all solutions of (2.1) approach zero as $t \to \infty$ and the origin is globally asymptotically stable.

The following will be our main stability result for (1.1).

Theorem 2.3. In addition to the basic assumptions imposed on the functions ψ, f, ϕ, g and h. Suppose that the following conditions are satisfied, where $\alpha_1, \cdots, \alpha_5$ are arbitrary positive constants and $\varepsilon, \varepsilon_o, \delta, \lambda, \rho, \rho_1, M$ and L are sufficiently small positive constants

i) $\alpha_1 > 0, \alpha_1\alpha_2 - \alpha_3 > 0, \alpha_5 > 0, (\alpha_1 \alpha_2 - \alpha_3)\alpha_3 - (\alpha_1 \alpha_4 - \alpha_5)\alpha_1 > 0.$

$$\delta_o = (\alpha_3 \alpha_4 - \alpha_2\alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 \alpha_4 - \alpha_5)^2 > 0, \tag{2.2}$$

and the following two inequalities

$$\Delta_1 = \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{\alpha_1 g'(y) - \alpha_5\} > 2\epsilon\alpha_2 \tag{2.3}$$

$$\Delta_2 = \frac{\alpha_3 \alpha_4 - \alpha_2\alpha_5}{\alpha_1 \alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5)\gamma(y)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\epsilon}{\alpha_1} > 0, \tag{2.4}$$

for all y and all $t \in \mathbb{R}^+$, where

$$\gamma(y) = \begin{cases} g(y)/y, & y \ne 0; \\ g'(0), & y = 0. \end{cases}$$

ii) $2\epsilon_o \le \psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \le \min\left\{\frac{\epsilon}{12\alpha_1^2}, \frac{\epsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{14\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\epsilon\alpha_4}{12\delta^2}\right\}.$

iii) $f(0) = 0, \frac{f(w)}{w} \ge \alpha_2;\ w \ne 0, \left|f'(w(s))\right| \le M$ and

$$\left[\frac{f(w)}{w} - \alpha_2\right]^2 \le \min\left\{\frac{2\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{21\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon^2\alpha_4}{9\delta^2}\right\}.$$

iv) $\phi(0) = 0, \frac{\phi(z)}{z} \ge \alpha_3;\ z \ne 0, \left|\phi'(z(s))\right| \le L$ and

$$\left[\frac{\phi(z)}{z} - \alpha_3\right]^2 \le \min\left\{\frac{\epsilon\epsilon_o\alpha_2}{7}, \frac{2\epsilon^2\alpha_2\alpha_4}{21\delta^2}\right\}.$$

v) $g(0) = 0, \frac{g(y)}{y} \ge \alpha_4, g'(y) - \frac{g(y)}{y} \le \frac{\alpha_5\delta_o}{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)};\ y \ne 0$ and

$$\left[\alpha_4 - g'(y)\right]^2 \le \frac{2\epsilon^2\alpha_2}{21}.$$

vi) $h(0)=0, h(x)\operatorname{sgn} x>0\,(x\neq 0), H(x)\equiv\int_0^x h(\xi)\,\mathrm{d}\xi\to\infty$ as $|x|\to\infty, h'(x)\le\alpha_5$ for all x, and

$$\left[\alpha_5-h'(x)\right]^2\le\min\left\{\frac{\epsilon^2\alpha_4}{9},\frac{2\epsilon^2\alpha_2\alpha_4}{21\delta^2}\right\}.$$

Then the zero solution of (1.1) is globally asymptotically stable, provided that

$$r<\min\left\{\frac{\epsilon\alpha_4}{6\delta(L+M)},\frac{\epsilon\alpha_2(\alpha_1\alpha_4-\alpha_5)}{7\alpha_4(L+M)(\alpha_1\alpha_2-\alpha_3)},\frac{\epsilon}{6\alpha_1(L+M)+12\lambda},\frac{\epsilon_o}{L+M+2\rho}\right\}.$$

Proof. We define the Lyapunov functional $V=V(x_t,y_t,z_t,w_t,u_t)$ as:

$$\begin{aligned}2V(x_t,y_t,z_t,w_t,u_t)&=u^2+2\alpha_1 uw+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz+2\delta uy+2\int_0^w f(\sigma)\,\mathrm{d}\sigma\\&+\left[\alpha_1^2-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]w^2+2\left[\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right]wz\\&+2\alpha_1\delta wy+2wg(y)+2wh(x)+2\alpha_1\int_0^z\phi(\zeta)\,\mathrm{d}\zeta\\&+\left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_4-\alpha_1\delta\right]z^2+2\delta\alpha_2 yz+2\alpha_1 zg(y)\\&-2\alpha_5 yz+2\alpha_1 zh(x)+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^y g(\eta)\,\mathrm{d}\eta\\&+(\delta\alpha_3-\alpha_1\alpha_5)y^2+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yh(x)+2\delta\int_0^x h(\xi)\,\mathrm{d}\xi\\&+2\lambda\int_{-r}^0\int_{t+s}^t w^2(\theta)\,\mathrm{d}\theta\mathrm{d}s+2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)\,\mathrm{d}\theta\mathrm{d}s,\end{aligned}\tag{2.5}$$

where ρ and λ are two positive constants, which will be determined later and δ is a positive constant defined by

$$\delta:=\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}+\epsilon.\tag{2.6}$$

Then it is convenient to rewrite the expression for the Lyapunov functional defined in (2.5) in the following form

$$\begin{aligned}2V&=\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}^2+\frac{\alpha_4\delta_o}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\frac{\alpha_5}{\alpha_4}y\right)^2\\&+\frac{\alpha_4(\alpha_1\alpha_4-\alpha_5)}{(\alpha_1\alpha_2-\alpha_3)\gamma}\left\{\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}h(x)+\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}\gamma y+\frac{\alpha_1}{\alpha_4}\gamma z+\frac{1}{\alpha_4}\gamma w\right\}^2\\&+\Delta_2(w+\alpha_1 z)^2+2\epsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+2\lambda\int_{-r}^0\int_{t+s}^t w^2(\theta)\,\mathrm{d}\theta\mathrm{d}s\\&+2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)\,\mathrm{d}\theta\mathrm{d}s+\sum_{i=1}^4 v_i,\end{aligned}\tag{2.7}$$

where

$$v_1:=2\delta\int_0^x h(\xi)\,\mathrm{d}\xi-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)\gamma}h^2(x),$$

$$v_2 := \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[2\int_0^y g(\eta)\mathrm{d}\eta - yg(y)\right]+\left[\delta\alpha_3-\alpha_1\alpha_5-\frac{\alpha_5^2\delta_o}{\alpha_4(\alpha_1\alpha_4-\alpha_5)^2}-\delta^2\right]y^2,$$

$$v_3 := \frac{\epsilon}{\alpha_1}w^2+2\int_0^w f(\sigma)\mathrm{d}\sigma-\alpha_2 w^2,$$

$$v_4 := 2\alpha_1\int_0^z \phi(\zeta)\mathrm{d}\zeta-\alpha_1\alpha_3 z^2.$$

For the component v_1, by using (2.6) and the definition of γ

$$\begin{aligned} v_1 &= 2\epsilon\int_0^x h(\xi)\mathrm{d}\xi+\frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^x h(\xi)\mathrm{d}\xi-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)\gamma}h^2(x) \\ &= 2\epsilon\int_0^x h(\xi)\mathrm{d}\xi+\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}\left\{2\alpha_5\int_0^x h(\xi)\mathrm{d}\xi-\frac{\alpha_4}{\gamma}h^2(x)\right\}, \end{aligned}$$

since $\frac{\alpha_4}{\gamma}\le 1$ by v), thus we obtain

$$2\alpha_5\int_0^x h(\xi)\mathrm{d}\xi-\frac{\alpha_4}{\gamma}h^2(x)=2\int_0^x\{\alpha_5-h'(\xi)\}h(\xi)\mathrm{d}\xi-h^2(0)\ge 0.$$

This is due to the fact that the integral on the right-hand side is non-negative by vi), therefore we get

$$v_1 \ge 2\epsilon\int_0^x h(\xi)\mathrm{d}\xi.$$

From the identity

$$yg(y)\equiv\int_0^y g(\eta)\mathrm{d}\eta+\int_0^y \eta g'(\eta)\mathrm{d}\eta,$$

therefore

$$v_2 \ge \int_0^y\left[\frac{\alpha_5\delta_o}{\alpha_4(\alpha_1\alpha_4-\alpha_5)}-2\epsilon\left\{\epsilon+\frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_3\right\}\right]\eta\mathrm{d}\eta \ge \frac{\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2,$$

and by using v) we find

$$v_2 \ge \int_0^y\left[\frac{\alpha_5\delta_o}{\alpha_4(\alpha_1\alpha_4-\alpha_5)}-2\epsilon\left\{\epsilon+\frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_3\right\}\right]\eta\mathrm{d}\eta \ge \frac{\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2,$$

provided that

$$\frac{\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}\ge\epsilon\left\{\epsilon+\frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_3\right\}.$$

$$v_3=\frac{\epsilon}{\alpha_1}w^2+2\int_0^w\left\{\frac{f(\sigma)}{\sigma}-\alpha_2\right\}\sigma\mathrm{d}\sigma\ge\frac{\epsilon}{\alpha_1}w^2,\quad \text{by (iii).}$$

From iv) we find

$$v_4=2\alpha_1\int_0^z\left\{\frac{\phi(\zeta)}{\zeta}-\alpha_3\right\}\zeta\mathrm{d}\zeta\ge 0.$$

Summing up the four inequalities obtained from $v_1,\cdots,v_4$ into (2.7), we have

$$2V \geq \left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}^2+\frac{\alpha_4\delta_o}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\frac{\alpha_5}{\alpha_4}y\right)^2$$
$$+\Delta_2(w+\alpha_1 z)^2+2\epsilon\int_0^x h(\xi)\mathrm{d}\xi+\frac{\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2+\frac{\epsilon}{\alpha_1}w^2 \tag{2.8}$$
$$+2\epsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+2\lambda\int_{-r}^{0}\int_{t+s}^{t}w^2(\theta)\mathrm{d}\theta\mathrm{d}s+2\rho\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)\mathrm{d}\theta\mathrm{d}s.$$

Clearly, it follows from the first six terms included in (2.8) that there exist sufficiently small positive constants $D_i,(i=1,\cdots,5)$, such that

$$2V \geq D_1 H(x)+2D_2 y^2+2D_3 z^2+D_4 w^2+D_5 u^2+2\epsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz$$
$$+2\lambda\int_{-r}^{0}\int_{t+s}^{t}w^2(\theta)\mathrm{d}\theta\mathrm{d}s+2\rho\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)\mathrm{d}\theta\mathrm{d}s. \tag{2.9}$$

Now we consider the terms

$$v_5 := D_2 y^2+2\epsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+D_3 z^2,$$

which are contained in (2.9) and by using the inequality $|yz|\leq\frac{1}{2}(y^2+z^2)$, we obtain

$$v_5 \geq D_2 y^2+D_3 z^2-\epsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)(y^2+z^2)\geq D_6(y^2+z^2),$$

for some $D_6>0, D_6=\frac{1}{2}\min\{D_2,D_3\}$, if

$$\epsilon\leq\frac{\alpha_1\alpha_4-\alpha_5}{2(\alpha_3\alpha_4-\alpha_2\alpha_5)}\min\{D_2,D_3\}.$$

By using the previous inequality, we get from (2.9) that

$$2V \geq D_1 H(x)+(D_2+D_6)y^2+(D_3+D_6)z^2+D_4 w^2+D_5 u^2$$
$$+2\lambda\int_{-r}^{0}\int_{t+s}^{t}w^2(\theta)\mathrm{d}\theta\mathrm{d}s+2\rho\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)\mathrm{d}\theta\mathrm{d}s. \tag{2.10}$$

As a result, since the integrals

$$2\lambda\int_{-r}^{0}\int_{t+s}^{t}w^2(\theta)\mathrm{d}\theta\mathrm{d}s \text{ and } 2\rho\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)\mathrm{d}\theta\mathrm{d}s$$

are non-negative, it is obvious that there exists a positive constant D_7 which satisfies the following inequality

$$V(x_t,y_t,z_t,w_t,u_t)\geq D_7\left\{H(x)+y^2+z^2+w^2+u^2\right\}, \tag{2.11}$$

where

$$D_7=\frac{1}{2}\min\{D_1,D_2+D_6,D_3+D_6,D_4,D_5\}.$$

Now by a direct calculation from (1.2) and (2.5) one finds

$$\begin{aligned}
\frac{dV}{dt}(x_t,y_t,z_t,w_t,u_t) = &-\left[\psi\left(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)\right)-\alpha_1\right]u^2 \\
&-\left[\alpha_1\frac{f(w)}{w}-\left\{\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right\}\right]w^2 \\
&-\left[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{\phi(z)}{z}-\left\{\delta\alpha_2+\left(\alpha_1 g'(y)-\alpha_5\right)\right\}\right]z^2 \\
&-\left[\delta\frac{g(y)}{y}-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}h'(x)\right]y^2-\alpha_1\left[\psi(x,y,z,w,u)-\alpha_1\right]wu \\
&-\left[\frac{\phi(z)}{z}-\alpha_3\right]uz-\left[\alpha_4-g'(y)\right]wz-\delta\left[\frac{\phi(z)}{z}-\alpha_3\right]yz \\
&-\alpha_1\left[\alpha_5-h'(x)\right]yz-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\psi(x,y,z,w,u)-\alpha_1\right]zu \\
&-\left[\alpha_5-h'(x)\right]wy-\delta\left[\psi(x,y,z,w,u)-\alpha_1\right]yu \\
&-\delta\left[\frac{f(w)}{w}-\alpha_2\right]wy-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\frac{f(w)}{w}-\alpha_2\right]wz \\
&+u\int_{t-r}^{t}f'(w(s))u(s)ds+u\int_{t-r}^{t}\phi'(z(s))w(s)ds \\
&+\alpha_1 w\int_{t-r}^{t}f'(w(s))u(s)ds+\alpha_1 w\int_{t-r}^{t}\phi'(z(s))w(s)ds \\
&+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t}f'(w(s))u(s)ds \\
&+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t}\phi'(z(s))w(s)ds \\
&+\delta y\int_{t-r}^{t}f'(w(s))u(s)ds+\delta y\int_{t-r}^{t}\phi'(z(s))w(s)ds \\
&+\rho u^2 r-\rho\int_{t-r}^{t}u^2(s)ds+\lambda w^2 r-\lambda\int_{t-r}^{t}w^2(s)ds.
\end{aligned} \tag{2.12}$$

Making use of the assumptions ii)-vi), (2.3) and (2.6), we get

$$\psi\left(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)\right)-\alpha_1\geq 2\epsilon_o,$$

$$\begin{aligned}
&\alpha_1\frac{f(w)}{w}-\left\{\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right\} \\
&=\alpha_1\left[\frac{f(w)}{w}-\alpha_2\right]+\left\{\alpha_1\alpha_2-\alpha_3+\delta-\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right\}\geq\epsilon,
\end{aligned}$$

$$\begin{aligned}
&\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{\phi(z)}{z}-\left\{\delta\alpha_2+\left(\alpha_1 g'(y)-\alpha_5\right)\right\} \\
&\geq\frac{(\alpha_4\alpha_3-\alpha_2\alpha_5)(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\left\{\alpha_1 g'(y)-\alpha_5\right\}-\epsilon\alpha_2 \\
&\geq 2\epsilon\alpha_2-\epsilon\alpha_2=\epsilon\alpha_2,
\end{aligned}$$

and

$$\left[\delta\frac{g(y)}{y}-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}h'(x)\right]\geq\epsilon\alpha_4+\frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}=\epsilon\alpha_4,$$

By v), vi) and (2.6).

By using the assumptions $\left|f'\left(w(s)\right)\right|\le M$ and $\left|\phi'\left(z(s)\right)\right|\le L$ of the theorem and inequality $2|ab|\le a^2+b^2$, we obtain the following inequalities

$$u\int_{t-r}^{t} f'\left(w(s)\right)u(s)\mathrm{d}s\le\frac{M}{2}ru^2(t)+\frac{M}{2}\int_{t-r}^{t}u^2(s)\mathrm{d}s,$$

$$\alpha_1 w\int_{t-r}^{t} f'\left(w(s)\right)u(s)\mathrm{d}s\le\frac{\alpha_1 M}{2}rw^2(t)+\frac{\alpha_1 M}{2}\int_{t-r}^{t}u^2(s)\mathrm{d}s,$$

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t} f'\left(w(s)\right)u(s)\mathrm{d}s\le\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{M}{2}rz^2(t)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{M}{2}\int_{t-r}^{t}u^2(s)\mathrm{d}s,$$

$$\delta y\int_{t-r}^{t} f'\left(w(s)\right)u(s)\mathrm{d}s\le\frac{\delta M}{2}ry^2(t)+\frac{\delta M}{2}\int_{t-r}^{t}u^2(s)\mathrm{d}s,$$

$$u\int_{t-r}^{t}\phi'\left(z(s)\right)w(s)\mathrm{d}s\le\frac{L}{2}ru^2(t)+\frac{L}{2}\int_{t-r}^{t}w^2(s)\mathrm{d}s,$$

$$\alpha_1 w\int_{t-r}^{t}\phi'\left(z(s)\right)w(s)\mathrm{d}s\le\frac{\alpha_1 L}{2}rw^2(t)+\frac{\alpha_1 L}{2}\int_{t-r}^{t}w^2(s)\mathrm{d}s,$$

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t}\phi'\left(z(s)\right)w(s)\mathrm{d}s\le\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{L}{2}rz^2(t)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{L}{2}\int_{t-r}^{t}w^2(s)\mathrm{d}s,$$

and

$$\delta y\int_{t-r}^{t}\phi'\left(z(s)\right)w(s)\mathrm{d}s\le\frac{\delta L}{2}ry^2(t)+\frac{\delta L}{2}\int_{t-r}^{t}w^2(s)\mathrm{d}s.$$

Replacing the last equality and the preceding inequalities into (2.12), we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\le&-\left[\frac{\epsilon\alpha_4}{6}-\frac{\delta(L+M)}{2}r\right]y^2-\left[\frac{\epsilon\alpha_2}{7}-\frac{\alpha_4(L+M)(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}r\right]z^2\\&-\left[\frac{\epsilon}{6}-\left\{\frac{\alpha_1(L+M)}{2}+\lambda\right\}r\right]\omega^2-\left[\epsilon_o-\left(\frac{L+M}{2}+\rho\right)r\right]u^2\\&-\left[\lambda-\left\{\frac{L}{2}+\frac{\alpha_1 L}{2}+\frac{\delta L}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}L\right\}\right]\int_{t-r}^{t}\omega^2(s)\mathrm{d}s\\&-\left[\rho-\left\{\frac{M}{2}+\frac{\alpha_1 M}{2}+\frac{\delta M}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}M\right\}\right]\int_{t-r}^{t}u^2(s)\mathrm{d}s-\sum_{k=6}^{15}v_k,\end{aligned}\tag{2.13}$$

where

$$v_6:=\frac{1}{8}(\psi-\alpha_1)u^2+\alpha_1(\psi-\alpha_1)wu+\frac{\epsilon}{6}w^2,$$

$$v_7:=\frac{1}{8}(\psi-\alpha_1)u^2+\left[\frac{\phi(z)}{z}-\alpha_3\right]uz+\frac{\epsilon\alpha_2}{7}z^2,$$

$$v_8:=\frac{\epsilon\alpha_4}{6}y^2+\delta\left[\frac{\phi(z)}{z}-\alpha_3\right]yz+\frac{\epsilon\alpha_2}{7}z^2,$$

$$v_9:=\frac{1}{8}(\psi-\alpha_1)u^2+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}(\psi-\alpha_1)uz+\frac{\epsilon\alpha_2}{7}z^2,$$

$$v_{10}:=\frac{1}{8}(\psi-\alpha_1)u^2+\delta(\psi-\alpha_1)uy+\frac{\epsilon\alpha_4}{6}y^2,$$

$$v_{11} := \frac{\epsilon}{6} w^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\left[\frac{f(w)}{w} - \alpha_2\right] wz + \frac{\epsilon\alpha_2}{7} z^2,$$

$$v_{12} := \frac{\epsilon}{6} w^2 + \delta\left[\frac{f(w)}{w} - \alpha_2\right] wy + \frac{\epsilon\alpha_4}{6} y^2,$$

$$v_{13} := \frac{\epsilon}{6} w^2 + \left[\alpha_4 - g'(y)\right] wz + \frac{\epsilon\alpha_2}{7} z^2,$$

$$v_{14} := \frac{\epsilon\alpha_4}{6} y^2 + \alpha_1\left[\alpha_5 - h'(x)\right] yz + \frac{\epsilon\alpha_2}{7} z^2$$

and

$$v_{15} := \frac{\epsilon}{6} w^2 + \left[\alpha_5 - h'(x)\right] wy + \frac{\epsilon\alpha_4}{6} y^2.$$

It is clear that the expressions given by $v_6, \cdots, v_{14}$ and v_{15} represent certain specific quadratic forms, respectively.

Making use of the basic information on the positive semi-definite of a quadratic form, one can easily conclude that $v_6 \geq 0$, $v_7 \geq 0$, $v_8 \geq 0$, $v_9 \geq 0$, $v_{10} \geq 0$, $v_{11} \geq 0$, $v_{12} \geq 0$, $v_{13} \geq 0$, $v_{14} \geq 0$ and $v_{15} \geq 0$ provided that

$$(\psi - \alpha_1) \leq \frac{\epsilon}{12\alpha_1^2}, \quad (\psi - \alpha_1) \leq \frac{\epsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{14\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \quad (\psi - \alpha_1) \leq \frac{\epsilon\alpha_4}{12\delta^2},$$

$$\left[\frac{\phi(z)}{z} - \alpha_3\right]^2 \leq \frac{2\epsilon^2\alpha_2\alpha_4}{21\delta^2}, \quad \left[\frac{\phi(z)}{z} - \alpha_3\right]^2 \leq \frac{\epsilon\epsilon_o\alpha_2}{7},$$

$$\left[\frac{f(w)}{w} - \alpha_2\right]^2 \leq \frac{2\epsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{21\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \quad \left[\frac{f(w)}{w} - \alpha_2\right]^2 \leq \frac{\epsilon^2\alpha_4}{9\delta^2},$$

$$\left[\alpha_5 - h'(x)\right]^2 \leq \frac{2\epsilon^2\alpha_2\alpha_4}{21\alpha_1^2}, \quad \left[\alpha_5 - h'(x)\right]^2 \leq \frac{\epsilon^2\alpha_4}{9}$$

and

$$\left[\alpha_4 - g'(y)\right]^2 \leq \frac{2\epsilon^2\alpha_2}{21},$$

respectively.

Thus in view of the above discussion and inequality (2.13), it follows that

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} \leq & -\left[\frac{\epsilon\alpha_4}{6} - \frac{\delta(L+M)}{2} r\right] y^2 - \left[\frac{\epsilon\alpha_2}{7} - \frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r\right] z^2 \\
& - \left[\frac{\epsilon}{6} - \left\{\frac{\alpha_1(L+M)}{2} + \lambda\right\} r\right] w^2 - \left[\epsilon_o - \left(\frac{L+M}{2} + \rho\right) r\right] u^2 \\
& - \left[\lambda - \left\{\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} L\right\}\right] \int_{t-r}^{t} w^2(s)\,\mathrm{d}s \\
& - \left[\rho - \left\{\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} M\right\}\right] \int_{t-r}^{t} u^2(s)\,\mathrm{d}s.
\end{aligned} \tag{2.14}$$

So we can choose the constants λ and ρ as the following

$$\lambda = \frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4 (\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} L$$

and

$$\rho = \frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4 (\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} M,$$

then the inequality in (2.14) implies that

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \le & -\left[\frac{\epsilon\alpha_4}{6} - \frac{\delta(L+M)}{2} r\right] y^2 - \left[\frac{\epsilon\alpha_2}{7} - \frac{\alpha_4 (L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r\right] z^2 \\ & -\left[\frac{\epsilon}{6} - \left\{\frac{\alpha_1 (L+M)}{2} + \lambda\right\} r\right] w^2 - \left[\epsilon_o - \left(\frac{L+M}{2} + \rho\right) r\right] u^2. \end{aligned} \tag{2.15}$$

Hence one can easily get from (2.15) that

$$\frac{\mathrm{d}V}{\mathrm{d}t}(x_t, y_t, z_t, w_t, u_t) \le -D_8 y^2 - D_9 z^2 - D_{10} w^2 - D_{11} u^2 \le 0, \tag{2.16}$$

for some positive constants $D_i, (i = 8, \cdots, 11)$, provided that

$$r < \min\left\{\frac{\epsilon\alpha_4}{6\delta(L+M)}, \frac{\epsilon\alpha_2 (\alpha_1\alpha_4 - \alpha_5)}{7\alpha_4 (L+M)(\alpha_1\alpha_2 - \alpha_3)}, \frac{\epsilon}{6\alpha_1 (L+M) + 12\lambda}, \frac{\epsilon_o}{L+M+2\rho}\right\}.$$

Finally, it follows that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ if and only if $x_t = y_t = z_t = w_t = u_t = 0$, $\frac{\mathrm{d}V(\varphi)}{\mathrm{d}t} < 0$ for $\varphi \ne 0$ and $V(\varphi) \ge u(|\varphi(0)|) \ge 0$.

Thus all the conditions of Theorem 2.2 are satisfied. This shows that the zero solution of (1.1) is globally asymptotically stable.

Then the proof of Theorem 2.3 is completed.

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