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β-Hausdorff Operator on Lipschitz Space in the Unit Polydisk*

Rong Hu, Chaofeng Zhang#

School of Mathematics and Finance-Economics, Sichuan University of Arts and Science, Dazhou, China
Email: #zbeyondiee@163.com

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Abstract

In this paper, we define β-Hausdorff operator on the unit polydisk and study the boundedness of the operator on Lipschitz space. Firstly, we translate the problem of coefficient into integral of weighted composition operator, then give the sufficient conditions of boundedness, and also obtain an upper bound for the operator norm on Lipschitz space.

Keywords

Unit Polydisk, Lipschitz Space, β-Hausdorff Operator, Weighted Composition Operator

1. Introduction

Let Δ be the forward difference operator defined on sequences $\{\mu_n\}_{n=0}^{\infty}$ by $\Delta\mu_n = \mu_n - \mu_{n+1}$. Let operator F_β^k be

$$F_\beta^k \mu_n = \Delta\left(F_\beta^{k-1}\mu_n\right), F_\beta^0 \mu_n = \mu_{\beta n}, k \in \mathbb{N}, \beta \in \mathbb{N}, \beta \ge 1.$$

Define the β-Hausdorff matrix $H_{\mu_n}^\beta$ as the lower triangular matrix $\left(c_{n,k}^\beta\right)$ with entries

$$c_{n,k}^\beta = \begin{pmatrix} n \\ k \end{pmatrix} F_\beta^{n-k}\mu_k, k \le n.$$

For $\beta = 1$, it is the Hausdorff matrix $H\left(\mu_n\right)$, see [1].

When μ_n is the moment sequence of a measure *i.e.* $\mu_n = \int_0^1 t^n \mathrm{d}\mu(t), n \in \mathbb{N}$, the matrix arising from a Borel measure μ is denoted by H_μ^β, a simple calculation then gives

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#Corresponding author.

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$$c_{n,k}^{\beta}=\binom{n}{k}\int_0^1 t^{\beta k}(1-t)^{n-k}\,\mathrm{d}\mu(t), k\le n.$$

Let U^n be the unit polydisk in the complex vector space $\mathbb{C}^n$, $H(U^n)$ be the space of all holomorphic functions on U^n, and $\mu_i, i=1,\cdots,n$ be the Borel measures on $(0,1)$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\cdots,\mu_n)$, $\mathrm{d}\mu(t)=\Pi_{j=1}^n \mathrm{d}\mu_j(t_j)$.

In [2], the Lipschitz space $Lip_\alpha(U^n)(0\le\alpha<1)$ is defined on U^n by

$$Lip_\alpha(U^n)=\left\{f \mid f\in H(U^n) \text{ and } \|f\|_\alpha<\infty\right\}$$

where $\|f\|_\alpha=|f(0)|+\sup_{z\in U^n}\sum_{i=1}^n\left|\frac{\partial f}{\partial z_i}(z)\right|\left(1-|z_i|^2\right)^{1-\alpha}$. It is easy to prove that $Lip_\alpha(U^n)$ is a Banach space under the norm $\|\cdot\|_\alpha$.

Let $f\in H(U^n)$, suppose $f(z)=\sum_{m_1,\cdots,m_n\ge0}a_{m_1,\cdots,m_n}z_1^{m_1}\cdots z_n^{m_n}$, and $H_{\mu_i}^\beta=\left(c_{n,k}^\beta(\mu_i)\right), i=1,2,\cdots,n$ be the β-Hausdorff matrices arising by Borel measures μ_i. The β-Hausdorff operator $\mathcal{H}_{\boldsymbol{\mu}}^\beta(\beta\in\mathbb{N},\beta\ge1)$ is defined as follows: $\mathcal{H}_{\boldsymbol{\mu}}^\beta(f)(z)=\sum_{m_1,\cdots,m_n\ge0}\sum_{0\le i_j\le m_j, j=1,\cdots,n}a_{i_1,\cdots,i_n}\prod_{j=1}^n c_{m_j,i_j}^\beta(\mu_j)z_1^{m_1}\cdots z_n^{m_n}$. For $\beta=1$, we obtain the classical Hausdorff operator $\mathcal{H}_{\boldsymbol{\mu}}$, see [3].

Hausdorff matrix and Hausdorff operator have studied on various space of holomorphic functions, see, e.g., [3]-[9]. In [3], the author obtained that the Hausdorff operator $\mathcal{H}_{\boldsymbol{\mu}}$ is bounded on Hardy space $H^p(1\le p<\infty)$, and in [4] we showed that this conclusion cannot be extended to the Bloch space directly. Then we try to study on the Lipschitz space, found that when the measure is common Lebesgue measure dt, the Hausdorff operator $\mathcal{H}_t$ is unbounded on Lipschitz space $Lip_\alpha(U^n)$, see the remark. In this paper, we study the operator which is got by amending the Hausdorff operator and called it β-Hausdorff operator. The results of this paper can be deemed as a continuation of the results in [3] on Lipschitz space.

2. Main Results

The main results in this paper is the following:

Theorem 1 Let $\mu_i(i=1,\cdots,n)$ be finite Borel measures on (0,1) and $H_{\mu_i}^\beta=\left(c_{n,k}^\beta(\mu_i)\right), i=1,\cdots,n$ be corresponding β-Hausdorff matrices, $\mathcal{H}_{\boldsymbol{\mu}}^\beta$ be β-Hausdorff operator. For $\beta\ge2$, $\mathcal{H}_{\boldsymbol{\mu}}^\beta$ is bounded on $Lip_\alpha(U^n)$ if

$$\int_{[0,1]^n}\prod_{j=1}^n\frac{1}{t_j}\sum_{k,l=1}^n\frac{1}{t_k(1-t_l^{2(\beta-1)})^{1-\alpha}}\mathrm{d}\mu(t)<\infty,$$

In this case, the operator norm satisfies

$$\left\|\mathcal{H}_{\boldsymbol{\mu}}^\beta\right\|\le\mu\left((0,1)^n\right)+C\int_{[0,1]^n}\prod_{j=1}^n\frac{1}{t_j}\sum_{k,l=1}^n\frac{1}{t_k\left(1-t_l^{2(\beta-1)}\right)^{1-\alpha}}\mathrm{d}\mu(t).$$

for some constant C.

In order to prove the main results, we need some auxiliary result.

Lemma 1 [2] Let $f\in Lip_\alpha(U^n)$, then $|f(z)|\le|f(0)|+\|f\|_\alpha\sum_{i=1}^n\left(1-|z_i|^2\right)^{\alpha-1}, z\in U^n$.

For each $t_j\in(0,1)$, we note the functions $\phi_{t_j}^\beta$ given by $\phi_{t_j}^\beta(z_j)=\dfrac{t_j^\beta z_j}{(t_j-1)z_j+1}$,

Lemma 2 Let $\mu_i, i=1,\cdots,n$ be finite Borel measures on $(0,1)$ and $H_{\mu_i}^\beta=\left(c_{n,k}^\beta(\mu_i)\right)$ be corresponding β-Hausdorff matrices. Suppose

$$f(z)=\sum_{m_1,\cdots,m_n\ge0}a_{m_1,\cdots,m_n}z_1^{m_1}\cdots z_n^{m_n}\in Lip_\alpha(U^n).$$

Then,
(a) The power series $\mathcal{H}_{\mu}^{\beta}(f)$ in (2) represents a holomorphic functions on U^n;
(b) $\mathcal{H}_{\mu}^{\beta}(f)$ can be written in terms of weighted composition operators as follows:

$$\mathcal{H}_{\mu}^{\beta}(f)(z)=\int_{[0,1]^n}\frac{1}{\prod_{j=1}^{n}\left(1+\left(t_j-1\right)z_j\right)}f\left(\phi_{t_1}^{\beta}(z_1),\cdots,\phi_{t_n}^{\beta}(z_n)\right)\mu(t).\text{ For each } z\in U^n.$$

Proof (a) Let $A_{m_1,\cdots,m_n}=\sum_{0\le i_j\le m_j,j=1,\cdots,n}a_{i_1,\cdots,i_n}\prod_{j=1}^{n}c_{m_j,i_j}^{\beta}(\mu_j)$. Since $Lip_{\alpha}(U^n)$ the sequence of Taylor coefficients of f is bounded by a constant $M<\infty$, then

$$\left|A_{m_1,\cdots,m_n}\right|\le\sum_{0\le i_j\le m_j,j=1,\cdots,n}\left|a_{i_1,\cdots,i_n}\right|\prod_{j=1}^{n}c_{m_j,i_j}^{\beta}(\mu_j)\le M\prod_{j=1}^{n}\int_0^1\sum_{i_j=0}^{m_j}\binom{m_j}{i_j}t_j^{\beta i_j}\left(1-t_j\right)^{m_j-i_j}\mu_j(t_j)\le M\prod_{j=1}^{n}\mu_j(0,1).$$

Hence the coefficients of the series (2) are bounded and consequently $\mathcal{H}_{\mu}^{\beta}(f)$ is defined and analytic on U^n.
(b) By the Schwarz lemma we have $\left|\phi_{t_j}^{\beta}(z_j)\right|\le\left|z_j\right|$ for each $t_j\in(0,1)$. Hence applying (3) we have

$$\sup_{0<t_i<1,i=1,\cdots,n}\left|f\left(\phi_{t_1}^{\beta}(z_1),\cdots,\phi_{t_n}^{\beta}(z_n)\right)\right|\le\sup_{|\zeta_i|\le|z_i|,i=1,\cdots,n}\left|f(\zeta_1,\cdots,\zeta_n)\right|\le\left|f(0)\right|+\|f\|_{\alpha}\sum_{i=1}^{n}\left(1-|z_i|^2\right)^{\alpha-1},$$

On the other hand,

$$\frac{1}{\left|\prod_{j=1}^{n}\left(1+\left(t_j-1\right)z_j\right)\right|}\le\frac{1}{\prod_{j=1}^{n}\left(1-\left|z_j\right|\right)},$$

Hence

$$G^{\beta}(z)=\int_{[0,1]^n}\frac{1}{\prod_{j=1}^{n}\left(1+\left(t_j-1\right)z_j\right)}f\left(\phi_{t_1}^{\beta}(z_1),\cdots,\phi_{t_n}^{\beta}(z_n)\right)\mu(t)$$

is finite and analytic on U^n.
Now we proof $\mathcal{H}_{\mu}^{\beta}(f)=G^{\beta}(z)$, in order to avoid tedious calculations, we may assume that $n=2$, For a fixed $z=(z_1,z_2)\in U^2$, we have

$$\begin{aligned}G^{\beta}(z)&=\int_{[0,1]^2}\frac{1}{\prod_{j=1}^{2}\left(1+\left(t_j-1\right)z_j\right)}f\left(\phi_{t_1}^{\beta}(z_1),\phi_{t_2}^{\beta}(z_2)\right)\mu(t)\\&=\int_{[0,1]^2}\sum_{k,l\ge0}a_{k,l}z_1^kz_2^l\frac{t_1^{\beta k}t_2^{\beta l}\mu_1(t_1)\mu_2(t_2)}{\left(1+\left(t_1-1\right)z_1\right)^{k+1}\left(1+\left(t_2-1\right)z_2\right)^{l+1}}\\&=\sum_{k,l\ge0}a_{k,l}z_1^kz_2^l\int_0^1\frac{t_1^{\beta k}\mu_1(t_1)}{\left(1+\left(t_1-1\right)z_1\right)^{k+1}}\int_0^1\frac{t_2^{\beta l}\mu_2(t_2)}{\left(1+\left(t_2-1\right)z_2\right)^{l+1}}.\end{aligned}$$

It easy to see that

$$\int_0^1\frac{t_j^{\beta k}\mu_j(t_j)}{\left(1+\left(t_j-1\right)z_j\right)^{k+1}}=\int_0^1\sum_{n=k}^{\infty}\binom{n}{k}t_j^{\beta k}\left(1-t_j\right)^{n-k}z_j^{n-k}\mu_j(t_j)=\sum_{n=k}^{\infty}c_{n,k}^{\beta}(\mu_j)z_j^{n-k},$$

Hence,

$$G^{\beta}(z)=\sum_{k,l\ge0}a_{k,l}z_1^kz_2^l\sum_{n=k}^{\infty}c_{n,k}^{\beta}(\mu_1)z_1^{n-k}\sum_{m=l}^{\infty}c_{m,l}^{\beta}(\mu_2)z_2^{m-l}=\sum_{m,n\ge0}\left(\sum_{k=0}^{n}\sum_{l=0}^{m}a_{k,l}c_{n,k}^{\beta}(\mu_1)c_{m,l}^{\beta}(\mu_2)\right)z_1^nz_2^m.$$

Denote T_t^{β} as follows

$$T_t^{\beta}(f)(z)=\frac{1}{\prod_{j=1}^{n}\left(1+\left(t_j-1\right)z_j\right)}f\left(\phi_{t_1}^{\beta}(z_1),\cdots,\phi_{t_n}^{\beta}(z_n)\right),$$

where $\phi_{t_i}^{\beta}(z_i)(1\le i\le n)$ is defined in (4).

Now we obtain estimates for the norms of the weighted composition operator T_t^{β} .

Lemma 3 Suppose $\beta\ge 2$, then T_t^{β} is bounded on $Lip_{\alpha}\left(U^n\right)$. Further more, there is a constant $C>0$ such that $\left\|T_t^{\beta}\right\|\le C\prod_{j=1}^{n}\frac{1}{t_j}\sum_{k,l=1}^{n}\frac{1}{t_k\left(1-t_l^{2(\beta-1)}\right)^{1-\alpha}}$. For each $t_i\in(0,1), i=1,\cdots,n$.

Proof Let $T_t^{\beta}(f)(z)={}_t(z)f\circ\phi_t^{\beta}(z)$, in which ${}_t(z)=\frac{1}{\prod_{j=1}^{n}\left(1+\left(t_j-1\right)z_j\right)}$, $\phi_t^{\beta}(z)=\left(\phi_{t_1}^{\beta}(z_1),\cdots,\phi_{t_n}^{\beta}(z_n)\right)$, and the function $\phi_{t_i}^{\beta}(z_i)(1\le i\le n)$ is defined in (4).

$$\begin{aligned}&\sum_{k=1}^{n}\left|\frac{\partial\left(T_t^{\beta}(f)\right)}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^{1-\alpha}\\&\le\sum_{k=1}^{n}\left|\frac{\partial_t(z)}{\partial z_k}\right|\left|f\left(\phi_t^{\beta}(z)\right)\right|\left(1-|z_k|^2\right)^{1-\alpha}+\left|{}_t(z)\right|\sum_{k,l=1}^{n}\left|\frac{\partial f}{\partial\omega_l}\left(\phi_t^{\beta}(z)\right)\frac{\partial\varphi_{t_l}^{\beta}(z_l)}{\partial z_k}\right|\left(1-|z_k|^2\right)^{1-\alpha}\\&\le\sum_{k=1}^{n}\left|\frac{\partial_t(z)}{\partial z_k}\right|\left(2\|f\|_{\alpha}\sum_{l=1}^{n}\left(1-\left|\phi_{t_l}^{\beta}(z_l)\right|^2\right)^{\alpha-1}\right)\left(1-|z_k|^2\right)^{1-\alpha}+\left|{}_t(z)\right|\sum_{k,l=1}^{n}\left|\frac{\partial\phi_{t_l}^{\beta}(z_l)}{\partial z_k}\right|\left(\frac{\left(1-|z_k|^2\right)}{1-\left|\phi_{t_l}^{\beta}(z_l)\right|^2}\right)^{1-\alpha}\|f\|_{\alpha}\\&\le\left[2\prod_{j=1}^{n}\frac{1}{t_j}\sum_{k,l=1}^{n}\frac{1}{t_k}\left(1-t_l^{2(\beta-1)}\right)^{\alpha-1}+\prod_{j=1}^{n}\frac{1}{t_j}\sum_{k=1}^{n}t_k^{\beta-2}\right]\|f\|_{\alpha}.\end{aligned}$$

and $\left|T_t^{\beta}(f)(0)\right|=|f(0)|\le\|f\|_{\alpha}$. Hence we obtain that

$$\left\|T_t^{\beta}(f)\right\|_{\alpha}\le C\|f\|_{\alpha}\prod_{j=1}^{n}\frac{1}{t_j}\sum_{k,l=1}^{n}\frac{1}{t_k\left(1-t_l^{2(\beta-1)}\right)^{1-\alpha}}.$$

Now we proof the main results.

The Proof of Theorem 1 For each $z\in U^n, f\in Lip_{\alpha}\left(U^n\right)$, by (5) we can obtain

$$\begin{aligned}\sum_{k=1}^{n}\left|\frac{\partial\left(\mathcal{H}_{\mu}^{\beta}(f)\right)}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^{1-\alpha}&=\sum_{k=1}^{n}\left|\frac{\partial}{\partial z_k}\int_{[0,1]^n}T_t^{\beta}(f)(z)\mu(t)\right|\left(1-|z_k|^2\right)^{1-\alpha}\\&\le\int_{[0,1]^n}\sum_{k=1}^{n}\left|\frac{\partial}{\partial z_k}\left(T_t^{\beta}(f)\right)(z)\right|\left(1-|z_k|^2\right)^{1-\alpha}\mu(t)\le\int_{[0,1]^n}\left\|T_t^{\beta}(f)\right\|_{\alpha}\mu(t).\end{aligned}$$

Then by (1) and (6),

$$\begin{aligned}\left\|\mathcal{H}_{\mu}^{\beta}(f)\right\|_{\alpha}&=\left|\mathcal{H}_{\mu}^{\beta}(f)(0)\right|+\sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial\left(\mathcal{H}_{\mu}^{\beta}(f)\right)}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^{1-\alpha}\\&\le\mu\left((0,1)^n\right)\|f\|_{\alpha}+C\|f\|_{\alpha}\int_{[0,1]^n}\prod_{j=1}^{n}\frac{1}{t_j}\sum_{k,l=1}^{n}\frac{1}{t_k\left(1-t_l^{2(\beta-1)}\right)^{1-\alpha}}\mathrm{d}\mu(t).\end{aligned}$$

from which the result follows.

Remark When the Borel measure $\mu(t)$ is the common Lebesgue measure $\mathrm{d}t$, the Hausdorff operator arising from measure $\mathrm{d}t$ is denoted as $\mathcal{H}_t$. $\mathcal{H}_t$ is bounded on Hardy space $H^p\left(U^n\right)(1\le p<\infty)$, see [3].

However, it is unbounded on Lipschitz space. For example, fix $k(1 \le k \le n)$, and let $f(z) = \ln(1 - z_k)$, it is easy to see that $f \in Lip(U^n)$, then

$$\mathcal{H}_t(f)(z) = \left(-\frac{1}{2z_k}\right)\ln^2(1-z_k)\prod_{\substack{j=1\\ j\neq k}}^{n}\left(-\frac{1}{z_j}\right)\ln(1-z_j)$$

From this it follows that

$$\left\|\mathcal{H}_t(f)\right\|_\alpha \ge \sup_{z\in U^n}\left|\frac{\partial(\mathcal{H}_t(f))}{\partial z_k}(z)\right|\left(1-|z_k|^2\right)^{1-\alpha} \ge \sup_{z_k=r\in(0,1)} C\left|\frac{\ln^2(1-r)}{2r^2}+\frac{\ln(1-r)}{r(1-r)}\right|\left(1-r^2\right)^{1-\alpha} = \infty.$$

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