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On a Class of Gronwall-Bellman Type Inequalities

Zareen A. Khan

Department of Mathematics, Princess Noura Bint Abdurehman University, Riyadh, KSA
Email: dr.zareenkhan@ymail.com

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Abstract

The aim of the present paper is to establish some new integral inequalities of Gronwall type involving functions of two independent variables which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential and integral equations.

Keywords

Integral Inequalities, Two Independent Variables, Partial Differential Equations, Nondecreasing, Nonincreasing

1. Introduction

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of the various types (please, see Gronwall [1] and Guiliano [2]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Numerous applications to the existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [4], Bihari [5], and Langenhop [6]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in one or two independent real variables [1]-[14]. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

In [14], Pachpatte investigated the following inequality:

Lemma: Let u, f and g be nonnegative continuous functions defined on R_+ and c be nonnegative constant. Let $w(t,r)$ be a nonnegative continuous function defined for $t \in R_+, 0 \le r < \infty$, and monotonic nondecreasing with respect to t for ant fixed $t \in R_+$. If

$$u^2(t) \le c^2 + 2\int_0^t u(s)\,w(s,u(s))\,\mathrm{d}s$$

For $t \in R_+$, then

$$u(t) \le r(t), \qquad t \in R_+$$

where $r(t)$ is the maximal solution of

$$r'(t) = w(t,r(t)), \qquad r(0) = c$$

For $t \in R_+$.

2. Main Results

Theorem 2.1: Let $u(x,y)$ and $a(x,y)$ be nonnegative continuous functions defined on $x,y \in R_+$. Let $w(x,y)$ be a positive continuous and nondecreasing functions in both variables and defined for $x,y \in R_+$. If

$$u(x,y) \le w(x,y) + \int_0^x\int_0^y u(s,t)\,\mathrm{d}t\mathrm{d}s + \int_0^x\int_0^y a(s,t)\,u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.1}$$

Then

$$u(x,y) \le E(x,y)\left[w(x,y) + r(x,y)\right], \quad \forall t \in R_+$$

where $0 < p < 1 \cdot p + q = 1, q > 0$ and

$$E(x,y) = \left[1 + (1-p)\int_0^x\int_0^y a(s,t)\,m^{-q}(s,t)\,\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-p}}, \tag{2.2}$$

and $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y) = E(s,t)\left[w(s,t) + r(s,t)\right], \quad r_x(x,0) = 0, \; r(0,y) = 0 \tag{2.3}$$

Proof: Define a function $m(x,y)$ by the right-hand side of (2.1).Then

$$m(x,y) = w(x,y) + \int_0^x\int_0^y u(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.4}$$

By using (2.4) in (2.1), we get

$$u(x,y) \le m(x,y) + \int_0^x\int_0^y a(s,t)\,u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.5}$$

Since $m(x,y)$ is a positive continuous and nondecreasing function, then

$$\frac{u(x,y)}{m(x,y)} \le 1 + \int_0^x\int_0^y a(s,t)\,m^{-q}(s,t)\left[\frac{u(s,t)}{m(s,t)}\right]^p \mathrm{d}t\mathrm{d}s \tag{2.6}$$

Let

$$\frac{u(x,y)}{m(x,y)} \le v(x,y) \tag{2.7}$$

From (2.6) and (2.7), we observe that

$$v(x,y) = 1 + \int_0^x\int_0^y a(s,t)\,m^{-q}(s,t)\,v^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.8}$$

And

$$v(0,y) = 1, \quad v(x,0) = 1, \quad \text{and} \quad v_x(x,0) = 0, \quad v_y(0,y) = 0 \tag{2.9}$$

Differentiating both sides of (2.8) with respect to x and y, we get

$$v_{xy}(x,y) \le a(x,y)m^{-q}(x,y)v^{p}(x,y)$$

$$\frac{\partial}{\partial y}\left[\frac{v_x(x,y)}{v^p(x,y)}\right] \le a(x,y)m^{-q}(x,y) \tag{2.10}$$

By keeping first x fixed in (2.10) and set $y=t$ and integrate from 0 to y then again keeping y fixed, set $x=s$ and integrate from 0 to x respectively and using (2.9), we get

$$v(x,y) \le \left[1+(1-p)\int_0^x\int_0^y a(s,t)m^{-q}(s,t)\,\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-p}} = E(x,y) \tag{2.11}$$

From (2.7) and (2.11), it is clear that

$$u(x,y) \le E(x,y)m(x,y)$$

From (2.4), it can be restated as

$$u(x,y) \le E(x,y)\left[w(x,y)+r(x,y)\right],$$

where $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y) = E(s,t)\left[w(s,t)+r(s,t)\right],\quad r_x(x,0)=0,\ r(0,y)=0$$

This completes the proof.

Theorem 2.2: Let $u(x,y), a(x,y), w(x,y)$ be defined as in Theorem 2.1. If

$$u(x,y) \le w(x,y)+\int_0^x\int_0^y u^p(s,t)\,\mathrm{d}t\mathrm{d}s+\int_0^x\int_0^y a(s,t)u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.12}$$

Then

$$u(x,y) \le E(x,y)\left[w(x,y)+r(x,y)\right],\quad \forall t \in R_+$$

where $0<p<1\cdot p+q=1,\ q>0$ and

$$E(x,y) = \left[1+(1-p)\int_0^x\int_0^y a(s,t)m^{-q}(s,t)\,\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-p}}, \tag{2.13}$$

where $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y) = \left[E(s,t)\left[w(s,t)+r(s,t)\right]\right]^{p},\quad r_x(x,0)=0,\ r(0,y)=0$$

Proof: Define a function $m(x,y)$ by the right-hand side of (2.12).Then

$$m(x,y) = w(x,y)+\int_0^x\int_0^y u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.14}$$

By using (2.14) in (2.12), we get

$$u(x,y) \le m(x,y)+\int_0^x\int_0^y a(s,t)u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.15}$$

By following the same steps of Theorem 2.1 from (2.5)-(2.11), we get

$$v(x,y) \le \left[1+(1-p)\int_0^x\int_0^y a(s,t)m^{-q}(s,t)\,\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-p}} = E(x,y) \tag{2.16}$$

From (2.7), (2.14) and (2.16), we observe that

$$u(x,y) \le E(x,y)\left[w(x,y)+r(x,y)\right],$$

where $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y)=\left[E(s,t)\left[w(s,t)+r(s,t)\right]\right]^p, \quad r_x(x,0)=0,\ r(0,y)=0$$

This completes the proof.

Theorem 2.3: Let $u(x,y)$, $a(x,y)$, $w(x,y)$ be defined as in Theorem 2.1. If

$$u^p(x,y) \le w(x,y)+\int_0^x\int_0^y u(s,t)\,\mathrm{d}t\mathrm{d}s+\int_0^x\int_0^y a(s,t)u^p(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.17}$$

Then

$$u(x,y) \le \left[E(x,y)\left[w(x,y)+r(x,y)\right]\right]^{\frac{1}{p}}, \quad \forall t \in R_+$$

where $p>1$. and

$$E(x,y)=\exp\left[\int_0^x\int_0^y a(s,t)\,\mathrm{d}t\mathrm{d}s\right], \tag{2.18}$$

where $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y)=E(s,t)\left[w(s,t)+r(s,t)\right], \quad r_x(x,0)=0,\ r(0,y)=0$$

Proof: Define a function $z(x,y)$ by the right-hand side of (2.17).Then

$$u^p(x,y) \le z(x,y) \Rightarrow u(x,y) \le z^{\frac{1}{p}}(x,y) \tag{2.19}$$

where $z(x,y)=m(x,y)+\int_0^x\int_0^y a(s,t)u^p(s,t)\,\mathrm{d}t\mathrm{d}s$

By using (2.19) in the above equation, we get

$$z(x,y) \le m(x,y)+\int_0^x\int_0^y a(s,t)z(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.20}$$

and

$$m(x,y)=w(x,y)+\int_0^x\int_0^y u(s,t)\,\mathrm{d}t\mathrm{d}s \tag{2.21}$$

Since $m(x,y)$ is a positive continuous and nondecreasing function, then from (2.20),

$$\frac{z(x,y)}{m(x,y)} \le 1+\int_0^x\int_0^y a(s,t)\frac{z(s,t)}{m(s,t)}\,\mathrm{d}t\mathrm{d}s \tag{2.22}$$

Let

$$\frac{z(x,y)}{m(x,y)} \le v(x,y) \tag{2.23}$$

where

$$v(x,y)=1+\int_0^x\int_0^y a(s,t)\frac{z(s,t)}{m(s,t)}\,\mathrm{d}t\mathrm{d}s \tag{2.24}$$

And

$$v(0,y)=1, \quad v(x,0)=1, \quad \text{and } v_x(x,0)=0, \quad v_y(0,y)=0 \tag{2.25}$$

Differentiating both sides of (2.24) with respect to x and y, and from (2.23), we get

$$v_{xy}(x,y) \le a(x,y)v(x,y)$$

$$\frac{\partial}{\partial y}\left[\frac{v_x(x,y)}{v(x,y)}\right]\le a(x,y) \tag{2.26}$$

By keeping first x fixed in (2.26) and set $y=t$ and integrate from 0 to y then again keeping y fixed, set $x=s$ and integrate from 0 to x respectively and using (2.25), we get

$$v(x,y)\le \exp\left[\int_0^x\int_0^y a(s,t)\,\mathrm{d}t\mathrm{d}s\right]=E(x,y) \tag{2.27}$$

From (2.19), (2.23) and (2.27), it is clear that

$$u(x,y)\le\left[E(x,y)\left[w(x,y)+r(x,y)\right]\right]^{\frac{1}{p}},$$

where $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y)=E(s,t)\left[w(s,t)+r(s,t)\right],\quad r_x(x,0)=0,\ r(0,y)=0$$

This completes the proof.

Application: As an application, let us consider the bound on the solution of a nonlinear hyperbolic partial differential equation of the form

$$u_{xy}(x,y)=A\left[x,y,u(s,t),v(s,t)\right] \tag{2.28}$$

with the given boundary conditions

$$u(x,y_0)=a_1(x),\ u(x_0,y)=a_2(y),\quad a_1(x_0)=a_2(y_0)=0 \tag{2.29}$$

where $u\in C[D,R],\ A\in C[D\times R,R],$ and $D=\{(x,y),x\ge 0,y\ge 0\}$ such that

$$\left|A(x,y,u,v)\right|\le|u|+|v| \tag{2.30}$$

$$|v|\le g(x,y)\left|u^p\right| \tag{2.31}$$

where $u(x,y)$ and $g(x,y)$ be nonnegative continuous functions defined on a domain D $0<p<1\cdot p+q=1,\ q>0$. The Equation (2.28) with (2.29) is equivalent to the integral equation

$$u(x,y)=a_1(x)+a_2(y)+\int_{x_0}^{x}\int_{y_0}^{y}A(s,t,u(s,t),v(s,t))\,\mathrm{d}t\mathrm{d}s \tag{2.32}$$

Let $u(x,y)$ be any solution of (2.28) with (2.29) and taking absolute values of both sides, we get

$$|u(x,y)|=|a_1(x)+a_2(y)|+\int_{x_0}^{x}\int_{y_0}^{y}\left|A(s,t,u(s,t),v(s,t))\right|\mathrm{d}t\mathrm{d}s, \tag{2.33}$$

Using (2.30)-(2.32) in (2.33) and assuming that $|a_1(x)|+|a_2(y)|\le w(x,y)$, where $w(x,y)$ be a positive continuous and nondecreasing function defined in the respective domain, we have

$$|u(x,y)|\le w(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}|u(s,t)|\mathrm{d}t\mathrm{d}s+\int_{x_0}^{x}\int_{y_0}^{y}g(s,t)\left|u^p(s,t)\right|\mathrm{d}t\mathrm{d}s$$

Then $|u(x,y)|\le E^*(x,y)\left[w(x,y)+r(x,y)\right],$

where $E^*(x,y)=\left[1+(1-p)\int_0^x\int_0^y a(s,t)m^{-q}(s,t)\,\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-p}}$, and $r(x,y)$ is the maximal solution of

$$r_{xy}(x,y)=E^*(s,t)\left[w(s,t)+r(s,t)\right],\quad r_x(x,0)=0,\ r(0,y)=0,$$

The remaining proof will be the same as the proof of Theorem 2.1 with suitable modifications.

We note that Theorem 2.1 can be used to study the stability, boundedness and continuous dependence of the solutions of (2.28).

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