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Winter Map Inverses

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Abstract

We demonstrate the functional inverse of a Winter map, which is an analog of the exponential map, for Lie algebras over fields of prime characteristic.

Keywords

Prime-Characteristic Lie Algebras, Prime-Characteristic Lie Groups

"Historically," note Strade and Farnsteiner in [1], "Lie algebras emerged from the study of Lie groups." In Section 1.1 of [1], they give a simple example of the close connection between Lie algebras and Lie groups. In prime characteristic, David Winter [2] has defined maps which mimic the zero-characteristic exponential maps. See also Lemma 1.2 of [3]. In this paper, we focus on the following "Winter maps": if x is an element of a characteristic-p Lie algebra L such that $(\mathrm{ad}_L x)^p = 0,$ we set

$$\xi(\mathrm{ad}_L x) = I + \mathrm{ad}_L x + \frac{(\mathrm{ad}_L x)^2}{2!} + \frac{(\mathrm{ad}_L x)^3}{3!} + \cdots + \frac{(\mathrm{ad}_L x)^{p-1}}{(p-1)!}$$

where I is the identity transformation of L. Such ad-nilpotent elements of degree less than p do exist in some graded Lie algebras, as can be seen from Lemma 2.3 and Proposition 2.7 of Chapter 4 of [1], as well as from Lemma 1 of [4]; of course, it is well known that non-zero-root vectors of simple classical-type Lie algebras are ad-nilpotent of degree less than or equal to four.

We will show here that for $x \in L$ such that $(\mathrm{ad}_L x)^p = 0,$ the inverse of $\xi(\mathrm{ad}_L x)$ as a linear transformation of L is $\xi(\mathrm{ad}_L(-x))$, so that such transformations generate a group G of linear transformations of L. We will also show that $\lambda(\xi(\mathrm{ad}_L x)) = \mathrm{ad}_L x,$ where, for g a linear transformation of L, and I as above, we define

$$\lambda(g) = (g - I) - \frac{(g-I)^2}{2} + \frac{(g-I)^3}{3} - \cdots \frac{(g-I)^{p-1}}{(p-1)!} \tag{1}$$

Thus, like $\ln(x)$ and $\exp(x)$, λ is, in a sense, the functional inverse of ξ.

Lemma 1 *If* x *and* c *are elements of* L *such that* $(\mathrm{ad}_L x)^p = 0,$ *and* $(\mathrm{ad}_L c)^p = 0,$ *then*

$$\xi(x)\xi(c) = \sum_{i=0}^{p-1}\sum_{j=0}^{i}\frac{(\mathrm{ad}_L x)^j (\mathrm{ad}\, c)^{i-j}}{j!(i-j)!} + \sum_{i=p}^{2p-2}\sum_{j=i-(p-1)}^{p-1}\frac{(\mathrm{ad}_L x)^j (\mathrm{ad}\, c)^{i-j}}{j!(i-j)!}$$

Proof. We group terms with respect to total degree in $\mathrm{ad}_L x$ and $\mathrm{ad}_L c$ □

Lemma 2 *Let* $a,b \in F$, *and suppose that* x *is an element of* L *such that* $(\mathrm{ad}_L x)^p = 0.$ *then*

$$\xi((a)\mathrm{ad}_L x)\xi((b)\mathrm{ad}_L x) = \xi((a+b)\mathrm{ad}_L x).$$

Proof. We have by Lemma 1 that $\xi((a)\mathrm{ad}_L x)\xi((b)\mathrm{ad}_L x)$ equals

$$\sum_{i=0}^{p-1}\sum_{j=0}^{i}\frac{a^j b^{i-j}(\mathrm{ad}_L x)^i}{j!(i-j)!} + 0$$

which we can write in terms of binomial coefficients as

$$\sum_{i=0}^{p-2}\frac{(\mathrm{ad}_L x)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}a^j b^{i-j}$$

By the Binomial Theorem, the above expression is equal to

$$\sum_{i=0}^{p-1}\frac{(\mathrm{ad}_L x)^i}{i!}(a+b)^i$$

which we can rewrite as

$$\sum_{i=0}^{p-1}\frac{(\mathrm{ad}_L (a+b)x)^i}{i!}$$

and recognize as $\xi((a+b)\mathrm{ad}_L x).$ □

Lemma 3 *For any integer* $n \geqq 2$ *and any integer* j, $0 < j < n$, *we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j = 0$$

Proof. We proceed by induction on n and j. When $n = 2$, we must have $j = 1$, and we have $0 - 2 + 2 = 0.$ For any $n > 2$, when $j = 1$, we have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^1 &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n}{k+1}(k+1)\\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\frac{n!}{(k+1)!(n-(k+1))!}(k+1)\\ &= (-n)\sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!}{k!((n-1)-k)!}\\ &= (-n)\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\\ &= (-n)\cdot(1-1)^{n-1} = 0.\end{aligned}$$

Now, for any $n \geqq 3$ and any positive integer j less than n, suppose that $\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^i = 0$ for all positive i less than j. Then we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j=\sum_{k=1}^{n}(-1)^k\frac{n!}{k!(n-k)!}k^j=(-n)\sum_{k=1}^{n}(-1)^{k-1}\frac{(n-1)!}{(k-1)!(n-k)!}k^{j-1}$$
$$=(-n)\sum_{k=1}^{n}(-1)^{k-1}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}k^{j-1}$$
$$=(-n)\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}k^{j-1}$$
$$=(-n)\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(k+1)^{j-1}$$
$$=(-n)\left\{\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}\sum_{i=1}^{j-1}\binom{j-1}{i}k^i+\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\right\}$$
$$=(-n)\sum_{i=1}^{j-1}\binom{j-1}{i}\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}k^i+0=(-n)\sum_{i=1}^{j-1}\binom{j-1}{i}\cdot 0=0$$

by induction, and the fact that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=(1-1)^n=0^n=0$ (the " $j=0$ case"). □

Lemma 4 *Let* x *be an element of* L *such that* $(\mathrm{ad}\,_L x)^p=0$. *Define*

$$\delta(\mathrm{ad}\,_L x)=\sum_{i=0}^{p-2}\frac{(\mathrm{ad}\,_L x)^{i+1}}{(i+1)!} \tag{2}$$

Then for any positive integer n *less than* p,

$$\left(\delta(\mathrm{ad}\,_L x)\right)^n=\sum_{t=0}^{p-n-1}\frac{(\mathrm{ad}\,_L x)^{t+n}}{(t+n)!}\left(\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j(n-j)^{t+n}\right) \tag{3}$$

Proof. We proceed by induction on n. Since when $n=1$, (3) is just (2), the initial step of the induction proof is established. Suppose (3) is true for $n=k\geqq 1$. Then $\left(\delta(\mathrm{ad}\,_L x)\right)^{k+1}$ equals

$$\left(\sum_{s=0}^{p-k-1}\frac{(\mathrm{ad}\,_L x)^{s+k}}{(s+k)!}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j(k-j)^{s+k}\right)\left(\sum_{i=0}^{p-2}\frac{(\mathrm{ad}\,_L x)^{i+1}}{(i+1)!}\right)$$

We group terms with respect to total degree ($t+k+1$, in this case) in $\mathrm{ad}\,_L x$ and get that

$$\left(\delta(\mathrm{ad}\,_L x)\right)^{k+1}=\sum_{t=0}^{p-(k+1)-1}\sum_{r=0}^{t}\frac{(\mathrm{ad}\,_L x)^{r+k}(\mathrm{ad}\,_L x)^{t-r+1}}{(r+k)!(t-r+1)!}\left(\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j(k-j)^{r+k}\right).$$

Rewriting the above expression using another binomial coefficient, we get that $\left(\delta(\mathrm{ad}\,_L x)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1}\frac{(\mathrm{ad}\,_L x)^{t+k+1}}{(t+k+1)!}\sum_{r=0}^{t}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^{r+k}\binom{t+k+1}{r+k}.$$

We change the order of summation to get

$$\left(\delta(\mathrm{ad}\,_L x)\right)^{k+1}=\sum_{t=0}^{p-(k+1)-1}\frac{(\mathrm{ad}\,_L x)^{t+k+1}}{(t+k+1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\sum_{r=0}^{t}\binom{t+k+1}{r+k}(k-j)^{r+k}.$$

We replace the index of summation r by $r-k$ to get

$$\left(\delta(\mathrm{ad}\,_L x)\right)^{k+1}=\sum_{t=0}^{p-(k+1)-1}\frac{(\mathrm{ad}\,_L x)^{t+k+1}}{(t+k+1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\sum_{r=k}^{t+k}\binom{t+k+1}{r}(k-j)^{r}.$$

Adding and subtracting terms, we get

$$\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1} = \sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \left\{ \sum_{r=0}^{t+k+1} \binom{t+k+1}{r} (k-j)^r - \sum_{r=0}^{k-1} \binom{t+k+1}{r} (k-j)^r - \sum_{r=t+k+1}^{t+k+1} \binom{t+k+1}{r} (k-j)^r \right\}$$

Setting $q = k - j$, we see, as in the proof of Lemma 3, that when $r \geq 1$,

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^r = (-1)^k k \sum_{q=1}^{k} (-1)^q \binom{k-1}{q-1} q^{r-1} = 0$$

by that same Lemma 3. Thus,

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \sum_{r=0}^{k-1} \binom{t+k+1}{r} (k-j)^r = \sum_{r=0}^{k-1} \binom{t+k+1}{r} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^r = \sum_{r=0}^{k-1} \binom{t+k+1}{r} \cdot 0 = 0,$$

so from the Binomial Theorem, we get that $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \left\{ (k-j+1)^{t+k+1} - 1 - (k-j)^{t+k+1} \right\}.$$

We now distribute to get that $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \left\{ \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j (k-j+1)^{t+k+1} + (-1)^k - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j (k-j)^{t+k+1} \right\}.$$

We replace the latter index of summation j by $j-1$ to get that $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \left\{ \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j (k-j+1)^{t+k+1} - \sum_{j=1}^{k} \binom{k}{j-1} (-1)^{j-1} (k+1-j)^{t+k+1} + (-1)^k \right\}.$$

We change the order of summation and factor to get that $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \left\{ (k+1)^{t+k+1} + \sum_{j=1}^{k-1} \left\{ \binom{k}{j} (-1)^j + \binom{k}{j-1} (-1)^j \right\} (k+1-j)^{t+k+1} - \binom{k}{k-1} (-1)^{k-1} - (-1)^{k-1} \right\}.$$

By binomial arithmetic $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \left\{ (k+1)^{t+k+1} + \sum_{j=1}^{k-1} \binom{k+1}{j} (-1)^j (k+1-j)^{t+k+1} - (k+1)(-1)^{k-1} \right\}.$$

The above displayed formula is just (3) for $n = k+1$; *i.e.*, $\left(\delta\left(\operatorname{ad}_L x\right)\right)^{k+1}$ equals

$$\sum_{t=0}^{p-(k+1)-1} \frac{\left(\operatorname{ad}_L x\right)^{t+k+1}}{(t+k+1)!} \left\{ \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} (k+1-j)^{t+k+1} \right\}.$$

Thus, the induction step is complete. □

Theorem *The linear transformation* $\xi\left(\operatorname{ad}_L x\right)$ *of* L *has* $\xi\left(\operatorname{ad}_L(-x)\right)$ *as its inverse, whereas the map* ξ *of* $\operatorname{ad} L$ *to the group of non-singular linear transformations of* L *has* λ *as its inverse, in the sense that* (a). $\xi(\mathrm{ad}_L x)\xi(\mathrm{ad}_L(-x)) = I$, and

(b). $\lambda(\xi(\mathrm{ad}_L x)) = \mathrm{ad}_L x$.

Proof. (a) If, in Lemma 2, we let $a = 1$ and $b = -1$, we see that (a) is true.
(b) Since $\xi(\mathrm{ad}_L x) - I$ equals the $\delta(\mathrm{ad}_L x)$ of Lemma 4, we have that $\lambda(\xi(\mathrm{ad}_L x))$ equals

$$(\delta(\mathrm{ad}_L x)) - \frac{(\delta(\mathrm{ad}_L x))^2}{2} + \frac{(\delta(\mathrm{ad}_L x))^3}{3} - \cdots - \frac{(\delta(\mathrm{ad}_L x))^{p-1}}{p-1}$$

which, by Lemma 4 equals

$$\sum_{n=1}^{p-1} \frac{(-1)^{n+1}}{n} \left\{ \sum_{t=0}^{p-n-1} \frac{(\mathrm{ad}_L x)^{t+n}}{(t+n)!} \left(\sum_{j=0}^{n-1} \binom{n}{j} (-1)^j (n-j)^{t+n} \right) \right\}$$

We replace the index t by $t-n$ to get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{n=1}^{p-1} \frac{(-1)^{n+1}}{n} \left\{ \sum_{t=n}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \left(\sum_{j=0}^{n-1} \binom{n}{j} (-1)^j (n-j)^t \right) \right\}$$

We change the order of summation to get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{n=1}^{t} \frac{(-1)^{n+1}}{n} \left\{ \sum_{j=0}^{n-1} \binom{n}{j} (-1)^j (n-j)^t \right\}$$

We replace the index j by $n-j$ to get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{n=1}^{t} \frac{(-1)^{n+1}}{n} \left\{ \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} j^t \right\}$$

We cancel an n and a j and combine the -1 factors to get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{n=1}^{t} \left\{ \sum_{j=1}^{n} \binom{n-1}{j-1} (-1)^{j-1} j^{t-1} \right\}$$

We replace the index n by $n+1$ and we replace the index j by $j+1$, and we get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{n=0}^{t-1} \left\{ \sum_{j=0}^{n} \binom{n}{j} (-1)^j (j+1)^{t-1} \right\}$$

We change the order of summation to get that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{j=0}^{t-1} (-1)^j (j+1)^{t-1} \sum_{n=j}^{t-1} \binom{n}{j}$$

We now appeal to a little more binomial arithmetic to observe that since $\binom{j}{j} = \binom{j+1}{j+1}$ and $\binom{t}{j} + \binom{t}{j+1} = \binom{t+1}{j+1}$, it follows by induction that

$$\sum_{n=j}^{t-1} \binom{n}{j} = \binom{t}{j+1}$$

from which we obtain that

$$\lambda(\xi(\mathrm{ad}_L x)) = \sum_{t=1}^{p-1} \frac{(\mathrm{ad}_L x)^t}{t!} \sum_{j=0}^{t-1} (-1)^j (j+1)^{t-1} \binom{t}{j+1}$$

We replace the index j by $j-1$ to get that

$$\lambda\left(\xi\left(\mathrm{ad}_{L}x\right)\right)=\sum_{t=1}^{p-1}\frac{\left(\mathrm{ad}_{L}x\right)^{t}}{t!}\sum_{j=1}^{t}\left(-1\right)^{j-1}\binom{t}{j}j^{t-1}$$

Finally, we use Lemma 3 to see that we are left with $\mathrm{ad}_{L}x$ □

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