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Convergence Theorem of Hybrid Iterative Algorithm for Equilibrium Problems and Fixed Point Problems of Finite Families of Uniformly Asymptotically Nonexpansive Semigroups

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Abstract

Throughout this paper, we introduce a new hybrid iterative algorithm for finding a common element of the set of common fixed points of a finite family of uniformly asymptotically nonexpansive semigroups and the set of solutions of an equilibrium problem in the framework of Hilbert spaces. We then prove the strong convergence theorem with respect to the proposed iterative algorithm. Our results in this paper extend and improve some recent known results.

Keywords

Hybrid Iterative Algorithm, Uniformly Asymptotically Nonexpansive Semigroups, Equilibrium Problem, Common Fixed Point

1. Introduction

Recall the following equilibrium problem. Let C be a closed convex subset of a real Hilbert space H with inner produce $\langle .,. \rangle$ and norm $\|.\|$. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for ϕ is to to find $x \in C$ such that

$$\phi(x, y) \geq 0, \qquad \forall y \in C,$$

the set of solutions is denoted by $EP(\phi)$.

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A mapping T of a normed space H into itself is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for each $x, y \in H$. We denote by $F(T)$ the set of fixed point of T. Given a mapping $T: C \to H$, let $\phi(x,y) = \langle Tx, y-x \rangle$ for all $y \in C$.Then $x \in EP(\phi)$ if and only if $\langle Tx, y-x \rangle \ge 0$ for all $y \in C$, *i.e.*, x is a solution of the variational inequality, there are several other problems, for example, the complementarity problem, minimax problems, the Nash equilibrium problem in noncooperative games, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP; see, for example ([1]-[3]) and references therein.

Iterative methods for finding fixed points of nonexpansivemappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem (see [4]-[7]), the split feasibility problem (see [8]-[10]) and image recovery and signal processing (see [6]).

In 1953, Mann [11] introduced the following iterative process to approximate a fixed point of a nonexpansive single valued mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) Tx_n, \quad \forall n \ge 1,$$

where the initial point x_0 is taken in C arbitrarily and α_n is a sequence in $[0,1]$. However, we note that Mann's iteration process has only weak convergence. To obtain strong converges for Mann iteration, Nakajo and Takahashi [12] and Takahashi *et al.* [13] introduce some hybrid iterative process. Motivated by Suzuki's result [14] and Nakajo-Takahashi's results [12].

On the other hand, Tada and Takahashi [15] introduce a new iterative method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H.

A family $\mathbb{T} := \{T(s): s \ge 0\}$ of mappings on a closed convex subset C of a Hibert space H is called a nonexpansive semigroup if it satisfies the following conditions:

1) $T(0)x = x$ for all $x \in C$;
2) $T(s+t) = T(s)T(t)$ for all $s, t \ge 0$;
3) $\|T(s)x - T(s)y\| \le \|x-y\|$ for all $x, y \in C$ and $s \ge 0$,
4) for all $x \in C$, $s \to T(s)x$ is continuous.

Takahashi and Chen [16] proved a strong convergence theorem for nonexpansive semigroups in Hilbert spaces by hybrid method in themathematical programming. Recently Saejung [17] improved the result in [16].

Takahashi's result gives us new idea that a finite family of uniformly asymptotically nonexpansive semigroups is introduced.

Definition 1.1 *A family* $\mathbb{T} := \{T(s): s \ge 0\}$ *of mappings on a closed convex subset* C *of a Hibert space* H *is called an uniformly asymptotically nonexpansive semigroup with sequence* k_n ($k_n \ge 1$ *and* $\lim_{n\to\infty} k_n = 1$) *if it satisfies the following conditions*:

1) $T(0)x = x$ for all $x \in C$;
2) $T(s+t) = T(s)T(t) = T(t)T(s)$ for all $s, t \ge 0$;
3) $\|T^n(s)x - T^n(s)y\| \le k_n \|x-y\|$ for all $x, y \in C$, $s \ge 0$, $n \ge 1$
4) for all $x \in C$, $s \to T(s)x$ is continuous.

In this paper, we introduce a new hybrid iterative process for finding a common element of the set of common fixed points of a finite family of uniformly asymptotically nonexpansive semigroups and the set of solutions of an equilibrium problem in the framework of Hilbert spaces. Then we prove some strong convergence theorems of the proposed iterative process. Our results generalize results of Tada and Takahashi [15], Takahashi *et al.* [13], He and Chen [16] and Saejung [17].

2. Preliminaries

Throughout the paper, we denote weak convergence of $\{x_n\}$ by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$. Let C be a closed convex subset of H, we use $\mathbb{T}$ to denote the common fixed points set of the semigroup $\mathbb{T} := \{T(s): s \ge 0\}$. *i.e.*, $F(\mathbb{T}) = \{x \in C : T(s)x = x, \forall s \ge 0\}$.

Next, We present an example of an uniformly asymptotically nonexpansive semigroup.

Example 2.1 *As an example, we consider the nonempty closed convex subset* $C=[,+\infty)$ *of a Hilbert space* $\mathbb{R}$. *define* $T(s)x=\exp(-s)x$. *Observe that* $T(s)$ *is an uniformly asymptotically nonexpansive semigroup.*

For every point $\in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x-P_C x\|\le\|x-y\|,\quad \forall y\in C,$$

that is, $P_C x=\inf_{y\in C}\|x-y\|$. P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, following the inequality holds:

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|,\quad \forall y\in H, y\ne x.$$

To prove our result, we recall the following Lemma.

Lemma 2.1 (*see* [18]). *Let* C *be a closed convex subset of* H. *Given* $x\in H$ *and a point* $z\in C$. *Then* $z=P_C x$ *if and only if* $\langle x-z,y-z\rangle\le 0$ *for all* $y\in C$.

Lemma 2.2 (*see* [12]). *Let* C *be a closed convex subset of* H. *Then for all* $x\in H$ *and* $y\in C$ *we have*

$$\|y-P_C\|^2+\|x-P_C\|^2\le\|x-y\|^2.$$

Lemma 2.3 (*see* [18]). *Let* H *be a real Hilbert space, there hold the following identities*:

1) $\|tx+(1-t)y\|=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|$, for all $t\in[0,1]$ and $x,y\in H$.

2) $\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle$, for all $x,y\in H$.

Lemma 2.4 (*see* [19]) *Let* H *be a real Hilbert space. For* $i,j=1,2,\cdots,N$,

$$\left\|\sum_{m=1}^{N}a_m\right\|^2=\sum_{m=1}^{N}a_m\|x_m\|^2-\sum_{i\ne j}a_i a_j\|x_i-x_j\|^2$$

$a_i,a_j\in[0,1]$ with $\sum_{m=1}^{N}a_m=1$.

For solving the equilibrium problem, let us assume the following conditions for a bifunction ϕ (see [1]):

1) $\phi(x,x)=0$, for all $x\in C$.

2) $\phi(x,y)+\phi(y,x)\le 0$, for all $x,y\in C$.

3) For each $x,y,z\in C$,

$$\limsup_{t\to 0^+}\phi(tz+(1-t)x,y)\le\phi(x,y).$$

4) $\phi(x,.)$ is convex and lower semicontinuous for each $x\in C$.

Lemma 2.5 (*see* [1]) *Let* C *be a nonempty closed convex subset of* H *and let* ϕ *be a bifunction of* $C\times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $r>0$ *and* $x\in H$. *Then, there exists* $z\in C$ *such that*

$$\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\quad \forall y\in C.$$

Lemma 2.6 *Let* $\phi:C\to C$ *satisfies* (A1)-(A4). *For* $r>0$ *and* $x\in H$, *define a mapping* $T_r:H\to C$ *as follows*:

$$T_r x=\left\{z\in C:\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\forall y\in C\right\}.$$

Then, the following holds:

1) T_r is single valued;

2) T_r is firmly nonexpansive, *i.e.*, for any $x,y\in H$, $\|T_r x-T_r y\|^2\le\langle T_r x-T_r y,x-y\rangle$;

3) $F(T_r)=EP(\phi)$;

4) $EP(\phi)$ is closed and convex.

In 2013, Mohammad, E. introduce a new hybrid iterative process for finding a common element of the set of common fixed points of a finite family of nonexpansive semigroups and the set of solutions of an equilibrium problem in the framework of Hilbert spaces. He then prove strong convergence of the proposed iterative process. In this paper, we improve Mohammad's result, and obtain follwing main results.

Mohammad's Theorem 3.1 (see [20]) about nonexpansive semigroups is the special case of our results. Our results improve chang's result in [21].

3. Main Results

First, we show the following theorem to our main results.

Theorem 3.1 *Let* C *be nonempty closed convex subset of* H. $T(s)(s\ge 0)$ *be an uniformly asymptotically nonexpansive semigroups with nonnegative real sequences* $\{k_n\}$ *with* $\{k_n\}\subset[1,\infty)$ *and* $k_n\to 0$ *(as* $n\to\infty$*), then* $F(\mathbb{T})$ *is a closed and convex subset of* C.

Proof. Let $\{x_n\}$ be a sequence in $F(\mathbb{T})$, such that $x_n\to x^*$. Since $T(s)(s\ge 0)$ be an uniformly asymptotically nonexpansive semigroups, we have

$$\|x_n-z\|=\|T(s)x_n-T(s)x^*\|\le k_1\|x_n-x^*\|$$

for $z=T(s)x^*$ and for all $n\in\mathbb{N}$. Therefore,

$$\|x^*-z\|=\lim_{n\to\infty}\|x_n-z\|\le\lim_{n\to\infty}k_1\|x_n-x^*\|=k_1\|x^*-x^*\|=0$$

We obtain $z=x^*$. Hence, $Tx^*=x^*$. So, we have $x^*\in F(\mathbb{T})$. This implies $F(\mathbb{T})$ is closed.

Let $p,q\in F(\mathbb{T})$ and $t\in(0,1)$, and put $w=tp+(1-t)q$. Next we prove that $w\in F(\mathbb{T})$. Indeed, in view of Lemma 2.3 2), let $z_n=T^n(s)w$, we have

$$\begin{aligned}\|w-z_n\|^2&=\|w\|^2-2\langle w,z_n\rangle+\|z_n\|^2\\&=\|w\|^2-2\langle tp+(1-t)q,z_n\rangle+\|z_n\|^2\\&=\|w\|^2+t\|p-z_n\|+(1-t)\|q-z_n\|-t\|p\|^2-(1-t)\|q\|^2.\end{aligned}\tag{1}$$

Since

$$\begin{aligned}&t\|p-z_n\|+(1-t)\|q,z_n\|\\&\le t(k_n-1)\|p-w\|+(1-t)(k_n-1)\|q-w\|\\&=t\left\{\|p\|^2-2\langle p,w\rangle+\|w\|^2+(k_n-1)\|q-w\|\right\}\\&\quad+(1-t)\left\{\|q\|^2-2\langle q,w\rangle+\|q\|^2+(k_n-1)\|q-w\|\right\}\\&=t\|p\|^2+(1-t)\|q\|^2-\|w\|^2+t(k_n-1)\|p-w\|+(1-t)(k_n-1)\|q-w\|.\end{aligned}\tag{2}$$

Substituting (1) into (2) and simplifying it we have

$$\|w-z_n\|\le t(k_n-1)\|p-w\|+(1-t)(k_n-1)\|q-w\|\to 0,\qquad(\text{as } n\to\infty)$$

Hence, we have $z_n\to w$. This implies that $z_{n+1}=TT^nw\to w$. Since $T(s)$ is closed, we have $T(s)w=w$, *i.e.*, $w\in F(\mathbb{T})$. This completes the proof of theorem 3.1.

Theorem 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* ϕ *be a bifunction of* $C\times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $\mathbb{T}_i:=\{T_i(s):s\ge 0\}$ $(i=1,2,\cdots,m)$ *be a finite family of uniformly asymptotically semigroups with sequence* $\{k_n^{(i)}\}$ ($k_n^{(i)}\ge 1$ *and* $\lim_{n\to\infty}k_n^{(i)}=1$). *Assume that* $\mathbb{F}:=\bigcap_{i=1}^m F(\mathbb{T}_i)\cap EP(\phi)\ne\varnothing$. *For an initial piont* $x_0\in C_0=C$, *let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases}\phi(z_n,y)+\dfrac{1}{r_n}\langle y-z_n,z_n-x\rangle\ge 0,\quad\forall y\in C\\ y_n=a_{n,0}z_n+a_{n,1}T_1^n(t_n)z_n+a_{n,2}T_2^n(t_n)z_n+\cdots+a_{n,m}T_m^n(t_n)z_n\\ C_{n+1}=\{u\in C_n:\|y_n-u\|\le k_n\|x_n-u\|\}\\ x_{n+1}=P_{C_{n+1}}x_0\quad(n=0,1,2,\cdots)\end{cases}\tag{3}$$

where $P_{D_{n+1}}$ is the metric projection of H onto D_{n+1}. If $\{a_{n,i}\}_{i=0}^m$, $\{r_n\}$, $\{k_n\}$ and $\{t_n\}$ satisfying the following conditions:

1) $k_n = \max_{1\le i\le m}\left\{k_n^{(i)},1\right\}$;

2) $a_{n,i}\in[a,1)$ (for $i=0,1,\cdots,m$) and $\sum_{i=0}^{m}a_{n,i}=1$;

3) $r_n\ge 0$ and $\liminf_{n\to\infty} r_n>0$;

4) $\liminf_{n\to\infty} t_n=0$, $\limsup_{n\to\infty} t_n>0$,

then, the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{\mathbb{F}}x_0$.

Proof. 1) First, we prove $\mathbb{F}:=\bigcap_{i=1}^{m}F(\mathbb{T}_i)\cap EP(\phi)\subset C_n$.

Indeed, $\mathbb{F}\subset C_0=C$ is obvious. Suppose that $\mathbb{F}\subset C_n$, then for $\forall p\in\mathbb{F}$ and $z_n=T_{r_n}x_n$, by Lemma 2.6 we have

$$\|z_n-p\|=\|T_{r_n}x_n-T_{r_n}p\|\le\|x_n-p\|. \tag{4}$$

Since $\mathbb{T}_i:=\{T_i(s):s\ge 0\}$ $(i=1,2,\cdots,m)$ be a finite family of uniformly asymptotically semigroups,we have

$$\begin{aligned}\|y_n-p\|&=\left\|a_{n,0}z_n+a_{n,1}T_1^n(t_n)z_n+a_{n,2}T_2^n(t_n)z_n+\cdots+a_{n,m}T_m^n(t_n)z_n-p\right\|\\&\le a_{n,0}\|z_n-p\|+a_{n,1}\left\|T_1^n(t_n)z_n-p\right\|+a_{n,2}\left\|T_2^n(t_n)z_n-p\right\|+\cdots+a_{n,m}\left\|T_m^n(t_n)z_n-p\right\|\\&\le a_{n,0}\|z_n-p\|+a_{n,1}k_n^{(1)}\|z_n-p\|+a_{n,2}k_n^{(2)}\|z_n-p\|+\cdots+a_{n,m}k_n^{(m)}\|z_n-p\|\\&\le k_n\|z_n-p\|\le k_n\|x_n-p\|,\end{aligned}$$

which implies that $p\in C_{n+1}$.Therefore we have $\mathbb{F}:=\bigcap_{i=1}^{m}F(\mathbb{T}_i)\cap EP(\phi)\subset C_n$ for all $n\ge 0$. Note C_n is closed and convex.this implies that $\{x_n\}$ is well defined. From Lemma 2.5, sequence $\{z_n\}$ is also well defined.

2) Next, we prove that $\lim\|x_n-x_0\|$ **exists.**

Since $\mathbb{F}$ is closed and convex subset of H, there exists a unique $w\in\mathbb{F}$ such that $w=P_{\mathbb{F}}x_0$. From $x_n=P_{c_n}x_0\in c_n$, we have

$$\|x_n-x_0\|\le\|x_{n+1}-x_0\|,\quad n\ge 0.$$

Since $w\in\mathbb{F}\subset C_n$, we get that

$$\|x_n-x_0\|\le\|w-x_0\|,\quad n\ge 0.$$

It follows that the sequence $\{x_n\}$ is bounded and non decreasing, this implies that $\lim\|x_n-x_0\|$ exists

3) Now we show that $\lim_{n\to\infty}x_n=q\in C$**,** $\lim_{n\to\infty}\left\|T_i^n(s)x_n-x_n\right\|=0\ (\forall s\ge 0)$**.**

Infact, from Lemma 2.2 we have

$$\|x_m-x_n\|\le\|x_m-x_0\|^2-\|x_n-x_0\|^2\to 0,\quad \text{as } m>n\to\infty.$$

witch implies that we get $\{x_n\}$ is Cauchy. Hence there exists $q\in C$ such that $\lim_{n\to\infty}x_n=q\in C$. Since $\|y_n-x_{n+1}\|\le\|x_n-x_{n+1}\|$, thus $\lim y_n=q\in C$. By Lemma 2.4, we have

$$\begin{aligned}\|y_n-p\|^2&=\left\|a_{n,0}z_n+a_{n,1}T_1^n(t_n)z_n+a_{n,2}T_2^n(t_n)z_n+\cdots+a_{n,m}T_m^n(t_n)z_n-p\right\|^2\\&\le a_{n,0}\|z_n-p\|^2+a_{n,1}\left\|T_1^n(t_n)z_n-p\right\|^2+a_{n,2}\left\|T_2^n(t_n)z_n-p\right\|^2\\&\quad+\cdots+a_{n,m}\left\|T_m^n(t_n)z_n-p\right\|^2-a_{n,0}a_{n,i}\left\|T_i^n(t_n)z_n-z_n\right\|^2\\&\le a_{n,0}\|z_n-p\|^2+a_{n,1}k_n^{(1)}\|z_n-p\|^2+a_{n,2}k_n^{(2)}\|z_n-p\|^2\\&\quad+\cdots+a_{n,m}k_n^{(m)}\|z_n-p\|^2-a_{n,0}a_{n,i}\left\|T_i^n(t_n)z_n-z_n\right\|^2\\&\le k_n\|z_n-p\|-a_{n,0}a_{n,i}\left\|T_i^n(t_n)z_n-z_n\right\|^2\\&\le k_n\|x_n-p\|^2-a_{n,0}a_{n,i}\left\|T_i^n(t_n)z_n-z_n\right\|^2.\end{aligned}\tag{5}$$

from condition ($C1$), so we have

$$a^2\left\|T_i^n(t_n)z_n - z_n\right\|^2 \le a_{n,0}a_{n,i}\left\|T_i^n(t_n)z_n - z_n\right\|^2 \le k_n\|x_n - p\|^2 - \|y_n - p\|^2,$$

this implies $\lim_{n\to\infty}\left\|T_i^n(t_n)z_n - z_n\right\| = 0$ for all $i \ge 1$. We know that $z_n = T_{r_n}x_n$, hence we have

$$\begin{aligned}\|z_n - p\|^2 &= \left\|T_{r_n}x_n - T_{r_n}p\right\|^2 \le \left\langle T_{r_n}x_n - T_{r_n}p, x_n - p\right\rangle \\ &= \langle z_n - p, x_n - p\rangle = \frac{1}{2}\left(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2\right),\end{aligned}$$

that is,

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2.$$

Using (3.5) we get that

$$\|y_n - p\|^2 \le k_n\|z_n - p\|^2 \le k_n\left(\|x_n - p\|^2 - \|x_n - z_n\|^2\right),$$

that is,

$$\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \frac{1}{k_n}\|y_n - p\|^2,$$

which implies $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Hence for all $i \ge 1$ we get that

$$\begin{aligned}\left\|T_i^n(t_n)x_n - x_n\right\| &\le \|x_n - z_n\| + \left\|T_i^n(t_n)z_n - z_n\right\| + \left\|T_i^n(t_n)z_n - T_i^n(t_n)x_n\right\| \\ &\le \left(1 + k_n^{(i)}\right)\|x_n - z_n\| + \left\|T_i^n(t_n)z_n - z_n\right\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Without loss of generality, as in Saejung's article [17], let $\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{\left\|T_i^n(t_n)x_n - x_n\right\|}{t_n} = 0$. For $s > 0$ and $1 \le i \le m$,

$$\begin{aligned}\left\|T_i^n(s)x_n - x_n\right\| &\le \sum_{k=0}^{\left[\frac{s}{t_n}\right]-1}\left\|T_i^n((k+1)t_n)x_n - T_i^n(kt_n)x_n\right\| + \left\|T_i^n\left(\left[\frac{s}{t_n}\right]t_n\right)x_n - T_i^n(s)x_n\right\| \\ &\le \left[\frac{s}{t_n}\right]k_n^{(i)}\left\|T_i^n(t_n)x_n - x_n\right\| + \left\|T_i^n\left(s - \left[\frac{s}{t_n}\right]t_n\right)x_n - T_i^n(s)x_n\right\| \\ &\le \frac{s}{t_n}k_n^{(i)}\left\|T_i^n(t_n)x_n - x_n\right\| + k_n\max\left\{\left\|T_i^n(t)x_n - x_n\right\|\right\}, \quad 0 < t \le t_n,\end{aligned}$$

where $[x]$ denotes the maximal integer that is not larger than x. Since for $\forall n \ge 0$ mapping $s \to T_i^n(s)x$ for a fixed $x \in C$ and $\lim_{n\to\infty}\left\|T_i^n(t_n)x_n - x_n\right\| = 0$, then $\lim_{n\to\infty}\left\|T_i^n(s)x_n - x_n\right\| = 0\ (\forall s \ge 0)$.

4) Now we prove that $q \in \mathbb{F} := \bigcap_{i=1}^m F(\mathbb{T}_i) \cap EP(\phi)$.

First, since $z_n = T_{r_n}x_n$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, by ($A2$) we get that

$$\frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge \phi(y, z_n) \quad \forall y \in C$$

and hence

$$\left\langle y - z_n, \frac{1}{r_n}(z_n - x_n)\right\rangle \ge \phi(y, z_n) \quad \forall y \in C.$$

Since $\lim_{n\to\infty}\frac{1}{r_n}(z_n-x_n)=0$, $\lim_{n\to\infty} z_n=q$ and $A(4)$, we get that

$$\phi(y,q)\le 0 \quad \forall y\in C.$$

If $t\in(0,1)$ and $y\in C$, let $y_t := ty+(1-t)q$, then $y_t\in C$. So, from (A1)-(A4) we have

$$0=\phi(y_t,y_t)=\phi(y_t,ty+(1-t)q)\le t\phi(y_t,y)+(1-t)\phi(y_t,q)\le t\phi(y_t,y)$$

which gives $\phi(y_t,y)\ge 0$ for all $y\in C$. Hence by (A3) we have

$$0\le \limsup_{t\to 0^+}\phi(y_t,q)\limsup_{t\to 0^+}\phi(ty+(1-t)q,y)\le \phi(q,y) \quad \forall y\in C,$$

which is $q\in EP(\phi)$.

For $i=1,2,\cdots,m$, we have

$$\begin{aligned}\|q-T_i(s)q\| &\le \|q-x_n\|+\|x_n-T_i^n(s)x_n\|+\|T_i^n(s)x_n-T_i(s)q\|\\ &\le \|q-x_n\|+\|x_n-T_i^n(s)x_n\|+k_n^{(i)}\|T_i^{n-1}(s)x_n-q\|\\ &\le \|q-x_n\|+\|x_n-T_i^n(s)x_n\|+k_n^{(i)}\left(\|T_i^{n-1}(s)x_n-T_i^{n-1}(s)x_{n-1}\|+\|T_i^{n-1}(s)x_{n-1}-q\|\right)\\ &\le \|q-x_n\|+\|x_n-T_i^n(s)x_n\|+k_n^{(i)}k_{n-1}^{(i)}\left(\|x_n-x_{n-1}\|+\|T_i^{n-1}(s)x_{n-1}-q\|\right)\end{aligned}$$

Since $\lim_{n\to\infty}\|T_i^{n-1}(s)x_{n-1}-q\|=0$, then $T_i(s)q=q$ *i.e.*, $q\in F(\mathbb{T}_i)$ for all $i=1,2,\cdots,m$ and thus $q\in\mathbb{F}:=\bigcap_{i=1}^m F(\mathbb{T}_i)\cap EP(\phi)$.

5) Now we prove that $q=P_{\mathbb{F}}x_0$.

Since $x_n=P_{C_n}x_0\in C_n$ and $\mathbb{F}\subset C_n$, we get that

$$\|x_n-x_0\|\le\|p-x_0\| \quad p\in\mathbb{F} \quad \forall n\ge 0.$$

Since $\lim_{n\to\infty} x_n=q$, we have

$$\|q-x_0\|\le\|p-x_0\| \quad p\in\mathbb{F} \quad \forall n\ge 0,$$

which implies $q=P_{\mathbb{F}}x_0$. The proof is completed.

From Theorem 3.1, taking $\phi\equiv 0$ and $r_n=1$, we obtain

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $\mathbb{T}_i:=\{T_i(s):s\ge 0\}$ $(i=1,2,\cdots,m)$ be a finite family of uniformly asymptotically semigroups with sequence $\{k_n^{(i)}\}$ ($k_n^{(i)}\ge 1$ and $\lim_{n\to\infty}k_n^{(i)}=1$). Assume that $\mathbb{F}:=\bigcap_{i=1}^m F(\mathbb{T}_i)\cap EP(\phi)\ne\varnothing$. For an initial piont $x_0\in C_0=C$, let $\{x_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} y_n=a_{n,0}x_n+a_{n,1}T_1^n(t_n)x_n+a_{n,2}T_2^n(t_n)x_n+\cdots+a_{n,m}T_m^n(t_n)x_n\\ C_{n+1}=\{u\in C_n:\|y_n-u\|\le k_n\|x_n-u\|\}\\ x_{n+1}=P_{C_{n+1}}x_0 \quad (n=0,1,2,\cdots)\end{cases} \tag{6}$$

where $P_{D_{n+1}}$ is the metric projection of H onto D_{n+1}. If $\{a_{n,i}\}_{i=0}^m$, $\{r_n\}$, $\{k_n\}$ and $\{t_n\}$ satisfying the following conditions:

1) $k_n=\max_{1\le i\le m}\{k_n^{(i)},1\}$;

2) $a_{n,i}\in[a,1)$ (for $i=0,1,\cdots,m$) and $\sum_{i=0}^m a_{n,i}=1$;

3) $\liminf_{n\to\infty}t_n=0$, $\limsup_{n\to\infty}t_n>0$,

then, the sequences $\{x_n\}$ converge strongly to $P_{\mathbb{F}}x_0$.

Competing Interests

The authors declare that they have no competing interests.

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