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# Nikodým-Type Theorems for Lattice Group-Valued Measures with Respect to Filter Convergence

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## Abstract

**We present some convergence and boundedness theorems with respect to filter convergence for lattice group-valued measures. We give a direct proof, based on the sliding hump argument. Furthermore we pose some open problems.**

## 1. Introduction

In the literature there have been several recent studies about limit theorems with respect to filter/ideal convergence for measures, taking values in abstract spaces, whose a particular case is the statistical convergence, related with asymptotic density of subsets of the set of natural numbers. Though in general it is not possible to give versions of limit theorems completely analogous to the corresponding classical ones in the filter/ideal setting (see also [1], Example 3.4), there are different kinds of results on these topics, whose an overview can be found in [2] and its bibliography, together with a historical survey on several types of kinds of such theorems and related topics since the beginning of the last century. Some classical like limit theorems for measures and integrals in the context of lattice groups or similar structures can be found, for instance, in [3]-[7]. In particular, in [1] [8], some versions of basic matrix theorems are given, extending results of [9] [10], which were proved in the normed space context and with respect to statistical convergence. In [11]-[17], some Schur, boundedness,

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decomposition and/or convergence theorems are given in the lattice group setting with respect to a suitable class of filters, extending some results of [18], while for positive measures it is possible to have some similar results even for a larger class of filters (see also [19]). Analogous results have been established also in the setting of topological group-valued measures in [20] [21]. In [22]-[24], some limit theorems are proved by means of the tool of uniform filter/ideal exhaustiveness. Moreover, in [25], some results about equivalence between Brooks-Jewett, Vitali-Hahn-Saks, Nikodým convergence and Dieudonné-type theorems are presented, using the Stone Isomorphism technique and extending results of [26], given in the classical case for topological group-valued measures. In this paper we use sliding hump-type techniques, similar to those used in the topological group-set-ting first by D. Candeloro and G. Letta in 1985 in [27]-[30] for proving limit and boundedness theorems for families of finitely additive group-valued measures defined on suitable Boolean algebras, and successively in [4] [5] [16] [17] [20] [21] in the setting of lattice groups and filter convergence. We prove some new further versions of Nikodým convergence, boundedness and Brooks-Jewett-type theorems for lattice group-valued measures, defined on a σ-algebra of an abstract nonempty set. The results and the proofs are direct and, differently than in [14] [16], without using Schur-type theorems proved for measures defined on $\mathcal{P}(\mathbb{N})$. Finally, we pose some open problems.

## 2. Preliminaries

A lattice group (shortly, $(\ell)$-group) $R$ is said to be *Dedekind complete* iff every nonempty subset of $R$, bounded from above, admits supremum in $R$. A Dedekind complete lattice group is said to be *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there exists a countable subset $A^* \subset A$, such that $\bigvee A = \bigvee A^*$.

Let $R$ be a Dedekind complete $(\ell)$-group. A sequence $(\sigma_p)_p$ of positive elements of $R$ is called an $(O)$-*sequence* iff it is decreasing and $\bigwedge_p \sigma_p = 0$. A sequence $(x_n)_n$ in $R$ is said to be *order convergent* (or $(O)$-*convergent*) to $x \in R$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that for every $p \in \mathbb{N}$ there is a positive integer $n_0$ with $|x_n - x| \leq \sigma_p$ for all $n \geq n_0$, and in this case we will write $(O)\lim_n x_n = x$. A bounded double sequence $(a_{t,l})_{t,l}$ in $R$ is called a $(D)$-*sequence* or a *regulator* iff for each $t \in \mathbb{N}$ the sequence $(a_{t,l})_l$ is an $(O)$-sequence. A sequence $(x_n)_n$ in $R$ is said to be $(D)$-*convergent* to $x \in R$ (and we write $(D)\lim_n x_n = x$) iff there exists a $(D)$-sequence $(a_{t,l})_{t,l}$ in $R$, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$. An $(\ell)$-group $R$ is *weakly $\sigma$-distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left( \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0$ for every $(D)$-sequence $(a_{t,l})_{t,l}$.

Observe that, if $\Sigma$ is the $\sigma$-algebra of all Borel subsets of $[0,1]$ and $\nu$ is the Lebesgue measure, then the space $L^0([0,1], \Sigma, \nu)$ of all $\nu$-measurable real-valued functions on $[0,1]$ (with identification up to $\nu$-null sets) is super Dedekind complete and weakly $\sigma$-distributive (see also [5], [6], Example 2.17).

We now recall the following theorem, which links $(O)$- and $(D)$-sequences in lattice groups.

**Theorem 2.1** ([25], Theorem 2.3, see also [31], Theorems 3.1 and 3.4) *Given any Dedekind complete $(\ell)$-group $R$ and any $(O)$-sequence $(\sigma_l)_l$ in $R$, the double sequence defined by $a_{t,l} := \sigma_l$, $t, l \in \mathbb{N}$, is a regulator, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, if $l = \varphi(1)$, then $\sigma_l \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$. Conversely, if $R$ is super Dedekind complete and weakly $\sigma$-distributive, then for every $(D)$-sequence $(a_{t,l})_{t,l}$ in $R$ there are an $(O)$-sequence $(b_k)_k$ in $R$ and a sequence $(\varphi_k)_k$ in $\mathbb{N}^{\mathbb{N}}$, with $\bigvee_{t=1}^{\infty} a_{t,\varphi_k(t)} \leq b_k$ for each k.*

Let $G$ be any nonempty set and $\Sigma \subset \mathcal{P}(G)$ be a $\sigma$-algebra. A bounded finitely additive measure $m : \Sigma \to \mathbb{R}$ is said to be $\sigma$-*additive* (resp. $(s)$-*bounded*) on $\Sigma$ iff $(D)\lim_n v(m)\left( \bigcup_{l=n}^{\infty} H_l \right) = 0$ (resp. $(D)\lim_n v(m)(H_n) = 0$) for each disjoint sequence $(H_n)_n$ in $\Sigma$, where $v(m)(E) := \bigvee \{ |m(A)| : A \in \Sigma, A \subset E \}$, $E \in \Sigma$, is the *semivariation* of $m$ on $\Sigma$. Moreover, if $G$ is a compact Hausdorff topological space and $\Sigma$ is its Borel $\sigma$-algebra, we say that a positive finitely additive measure $m : \Sigma \to R$ is *regular* iff for every

$E \in \Sigma$ there exists a $(D)$-sequence $\left(a_{i,j}\right)_{i,j}$, depending on $E$, such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are a compact set $K$ and an open set $U$ with $K \subset E \subset U$ and $m(U \setminus K) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$. We recall the following.

**Theorem 2.2** ([32], Theorem 2.2) *Assume that* $R$ *is a Dedekind complete and weakly* $\sigma$-*distributive Riesz space, and let* $m : \Sigma \to R$ *be a regular measure defined on the Borel* $\sigma$-*algebra of a compact Hausdorff space* $G$. *Then* $m$ *is* $\sigma$-*additive.*

The following result (Fremlin lemma, see [33], Lemma 1C) allows us to replace a countable family or a "series" of $(D)$-sequences with a single regulator.

**Lemma 2.3** *Let* $R$ *be any Dedekind complete* $(\ell)$-*group and* $\left(a_{t,l}^{(n)}\right)_{t,l}$, $n \in \mathbb{N}$, *be a sequence of regulators in* $R$. *Then for every* $u \in R$, $u \geq 0$ *there exists a* $(D)$-*sequence* $\left(a_{t,l}\right)_{t,l}$ *in* $R$ *with*

$$u \wedge \left( \sum_{n=1}^{q} \left( \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for every } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now give some basic properties of filters, which will be useful in the sequel. Let $Q$ be a countable set and $\mathcal{F}$ be a filter of $Q$. A subset of $Q$ is $\mathcal{F}$-*stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^{*}$ the family of all $\mathcal{F}$-stationary subsets of $Q$.

A filter $\mathcal{F}$ of $Q$ is said to be *diagonal* iff for every sequence $\left(A_n\right)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^{*}$ there exists a set $J \subset I$, $J \in \mathcal{F}^{*}$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$ (see also [15] [16] [18]). Given an infinite set $I \subset Q$, a *blocking* of $I$ is a countable partition $\{D_k : k \in \mathbb{N}\}$ of $I$ into nonempty finite subsets. A filter $\mathcal{F}$ of $Q$ is said to be *block-respecting* iff for every $I \in \mathcal{F}^{*}$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of $I$ there is a set $J \in \mathcal{F}^{*}$, $J \subset I$ with $\#(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\#$ denotes the number of elements of the set into brackets. A particular class of filters, which are block-respecting and diagonal at the same time, is that of the *category respecting* filters. A filter $\mathcal{F}$ of $\mathbb{N}$ is said to be *category respecting* iff for every compact metric space $K$ and for every family of closed subsets $\left(F_A\right)_{A \in \mathcal{F}}$ of $K$, if $F_A \subset F_B$ whenever $B \subset A$ in $\mathcal{F}$ and $K = \bigcup_{A \in \mathcal{F}} F_A$, then there is a set $B \in \mathcal{F}$ such that the interior of $F_B$ is non-empty (see also [18], Theorem 4.3).

Let $D = \left(D_n\right)_n$ be a disjoint partition of $\mathbb{N}$ into infinite subsets. For each sequence $C = \left(C_n\right)_n$ of finite subsets $C_n \subset D_n$ and every $q \in \mathbb{N}$, set $B_{q,C} := \bigcup_{n=q}^{\infty} \left( D_n \setminus C_n \right)$ The filter $\mathcal{F}_D$ generated by the sets of type $B_{q,C}$ is a non-diagonal and block-respecting filter. Furthermore, note that the filter of all subsets of $\mathbb{N}$ having asymptotic density one is a diagonal and not block-respecting filter (see also [18]).

If $I \in \mathcal{F}^{*}$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on $I$ is the family $\{A \cap I : A \in \mathcal{F}\}$. It is not difficult to see that $\mathcal{F}(I)$ is a filter of $I$ (see also [21]).

**Remark 2.4** Observe that, if $\mathcal{F}$ is a block-respecting filter of $\mathbb{N}$, then $\mathcal{F}(I)$ is a block-respecting filter of $I$ for every $I \in \mathcal{F}^{*}$ (see also [20], Proposition 2.1, [21], Proposition 2.3).

We now recall some main properties of filter convergence in the lattice group setting (see also [1] [16]).

Let $\mathcal{F}$ be a filter of $\mathbb{N}$. A sequence $\left(x_n\right)_n$ in $R$ $(D\mathcal{F})$-*converges to* $x \in R$ iff there is a $(D)$-sequence $\left(a_{t,l}\right)_{t,l}$ with the property that $\left\{n \in \mathbb{N} : |x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F}$ for each $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Let $\Xi$ be any arbitrary nonempty set. A family $\left(\beta_{\xi,n}\right)_{\xi \in \Xi, n \in \mathbb{N}}$ is said to be $(RD\mathcal{F})$-*convergent* to a family $\left(\beta_{\xi}\right)_{\xi \in \Xi}$ with respect to $\xi \in \Xi$ iff there is a regulator $\left(a_{t,l}\right)_{t,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\xi \in \Xi$ we get

$$\left\{ n \in \mathbb{N} : \left| \beta_{\xi,n} - \beta_{\xi} \right| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right\} \in \mathcal{F}.$$

Given $a < b \in R$, set $[a,b] = \{x \in R : a \leq x \leq b\}$. For $A, B \subset R$, $n \in \mathbb{N}$, put $A + B = \{a + b : a \in A, b \in B\}$, $nA = \{a + \cdots + a\}$ ($n$ times). Let $U_n = [-u_n, u_n]$, $n \in \mathbb{N}$, be such that $0 < u_n \leq u_{n+1}$ for every $n \in \mathbb{N}$. A set $\{x_n : n \in \mathbb{N}\} \subset R$ is said to be $(PR)$-$\mathcal{F}$-*bounded* by $\left(U_n\right)_n$, iff $\{n \in \mathbb{N} : x_n \in U_n\} \in \mathcal{F}$, and $(PR)$-*eventually bounded* by $\left(U_n\right)_n$ iff it is $(PR)$-$\mathcal{F}_{\mathrm{cofin}}$-bounded by $\left(U_n\right)_n$ (see also [1] [15] [16] [34]).

## 3. The Main Results

We begin with recalling the following

**Lemma 3.1** ([16], Lemma 2.3) *Let* $\mathcal{F}$ *be a diagonal filter of* $\mathbb{N}$, $\left(a_i\right)_i$ *be a sequence in* $R$ *with*

$(O\mathcal{F})\lim_i a_i = 0$ *with respect to an* $(O)$*-sequence* $\left(\sigma_p\right)_p$. *Then for every* $I \in \mathcal{F}^*$ *there exists* $J \in \mathcal{F}^*$ *such that* $J \subset I$ *and* $(O)\lim_{i \in J} a_i = 0$ *with respect to the same* $(O)$*-sequence* $\left(\sigma_p\right)_p$.

We now prove the main result, by means of sliding hump-type techniques.

**Theorem 3.2** *Let* $R$ *be a Dedekind complete* $(\ell)$*-group,* $\mathcal{F}$ *be a block-respecting filter of* $\mathbb{N}$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, *be a sequence of equibounded* $\sigma$*-additive measures,* $\left(C_k\right)_k$ *be a disjoint sequence in* $\Sigma$, *with*

(i) $(D)\lim_n m_n\left(C_k\right) = 0$ *for any* $k \in \mathbb{N}$, *and*

(ii) $(RD\mathcal{F})\lim_n m_n\left(\bigcup_{k \in P} C_k\right) = 0$ *with respect to* $P \in \mathcal{P}(\mathbb{N})$. *Then,*

3.2.1) *for every strictly increasing sequence* $\left(k_n\right)_n$ *in* $\mathbb{N}$ *we get*

$$(D\mathcal{F})\lim_n m_n\left(C_{k_n}\right) = 0; \tag{1}$$

3.2.2) *if* $\mathcal{F}$ *is also diagonal and* $R$ *is super Dedekind complete and weakly* $\sigma$*-distributive, then the only condition* (ii) *is sufficient to get* (1).

**Proof:** For each $n \in \mathbb{N}$, set $H_n := C_{k_n}$. Let $u := \bigvee_{A \in \Sigma, n \in \mathbb{N}} \left|m_n\left(A\right)\right|$: such an element does exist in $R$, thanks to equiboundedness of the $m_n$'s. For each $n \in \mathbb{N}$ let $\left(a_{t,l}^{(n)}\right)_{t,l}$ be a $(D)$-sequence related with $\sigma$-additivity of $m_n$ and the sequence $\left(H_n\right)_n$. For every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ there is $\bar{l} \in \mathbb{N}$, with

$$\left|m_n\left(A\right)\right| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \quad \text{for all } A \in \Sigma, A \subset \bigcup_{l=\bar{l}}^{\infty} H_l. \tag{2}$$

By the Fremlin Lemma 2.3 there is a $(D)$-sequence $\left(a_{t,l}\right)_{t,l}$ with

$$u \wedge \left(\sum_{n=1}^{q}\left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}\right)\right) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3}$$

for each $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$. From (2) and (3) it follows that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ there exists $\bar{l} \in \mathbb{N}$ with $\left|m_n\left(A\right)\right| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $A \in \Sigma, A \subset \bigcup_{l=\bar{l}}^{\infty} H_l$.

Let $\left(b_{t,l}\right)_{t,l}$ be a regulator, satisfying the condition of $(RD\mathcal{F})$-convergence as in (ii).

Since $(D)\lim_n m_n\left(C_k\right) = 0$ for every $k \in \mathbb{N}$, then for each $k \in \mathbb{N}$ there exists a regulator $\left(c_{t,l}^{(k)}\right)_{t,l}$, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\bar{n} \in \mathbb{N}$ with $\left|m_n\left(C_k\right)\right| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t+k)}^{(k)}$ for all $n \ge \bar{n}$. Since the $m_n$'s are equibounded, arguing analogously as above, by the Fremlin Lemma 2.3 we find a regulator $\left(c_{t,l}\right)_{t,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ with $\left|m_n\left(C_k\right)\right| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$ for all $n \ge \bar{n}$.

Again by Lemma 2.3, there are two $(D)$-sequences $\left(d_{t,l}\right)_{t,l}$, $\left(e_{t,l}\right)_{t,l}$, with

$$u \wedge \left(\sum_{k=1}^{q}\left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+k+1)}\right)\right) \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}, \tag{4}$$

$$u \wedge \left(\sum_{k=1}^{q}\left(\bigvee_{t=1}^{\infty} c_{t,\varphi(t+k)}\right)\right) \le \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} \tag{5}$$

for every $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$. For every $t,l \in \mathbb{N}$, set

$$z_{t,l} = 2\left(b_{t,l} + d_{t,l} + e_{t,l}\right). \tag{6}$$

We prove that the $(D)$-sequence $\left(z_{t,l}\right)_{t,l}$ satisfies the condition of $(D\mathcal{F})$-convergence in 3.21). Otherwise there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that $C := \left\{n \in \mathbb{N} : \left|m_n\left(H_n\right)\right| \le \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}\right\} \notin \mathcal{F}$.

We get that $I := \mathbb{N} \setminus C = \left\{n \in \mathbb{N} : \left|m_n\left(H_n\right)\right| \nleq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}\right\} \in \mathcal{F}^*$: otherwise, there would be $F \in \mathcal{F}$ with $I \cap F = \varnothing$, that is $F \subset C$ and hence $C \in \mathcal{F}$, a contradiction.

Let $N_0 = 1$. By $\sigma$-additivity of $m_1$, there is a cofinite subset $P_1 \subset \mathbb{N}$, with

$$N_0 < p_1 := \min P_1, \text{ and } \ v(m_1)(F_1) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+1)},$$

where $F_1 := \bigcup_{l \in P_1} H_l$. By (i) there is an integer $N_1 > p_1$ with $\left| m_n(H_l) \right| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t+1)}$ whenever $n \geq N_1$ and $l = 1, \cdots, p_1$. By $\sigma$-additivity of $m_1$, $m_2, \cdots, m_{N_1}$, there is a cofinite subset $P_2 \subset P_1$, with $N_1 < p_2 := \min P_2$, and $v(m_n)(F_2) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+2)}$ for every $n = 1, 2, \cdots, N_1$, where $F_2 := \bigcup_{l \in P_2} H_l$. Proceeding analogously as above, we find an integer $N_2 > p_2$ with $\left| m_n(H_l) \right| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t+2)}$ whenever $n \geq N_2$ and $l = 1, \cdots, p_2$.

By induction, it is possible to find: a strictly decreasing sequence $(P_k)_k$ of cofinite subsets of $\mathbb{N}$, a strictly decreasing sequence $(F_k)_k$ in $\Sigma$ and two strictly increasing sequences $(N_k)_k$, $(p_k)_k$ in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$, $N_k > p_k, p_{k+1} > N_k p_k = \min P_k; F_k = \bigcup_{l \in P_k} H_l$;

$$v(m_n)(F_{k+1}) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+k+1)}, \text{ and so } v(m_n)(F_{k+1}) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}, \text{ for any } n = 1, \cdots, N_k; \tag{7}$$

$$\left| m_n(H_l) \right| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t+k)}, \text{ and so } \left| m_n(H_l) \right| \leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} \text{ for all } n \geq N_k \text{ and } l = 1, \cdots, p_k. \tag{8}$$

Since $\mathcal{F}$ is block-respecting, there is $J := \{j_1, j_2, \cdots\} \in \mathcal{F}^*$, $J \subset I$, with $N_k \leq j_k < N_{k+1}$ for every $k \in \mathbb{N}$. As $J \in \mathcal{F}^*$, then either $J_1 := \{j_1, j_3, j_5, \cdots\} \in \mathcal{F}^*$ or $J_2 := \{j_2, j_4, j_6, \cdots\} \in \mathcal{F}^*$. Without loss of generality, let $J_1 \in \mathcal{F}^*$ (see also [15] [16] [18]). Put $A := \bigcup_{h=1}^{\infty} H_{j_{2h-1}}$. We have:

$$\begin{aligned}
&m_{j_1}(A) = m_{j_1}(H_{j_1}) + m_{j_1}(H_{j_3} \cup H_{j_5} \cup \cdots); \\
&m_{j_{2h-1}}(A) = m_{j_{2h-1}}(H_{j_1} \cup H_{j_3} \cup \ldots \cup H_{j_{2h-3}}) + m_{j_{2h-1}}(H_{j_{2h-1}}) + m_{j_{2h-1}}(H_{j_{2h-1}} \cup H_{j_{2h+3}} \cup \cdots), \quad h \geq 2.
\end{aligned} \tag{9}$$

Since $j_{2h-1} < N_{2h-1} < p_{2h}$ and

$$H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \cdots \subset \bigcup_{l=p_{2h+1}}^{\infty} H_l = F_{2h+1} \quad \text{for every } h \in \mathbb{N}, \tag{10}$$

from (7) and (10) we get

$$\left| m_{j_{2h-1}}(H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \ldots) \right| \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{11}$$

Moreover, since $j_{2h-3} < N_{2h-3} < p_{2h-2} < p_{2h-1}$ for every $h \geq 2$, from (8) we obtain

$$\left| m_{j_{2h-1}}(H_{j_1} \cup H_{j_3} \cup \cdots \cup H_{j_{2h-3}}) \right| \leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}. \tag{12}$$

If $\left| m_{j_{2h-1}}(A) \right| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$, then from (9), (11) and (12) we have $\left| m_{j_{2h-1}}(H_{j_{2h-1}}) \right| \leq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}$.

But we know that $\left| m_{j_{2h-1}}(H_{j_{2h-1}}) \right| \not\leq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}$, and so we get a contradiction.

Thus $\left| m_{j_{2h-1}}(A) \right| \not\leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for all $h \in \mathbb{N}$, and hence $L := \left\{ l \in \mathbb{N} : \left| m_l(A) \right| \not\leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \right\} \in \mathcal{F}^*$.

Since, by (ii), $\mathbb{N} \setminus L \in \mathcal{F}$, we obtain that $L \bigcap (\mathbb{N} \setminus L) \neq \varnothing$, which is absurd. This proves 3.2.1).

3.2.2) Put $H_n := C_{k_n}$, $n \in \mathbb{N}$, and let $(b_{t,l})_{t,l}$ be a $(D)$-sequence, satisfying $(RD\mathcal{F})$-convergence in condition (ii). Of course, for every $\mathcal{F}$-stationary set $J$, the regulator $(b_{t,l})_{t,l}$ satisfies (ii) also with respect to $(RD\mathcal{F}(J))$-convergence. Since $R$ is super Dedekind complete and weakly $\sigma$-distributive, by Theorem 2.1 there is an $(O)$-sequence $(\sigma_p)_p$, satisfying condition (ii), when $(RD\mathcal{F})$-convergence is replaced with $(RO\mathcal{F})$- or $(RO\mathcal{F}(J))$-convergence. For every $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, $J \subset I$, with $(RO)\lim_{n \in J} m_n(C_k) = 0$, $k \in \mathbb{N}$, with respect to $(\sigma_p)_p$ (see also Lemma 3.1). From this and Theorem 2.1 it follows that there is a regulator $(c_{t,l})_{t,l}$, such that for every $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with

$$(RD)\lim_{n \in J} m_n\left(C_k\right) = 0, \ \ k \in \mathbb{N},\tag{13}$$

with respect to $\left(c_{t,l}\right)_{t,l}$. Let now $\left(a_{t,l}^{(n)}\right)_{t,l}$, $n \in \mathbb{N}$, be regulators associated to $\sigma$-additivity of the $m_n$'s, $u$ be as in the proof of 3.2.1), $\left(a_{t,l}\right)_{t,l}$ be as in (3) and $\left(d_{t,l}\right)_{t,l}$, $\left(e_{t,l}\right)_{t,l}$, $\left(z_{t,l}\right)_{t,l}$ be as in (4), (5), (6) respec- tively. We prove that the regulator $\left(z_{t,l}\right)_{t,l}$ satisfies 3.2.2). Otherwise, by proceeding analogously as in the proof of 3.2.1), we find $I \in \mathcal{F}^*$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ with $\left|m_n\left(H_n\right)\right| \not\leq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)}$ for each $n \in I$. In correspondence with $I$, there is $J \in \mathcal{F}^*$, $J \subset I$, satisfying (13). Note that the sequence $m_n\left(H_n\right)$, $n \in J$, does not $\left(\mathcal{F}(J)\right)$- converge to 0 (see also [18]). Since $J \in \mathcal{F}^*$ and $\mathcal{F}$ is block-respecting, then, by Remark 2.4, $\mathcal{F}(J)$ is block-respecting too. By 3.2.1) used with $m_n$, $n \in J$, and $\mathcal{F}(J)$, it follows that $\left(\mathcal{F}(J)\right)\lim_{n \in J} m_n\left(H_n\right) = 0$, getting a contradiction. This proves 3.2.2). $\square$

A result analogous to Theorem 3.2 holds in the setting of finitely additive measures.

**Theorem 3.3** *Let* $R$ *be a Dedekind complete* $(\ell)$*-group,* $\left(C_k\right)_k$ *be as in Theorem 3.2,* $\mathcal{F}$ *be a block-respecting filter of* $\mathbb{N}$*,* $m_n : \Sigma \to R$*,* $n \in \mathbb{N}$*, be an equibounded sequence of finitely additive measures, and assume that*

(i) $(D)\lim_n m_n\left(C_k\right) = 0$ *for any* $k \in \mathbb{N}$;
(ii) $(RD\mathcal{F})\lim_n \sum_{k \in P} m_n\left(C_k\right) = 0$ *with respect to* $P \in \mathcal{P}(\mathbb{N})$.
*Then for every strictly increasing sequence* $\left(k_n\right)_n$ *in* $\mathbb{N}$ *we get*

$$(D\mathcal{F})\lim_n m_n\left(C_{k_n}\right) = 0.\tag{14}$$

*If* $\mathcal{F}$ *is also diagonal and* $R$ *is super Dedekind complete and weakly* $\sigma$*-distributive, then the only condition* (ii) *is enough to get* (14).

Indeed, it will be enough to apply Theorem 3.2 to the measures $\mu_n$, defined by

$$\mu_n\left(P\right) = \sum_{k \in P} m_n\left(C_k\right), \quad P \subset \mathbb{N}, n \in \mathbb{N}. \ \square$$

Analogously as Theorem 3.2 it is possible to prove a Nikodým boundedness-type theorem in the context of $(\ell)$-groups and filter convergence, extending [34], Theorem 4.6 (see also [15] Lemma 3.4).

**Theorem 3.4** *Let* $R$ *be any Dedekind complete* $(\ell)$*-group,* $u \in R$*,* $u > 0$*,* $U = [-u, u]$*,* $\mathcal{F}$ *be a block-respecting filter of* $\mathbb{N}$*,* $m_j : \Sigma \to R$*,* $j \in \mathbb{N}$*, be a sequence of finitely additive measures, and assume that*

3.4.1) *for every disjoint sequence* $\left(C_n\right)_n$ *in* $\Sigma$ *and* $j \in \mathbb{N}$ *there is a cofinite set* $Q_j \subset \mathbb{N}$ *with* $\sum_{n \in Q} m_j\left(C_n\right) \in U$ *for each* $Q \subset Q_j$.

*Let* $\left(C_n\right)_n$ *be a disjoint sequence in* $\Sigma$ *and* $\left(w_n\right)_n$ *be an increasing sequence of positive elements of* $R$. *For each* $n \in \mathbb{N}$*, set* $W_n := [-w_n, w_n]$ *and* $V_n := nW_n + U$. *Moreover suppose that:*

(i) *the set* $\left\{m_n\left(C_p\right) : n \in \mathbb{N}\right\}$ *is* $(PR)$*-eventually bounded by* $\left(W_n\right)_n$ *for each* $p \in \mathbb{N}$;

(ii) *the set* $\left\{\sum_{p \in P} m_j\left(C_p\right) : n \in \mathbb{N}\right\}$ *is* $(PR)$*-$\mathcal{F}$-bounded by* $\left(W_n\right)_n$ *for each* $P \in \mathcal{P}(\mathbb{N})$.

*Then we get:*

3.4.2) *for every strictly increasing sequence* $\left(l_n\right)_n$ *in* $\mathbb{N}$*, the set* $D := \left\{m_n\left(C_{l_n}\right) : n \in \mathbb{N}\right\}$ *is* $(PR)$*-$\mathcal{F}$-bounded by* $\left(V_n\right)_n$;

3.4.3) *if* $\mathcal{F}$ *is also diagonal, then the only condition* (ii) *is enough in order that* $D$ *is* $(PR)$*-$\mathcal{F}$-bounded by* $\left(V_n\right)_n$.

**Proof:** For every $n \in \mathbb{N}$, let $H_n := C_{l_n}$. If the thesis of the theorem is not true, then $I := \left\{n \in \mathbb{N} : m_n\left(H_n\right) \notin V_n\right\} \in \mathcal{F}^*$. Set $n_0 := 1$. By 3.4.1) there is a cofinite set $P_1 \subset \mathbb{N}$, with $1 < p_1 = \min P_1$ and $\sum_{n \in P} m_1\left(H_r\right) \in U$ for each $P \subset P_1$. By (i) there is $n_1 > p_1$ with $m_j\left(H_l\right) \in W_j$ for each $j \geq n_1$ and $l = 1, \cdots, n_1$. By induction, there are a strictly decreasing sequence $\left(P_k\right)_k$ of subsets of $\mathbb{N}$ and two strictly increasing sequences $\left(n_k\right)_k$, $\left(p_k\right)_k$ of positive integers such that, for each $k \in \mathbb{N}$,

- $n_k > p_k$; $p_{k+1} > n_k$; $\sum_{n \in P} m_r\left(H_n\right) \in U$ for every $r = 1, \cdots, n_k$ and $P \subset P_k$;

- $m_j\left(H_l\right) \in W_j$ for any $j \geq n_k$ and $l = 1, \cdots, p_k$.

As $\mathcal{F}$ is block-respecting, proceeding analogously as in the proof of Theorem 3.2, we find a set

$J_1 := \{j_1, j_3, j_5, \cdots\} \in \mathcal{F}^*$, $J_1 \subset I$, with $n_k \le j_k < n_{k+1}$ for every $k \in \mathbb{N}$. For any $h \in \mathbb{N}$ we have:

$$\sum_{q=1}^{\infty} m_{j_{2h-1}}\left(H_{j_{2h+2q-1}}\right) \in U, \tag{15}$$

$m_{j_{2h-1}}\left(H_{j_l}\right) \in W_{2h-1}$, $h \ge 2$, $l = 1, 3, \cdots, 2h-3$, and

$$m_{j_{2h-1}}\left(H_{j_1} \bigcup H_{j_3} \bigcup \cdots \bigcup H_{j_{2h-3}}\right) \in (h-1)W_{2h-1}. \tag{16}$$

Put $P^* := \{j_{2q-1} : q \in \mathbb{N}\}$. If $\sum_{p \in P^*} m_{j_{2h-1}}\left(H_p\right) \in W_{j_{2h-1}}$, then from (15) and (16) we get $m_{j_{2h-1}}\left(H_{j_{2h-1}}\right) \in hW_{2h-1} + U \subset j_{2h-1}W_{j_{2h-1}} + U = V_{j_{2h-1}}$ and $m_{j_1}\left(H_{j_1}\right) \in W_{j_1} + U \subset j_1 W_{j_1} + U = V_{j_1}$. This contradicts $m_{j_{2h-1}}\left(H_{j_{2h-1}}\right) \notin V_{j_{2h-1}}$. Thus $\sum_{p \in P^*} m_{j_{2h-1}}\left(H_p\right) \notin W_{j_{2h-1}}$ for every $h \in \mathbb{N}$, and hence $\{l \in \mathbb{N} : m_l(A) \notin W_l\} \in \mathcal{F}^*$. From this, arguing as in 3.2.1), we obtain a contradiction, and this proves 3.4.2). From 3.4.2), proceeding analogously as in the proof of Theorem 3.2, we get 3.4.3). □

**Open problems:** (a) Find some versions of limit theorems with respect to some other classes of filters and/or algebras satisfying suitable properties.

(b) Find some convergence and boundedness theorems with respect to other kinds of $(s)$-boundedness, $\sigma$-additivity or boundedness.

## 3. Conclusions

The Schur, Nikodým convergence and boundedness, Vitali-Hahn-Saks and Dieudonné limits have been widely investigated in the literature since the beginning of the last century, and there are several extensions of them along different directions, concerning for example the set of definition of the involved set functions, their properties and the structure of their range. The novelties in our context are both the structure of the space in which the considered measures take values and the types of convergence: indeed we deal with filter/ideal convergence in lattice groups introduced in [1]. Moreover we use a new technique in the filter setting, inspired by those in [27]-[30] and [18], and similar to that in [20], which allows us to prove some Nikodým-type convergence and boundedness theorems with respect to filter convergence for measures defined on a $\sigma$-algebra of parts of an abstract nonempty set $G$ with a direct approach, without using earlier Schur-type results for measures defined on $\mathcal{P}(\mathbb{N})$. In this context the main properties of diagonal and block-respecting filters are used, which allow us to apply the sliding hump argument to filter convergence. So it is possible to pose the question of finding some other classes of filters for which similar theorems hold or for which they are not valid, for measures, or even not necessarily finitely additive set functions, taking values in different types of abstract structures, like topological or lattice groups or metric semigroups, and defined in algebras, which satisfy similar properties and are not necessarily $\sigma$-algebras. Note that, in general, filter convergence is not inherited by subsequences. In this context, another problem that one can pose is to find similar results on convergence and boundedness theorems for non-additive set functions, or results like some kinds of uniform $(s)$-boundedness of $\sigma$-additivity when the limits of the involved sequences are intended in the filter sense, and/or with respect some other kinds of convergence in the lattice group-context, like order convergence (see also [2]).

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