Scientific
Research

A Comparison of Sufficiency Condtions for the Goldbach and the Twin Primes Conjectures

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Abstract

It is generally known that under the generalized Riemann hypothesis one could establish the twin primes conjecture by the circle method, provided one could obtain the estimate $o\left(n\log^{-2}n\right)$ for the integral of the representation function over the minor arcs. One of the new results here is that the assumption of GRH can be removed. We compare this and other such sufficiency results with similar results for the Goldbach conjecture.

1. Introduction

Let $n \ge n_0$ and $\log^{15} n \le P \le n^\epsilon$ for $\epsilon > 0$. Let $x_0 = x_0(n) \to 0$ rapidly. When $0 < h \le q \le P$ and $(h, q) = 1$ let $M(h, q)$ denote the closed interval $\left[\frac{h}{q} - x_o, \frac{h}{q} + x_0\right]$, a so-called major arc.

It is easily shown, for any choice of P, that all the $M(h, q)$ are disjoint and contained in the closed interval $[x_0, 1 + x_0]$.

For each n let $m(n)$ be those points in $[x_0, 1 + x_0]$ which are not in any closed neighborhood (major arc) of radius x_0 about any rational number $\frac{h}{q}$, where $(h, q) = 1$ and $q \le P$.

For each n let $m^*(n)$ be those points in $[x_0, 1 + x_0]$ which are not in any closed neighborhood (major arc)

of radius x_0 about any rational number $\dfrac{h}{q}$, where $(h,q)=1$, $(q,n)=1$ and $q \le \log^{15} n$.

Let $r(n)$ denote the number of ways n (even) can be represented as a sum of two primes.
Let $r'(n)$ denote the number of twin primes less than or equal to n.

$$\text{Let } f(x,v) = \sum_{p \le v} \epsilon(px) \text{ where } \epsilon(z) = \epsilon^{2\pi i z} \; (v \ge 0).$$

$$\text{Let } g(x,v) = \sum_{2 \le m \le v} \frac{\epsilon(mx)}{\log m} (v \ge 2), g(x,v) = 0 (v < 2).$$

$$Ls[v] = \sum_{2 \le m \le v} \frac{1}{\log m} = g(0,v) \sim \left(\frac{v}{\log v}\right), Ls[0] = Ls[1] = 0.$$

In [1] the following two theorems are established:

<u>Theorem 1.1</u> Under the generalized Riemann hypothesis with $P = n^\epsilon$ and $x_0 = \dfrac{\log_n^{15}}{n}$ if

$$\int_{m(n)} f^2(x,n)\epsilon(-nx)\,dx = 0(n\log^{-2} n), \text{ then } r(n) > 0 \text{ for all even } n \ge n_0.$$

<u>Theorem 1.2</u> Let $P = \log^{15} n$ and $x_0 = \dfrac{\log^{15} n}{n}$ if

$$\int_{m^*(n)} f^2(x,n)\epsilon(-nx)\,dx = 0(n\log^{-2} n), \text{ then } r(n) > 0 \text{ for all even } n \ge n_0.$$

In [2] the following two theorems are established:

<u>Theorem 1.3</u> Under the assumption that Siegel zeros do not exist with $P = \exp(c\log^{1/2} n)$ and $x_0 = \dfrac{P}{n}$ if

$$\int_{m(n)} f^2(x,n)\epsilon(-nx)\,dx = o(n\log^{-2} n), \text{ then } r(n) > 0 \text{ for all even } n \ge n_0.$$

<u>Theorem 1.4</u> Let $P = \exp(c\log^{1/2} n)$ and $x_0 = \dfrac{P}{n}$. If

$$\int_{m(n)} f^2(x,n)\epsilon(-nx)\,dx = o(nP^{-1/32}\log^{-2} n), \text{ then } r(n) > 0 \text{ for all even } n \ge n_0.$$

The proof of Theorem 1.3 and, in particular, the proof of Theorem 1.4 is very complicated.

In Section 6 of [2] it is shown, by a very complicated argument, that a particularly natural approach for eliminating the condition $(q,n)=1$ in Theorem 1.2 does not work.

As we will now see, the situation with regard to the twin prime conjecture is significantly, very less complicated. The reason is primarily because we only need to consider the Ramanujan sums $C_q(2)$ rather than the sums $C_q(n)$, which appear in all of the theorems above, related to the Goldbach conjecture.

In Section 2 we will establish

<u>Theorem 1.5</u> Let $P = \log^{15} n$ and $x_0 = \dfrac{\log^{15} n}{n}$. $r'(n) \to \infty$ if $\int_{m(n)} |f(x,n)|^2 \epsilon(-2x)\,dx = o(n\log^{-2} n)$ as n goes to infinity in some suitable sequence.

2. A Proof of Theorem 1.5

$$r'(n) = \int_{x_0}^{x_0+1} |f(x,n)|^2 \epsilon(-2x)\,dx.$$

We decompose the above integral

$$r'(n) = \int_{m(n)} |f(x,n)|^2 \epsilon(-2x)\,dx + \int_{M(n)} |f(x,n)|^2 \epsilon(-2x)\,dx.$$

$$r'(n) = A(n) + B(n).$$

It is immediate by the prime numbers theorem that

$$A(n) = O\left(n\log^{-1} n\right).$$

By definition

$$\int_{M(n)} \left|f\left(x,n\right)\right|^2 \epsilon\left(-2x\right)\mathrm{d}x = \sum_{\substack{q \leq \log^{15} n}} \sum_{\substack{0 < h \leq q \\ (h,q)=1}} T\left(h,q\right)$$

where

$$T\left(h,q\right) = \int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} \left|f\left(x,n\right)\right|^2 \epsilon\left(-2x\right)\mathrm{d}x.$$

<u>Lemma 2.1</u> Let $q \leq \log^{15} n$, $\left|y\right| \leq x_0, (h,q)=1$ and $n \geq n_0$. Then

$$\left|f\left(\frac{h}{q}+y,n\right) - \frac{u(q)}{\phi(q)}g\left(y,n\right)\right| \leq n\log^{-69} n$$

This is Theorem 58 in [3].

<u>Lemma 2.2</u> Under the hypothesis of Lemma 2.1 we have

$$\left|f^2\left(\frac{h}{q}+y,n\right) - \frac{u^2(q)}{\phi^2(q)}g^2\left(y,n\right)\right| \leq C_1 n^2 \log^{-69} n.$$

Proof. This follows immediately from Lemma 2.1 and the trivial inequalities $\left|f\left(x,n\right)\right| \leq n$ and $\left|g\left(y,n\right)\right| \leq n$ and the fact that if $\left|a\right| \leq n$ and $b \leq n$, then $\left|a^2 - b^2\right| \leq 2n\left|a-b\right|$.

Hence it is immediate that

$$\left|\left|f\left(\frac{h}{q}+y,n\right)\right|^2 - \frac{u^2(q)}{\phi^2(q)}\left|g\left(y,n\right)\right|^2\right| \leq C_1 n^2 \log^{-69} n.$$

By the change of variable $y = \left(x - \dfrac{h}{q}\right)$ we have

$$T\left(h,q\right) = \epsilon\left(-\frac{2h}{q}\right) \int_{-x_0}^{x_0} \left|f\left(\frac{h}{q}+y,n\right)\right|^2 \epsilon\left(-2y\right)\mathrm{d}y.$$

However, by (2.1) we have

$$\left|\epsilon\left(-\frac{2h}{q}\right) \int_{-x_0}^{x_0} \left|f\left(\frac{h}{q}+y,n\right)\right|^2 \epsilon\left(-2y\right)\mathrm{d}y - \frac{u^2(q)}{\phi^2(q)}\epsilon\left(-2\frac{h}{q}\right) \int_{-x_0}^{x_0} \left|g\left(y,n\right)\right|^2 \epsilon\left(-2y\right)\mathrm{d}y\right|$$

$$\leq \int_{-x_0}^{x_0} \left|\left|f\left(\frac{h}{q}+y,n\right)\right|^2 - \frac{u^2(q)}{\phi^2(q)}\left|g\left(y,n\right)\right|^2\right|\mathrm{d}y \leq \int_{-x_0}^{x_0} C_1 n^2 \log^{-69} n\mathrm{d}y$$

$$= 2x_0 C_1 n^2 \log^{-69} n = C_2 n\log^{-54} n.$$

Let $T_1(n) = \int_{-x_0}^{x_0} \left|g^2\left(y,n\right)\right|^2 \epsilon\left(-2y\right)\mathrm{d}y$ so that if $(h,q)=1$, $q \leq \log^{15} n$, then

$$\left|T\left(h,q\right) - \frac{u^2(q)}{\phi^2(q)}T_1(n)\epsilon\left(-\frac{2h}{q}\right)\right| \leq C_2 n\log^{-54} n.$$

Let $T(n) = \sum_{m_1,m_2} \log^{-1} m_1$, $\log^{-1} m_2$ with the condition of summation $n \geq m_1 \geq 2, n \geq m_2 \geq 2$ and $m_2 - m_1 = 2$. It is easy to see that

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q(y,n)\right|^2 \epsilon\left(-2y\right)\mathrm{d}y \text{ and } \frac{1}{3}n\log^{-2} n \leq T(n) \leq n.$$

For $(h,q)=1, q \le \log^{15} n$ we have

$$\left| \epsilon\left(-\frac{2h}{q}\right) \right| \left| \frac{u^2(q)}{u^2(q)} \right| \left| T(n)-T_1(n) \right| \le \frac{1}{\phi^2(q)}\left(2n\log^{-15} n\right).$$

Hence for $(h,q)=1$ and $q \le \log^{15} n$

$$\left| (h,q) - \frac{u^2(q)}{\phi^2(q)} T(n)\epsilon\left(-\frac{2h}{q}\right) \right| \le C_2 n \log^{-54} n + \frac{1}{\phi^2(q)}\left(2n\log^{-15} n\right);$$

so that for some fixed $q \le \log^{15} n$ we have:

$$\left| \sum_{\substack{0<h\le q \\ (h,q)=1}} T(h,q) - \frac{u^2(q)}{\phi^2(q)} T(n) \sum_{\substack{0<h\le q \\ (h,q)=1}} \epsilon\left(-\frac{2h}{q}\right) \right| \le \left(c_2 n \log^{-54} n\right)\phi(q) + \frac{1}{\phi^{4/3}(q)}\left(2n\log^{-15} n\right)\phi^{1/3}(q).$$

But $\phi(q) \le \log^{15} n$ and by definition

$$\sum_{\substack{0<h\le q \\ (h,q)=1}} \epsilon\left(-\frac{2h}{q}\right) = C_q(2);$$

so that it follows immediately that

$$\left| \sum_{\substack{0<h\le q \\ (h,q)=1}} T(h,q) - \frac{u^2(q)}{\phi^2(q)} T(n) c_q(2) \right| \le C_2 n \log^{-39} n + \frac{1}{\phi^{4/3}(q)}\left(2n\log^{-10} n\right).$$

Now summing over all $q \le \log^{15} n$ we have

$$\left| \sum_{q \le \log^{15} n} \sum_{\substack{0<h\le q \\ (h,q)=1}} T(h,q) - T(n) \sum_{q \le \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(2) \right| \le \left(C_2 n \log^{-39} n\right)\left(\log^{15} n\right) + \sum_{q < \log^{15} n} \frac{1}{\phi^{4/3}(q)}\left(2n\log^{-10} n\right)$$

$$\le C_2 n \log^{-24} n + C_3\left(2n\log^{-10} n\right) \le C_4 n \log^{-10} n.$$

since by Theorem 327 page 267 in [3]

$$\sum_{q \le \log^{15} n} \frac{1}{\phi^{4/3}(q)} \le C_3 \left(C_3 \text{ independent of } n\right).$$

Hence

$$\left| r'(n) - T(n) \sum_{q \le \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(2) \right| \le k(n) n \log^{-2} n + C_4 n \log^{-10} n,$$

where $k(n) \to 0$. Now let

$$R(n) = \sum_{q \le \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(2)$$

$$S = \sum_{q=1}^{\infty} \frac{u^2(q)}{\phi^2(q)} C_q(2)$$

<u>Lemma 2.3</u>

$$\sum_{q=1}^{\infty} \frac{u^2(q)}{\phi^2(q)} C_q(n) - \sum_{q \le y} \frac{u^2(q)}{\phi^2(q)} C_q(n) = 0\left(d(n)(\log\log 3y)^2 y^{-1}\right).$$

Proof. [4] page 211.
It is immediate by Lemma 2.3 that

$$S - R(n) = 0(1).$$

What remains to be done is to show S is bounded away from 0.
Let

$$f(q) = \frac{u(q)}{\phi(q)} C_q(2).$$

Since $u(q)$, $\phi(q)$ and $C_q(2)$ are all multiplicative functions of q, f is a multiplicative function of q. Also, by means of the trivial estimate on $C_q(2)$, namely 2, and a direct application of Theorem 327 page 267 in [5] we have

$$\sum_{q=1}^{\infty} |f(q)| \le 2 \sum_{q=1}^{\infty} \frac{1}{\phi^2(q)} < \infty;$$

so that by Theorem 2 [3] page 3 we have

$$S = \prod_p \sum_{m=0}^{\infty} f(p^m)$$

$$m = 0 \quad f(p^0) = f(1) = 1$$

$$m = 1 \quad f(p) = \frac{u^2(p)}{\phi^2(p)} C_p(2) = \frac{1}{(p-1)^2} C_p(2)$$

$$m \ge 2 \quad f(p) = 0$$

Hence

$$S = \prod_p \left(1 + \frac{1}{(p-1)^2} C_p(2)\right) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) > 2 \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^2}\right) = 1.$$

3. A Primitive Formulation of the Circle Method

1) Part I
We assume n (even) $> C_0$. $x_0 = \left(\frac{\log^{2+\epsilon} n}{n}\right)$.
For each n let $E_n = [x_0, 1-x_0]$.
Let $r(n)$ be the number of representations of n as the sum of two primes, each of which is less than n. Clearly,

$$r(n) = \int_{x_0}^{1+x_0} f^2(x,n) \epsilon(-nx) dx, \text{ for each even } n.$$

We decompose this integral

$$r(n) = m(n) + T(1,1),$$

where

$$m(n) = \int_{E_n} f^2(x,n) \epsilon(-nx) dx, \quad T(1,1) = \int_{1-x_0}^{1+x_0} f^2(x,n) \epsilon(-nx) dx.$$

Clearly, by Theorem 55 in [3] and the last paragraph on page 63 in [3] we have

$$|f(1,v) - g(0,v)| \le n \log^{-100} n, \text{ uniformly } (0 \le v \le n).$$

By direct application of the easily established Equation (151) in [3]

$$f(x_1 + x_2, v) = \epsilon(vx_2) f(x_1, v) - 2\pi i x_2 \int_0^v \epsilon(ux_2) f(x_1, u) du, \tag{3.0}$$

and the Equation (204) in [3]

$$g\left(x_1 + x_2, v\right) = \epsilon\left(vx_2\right)g\left(x_1, v\right) - 2\pi i x_2 \int_0^v \epsilon\left(ux_2\right)g\left(x_1, u\right)du. \tag{3.1}$$

We have for $\left|y\right| \le x_0$

$$\left|f\left(1 + y, n\right) - g\left(y, n\right)\right| \le \left(n\log^{-100} n\right)$$
$$\left(1 + 2\pi x_0 n\right) \le C_1 n \log^{-98+\varepsilon} n. \tag{3.2}$$

By the trivial inequalities $\left|f\left(x, n\right)\right| \le n$ and $\left|g\left(y, n\right)\right| \le n$ and the fact that if $\left|a\right| \le n$ and $\left|b\right| \le n$, then $\left|a^2 - b^2\right| \le 2n\left|a - b\right|$ with $a = f\left(1 + y, n\right)$ and $b = g\left(y, n\right)$, we have for $\left|y\right| \le x_0$

$$\left|f^2\left(1 + y, n\right) - g^2\left(y, n\right)\right| \le C_2 n^2 \log^{-98+\epsilon} n. \tag{3.3}$$

By the change of variable $y = \left(x - 1\right)$, we have

$$T\left(1, 1\right) = \int_{-x_0}^{x_0} f^2\left(1 + y, n\right)\epsilon\left(-ny\right)dy;$$

so that

$$\left|\int_{-x_0}^{x_0} f^2\left(1 + y, n\right)\epsilon\left(-ny\right)dy - \int_{-x_0}^{x_0} g^2\left(y, n\right)\epsilon\left(-ny\right)dy\right| \le \int_{-x_0}^{x_0}\left|f^2\left(1 + y, n\right) - g^2\left(y, n\right)\right|dy \le C_3 n\log^{-96+2\epsilon} n.$$

$$\text{Let } T_1\left(n\right) = \int_{-x_0}^{x_0} g^2\left(y, n\right)\epsilon\left(-ny\right)dy$$

$$\left|T\left(1, 1\right) - T_1\left(n\right)\right| \le C_3 n\log^{-96+2\epsilon} n$$

$$\text{Let } T\left(n\right) = \sum_{\substack{m_1, m_2 \\ m_1 \ge 2, m_2 \ge 2 \text{ and } m_1 + m_2 = n}} \log^{-1} m_1 \log^{-1} m_2. \tag{3.4}$$

Clearly

$$T\left(n\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g^2\left(y, n\right)\epsilon\left(-ny\right)dy.$$

Also, the number of terms on the right hand side of (3.4) is $\left(n - 3\right)$ and each term is greater than $\log^{-2} n$ and less than 1 so that

$$\frac{1}{3}n\log^{-2} n < T\left(n\right) < n.$$

$$\left|\sum_{m=2}^{m_1} \epsilon\left(m, y\right)\right| \le \frac{1}{\left|\sin \pi y\right|} \le \frac{1}{2\left|y\right|}; \left(m_1 \ge 2, 0 < \left|y\right| \le \frac{1}{2}\right).$$

Hence by definition of $g\left(y, n\right)$ and Abel's lemma we have

$$\left|g\left(y, n\right)\right| < \left|y\right|^{-1}, \left(0 < \left|y\right| \le \frac{1}{2}\right);$$

so that

$$\left|T\left(n\right) - T_1\left(n\right)\right| \le 2\int_{x_0}^{\frac{1}{2}} y^{-2}dy \le C_4 x_0^{-1} = C_4 n\log^{-(2+\epsilon)} n. \tag{3.5}$$

Hence,

162

$$\left|T(1,1)-T(n)\right| \le C_3 n \log^{-96+2\epsilon} n + C_4 n \log^{-(2+\epsilon)} n. \tag{3.6}$$

Let $S(n)=1$. Then $\left|T(n)S(n)-T(n)\right|=0$
so that

$$\left|r(n)-T(n)S(n)\right| \le C_5 n \log^{-(2+\epsilon)} n + \left|m(n)\right|;$$

so that we have

$$r(n) > 0 \text{ if } m(n)=o\left(n\log^{-2}n\right) \text{ as } n \text{ (even) goes to infinity.}$$

Remark. Unfortunately, the integral $m(n)$ cannot be $o\left(\dfrac{n}{\log^2 n}\right)$; since for almost all n, the integral is asymptotically $\left(S(n)-1\right)\left(\dfrac{n}{\log^2 n}\right)$ where $S(n)$ is the usual singular series.

We assume $n > C_0$. For each n let $E_n = \left[x_0, 1-x_0\right]$.
Let $r(n)$ be the number of twin primes, each of which is less than n.
Clearly,

$$r(n) = \int_{x_0}^{1+x_0} \left|f(x,n)\right|^2 \epsilon(-2x)\,dx.$$

We decompose this integral

$$r(n) = m(n) + T(1,1),$$

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where

$$m(n) = \int_{E_n} \left|f(x,n)\right|^2 \epsilon(-2x)\,dx, \quad T(1,1) = \int_{1-x_0}^{1+x_0} \left|f(x,n)\right|^2 \epsilon(-2x)\,dx.$$

Immediately, from (3.3) we have

$$\left|\left|f(1+y,n)\right|^2 - \left|g(y,n)\right|^2\right| \le \left|f^2(1+y,n) - g^2(y,n)\right| \le C_2 n^2 \log^{-98+\epsilon} n.$$

By the change of variable $y=(x-1)$, we have

$$T(1,1) = \int_{-x_0}^{x_0} \left|f(1+y,n)\right|^2 \epsilon(-2y)\,dy;$$

so that

$$\left|\int_{-x_0}^{x_0} \left|f(1+y,n)\right|^2 \epsilon(-2y)\,dy - \int_{-x_0}^{x_0} \left|g(y,n)\right|^2 \epsilon(-2y)\,dy\right| \le \int_{-x_0}^{x_0} \left|\left|f(1+y,n)\right|^2 - \left|g(y,n)\right|^2\right|\,dy < C_6 n \log^{-96+2\epsilon} n.$$

Let

$$T_1(n) = \int_{-x_0}^{x_0} \left|g(y,n)\right|^2 \epsilon(-2y)\,dy.$$

$$\left|T(1,1)-T_1(n)\right| \le C_6 n \log^{-96+2\epsilon} n.$$

Let

$$T(n) = \sum_{\substack{m_1 m_2 \\ m_1 \ge m_2, m_2 \ge 2 \text{ and } m_2-m_1=2.}} \log^{-1} m_1 \log^{-1} m_2. \tag{3.7}$$

163

Clearly,

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(y,n)|^2 \, \epsilon(-2y) \, \mathrm{d}y.$$

Also, the number of terms on the right hand side of (3.7) is $(n-3)$ and each term is greater than $\log^{-2} n$ and less than 1 so that

$$\frac{1}{3} n \log^{-2} n < T(n) < n.$$

Hence by definition of $g(y,n)$ and Abel's lemma we have

$$|g(y,n)| < |y|^{-1}, \left(0 < |y| < \frac{1}{2}\right);$$

so that

$$|T(n) - T_1(n)| \leq 2 \int_{x_0}^{\frac{1}{2}} y^{-2} \mathrm{d}y \leq C_4 n \log^{-(2+\epsilon)} n. \tag{3.8}$$

Hence

$$|T(1,1) - T(n)| \leq C_6 n \log^{-96+2\epsilon} n + C_4 n \log^{-(2+\epsilon)} n. \tag{3.9}$$

Let $S(n) = 1$. Then $|T(n)S(n) - T(n)| = 0$;
so that

$$|r(n) - T(n)S(n)| \leq C_7 n \log^{-(2+\epsilon)} n + |m(n)|;$$

so that we have $r(n) \to \infty$ if $m(n) = o(n/\log^2 n)$ as n goes to infinity in some suitable sequence.

2) Part II

For each (even) $n > C_0$ let p_n be a prime in $\left[2\left(\log^\theta n + \frac{1}{2}\right), 4\left(\log^\theta n + \frac{1}{2}\right)\right]$. Let E_n be those points in $[x_0, 1+x_0]$ which are not in $[1-x_0, 1+x_0]$ and not in any closed interval of radius x_0 about any rational number $\frac{h}{p_n}$ where

$$0 < h < p_n.$$

$$\text{Let } m = \left[\log^{\theta_1} n\right].$$

$$\text{Let } f_1(x,v) = \sum_{\substack{p \leq v \\ p \equiv 1 \bmod p_n \\ p \equiv 2 \bmod p_n \\ \vdots \\ p \equiv m \bmod p_n}} \epsilon(px).$$

Let $r(n)$ be the number of representations of n as the sum of two primes, which are limited to those primes in the m arithmetic progressions mod p_n.

Clearly,

$$r(n) = \int_{x_0}^{1+x_0} f_1^2(x,n) \epsilon(-nx) \, \mathrm{d}x.$$

We decompose this integral

$$r(n) = m(n) + T^*(1,1) + \sum_{0 < h < p_n} T(h, p_n),$$

where

$$m(n) = \int_{E_n} f_1^2(x,n)\epsilon(-nx)\,\mathrm{d}x,$$

$$T^*(1,1) = \int_{1-x_0}^{1+x_0} f_1^2(x,n)\epsilon(-nx)\,\mathrm{d}x,$$

$$T(h,p_n) = \int_{\frac{h}{p_n}-x_0}^{\frac{h}{p_n}+x_0} f_1^2(x,n)\epsilon(-nx)\,\mathrm{d}x, (0 < h < p_n),$$

$$|m(n)| \le \int_{x_0}^{1+x_0} |f_1(x,n)|^2\,\mathrm{d}x \le C_8\frac{m^2 n}{\phi(p_n)\log n} \le C_9\frac{n}{\log^{(\theta+1-2\theta_1)}n}$$

$$T^*(1,1) = T(1,1) + \int_{1-x_0}^{1+x_0}\left(f_1^2(x,n)-f^2(x,n)\right)\epsilon(-nx)\,\mathrm{d}x.$$

Conjecture 1.

$$I(n) = \int_{1-x_0}^{1+x_0}\left(f_1^2(x,n)-f^2(x,n)\right)\epsilon(-nx)\,\mathrm{d}x = o\left(\frac{n}{\log^2 n}\right) \text{ as } n \text{ (even) goes to infinity.}$$

We now estimate $T(h,p_n), 0 < h < p_n$.

$$f\left(\frac{h}{p_n},v\right) = \sum_{\substack{p \le v \\ p \equiv 1 \bmod p_n \\ \vdots \\ p \equiv m \bmod p_n}} \epsilon\left(\frac{ph}{p_n}\right) = \sum_{\substack{p \le v \\ p \equiv 1 \bmod p_n}} \epsilon\left(\frac{ph}{p_n}\right) + \cdots + \sum_{\substack{p \le v \\ p \equiv m \bmod p_n}} \epsilon\left(\frac{ph}{p_n}\right)$$

$$= \epsilon\left(\frac{h}{p_v}\right)\pi\left([v],p_n,1\right) + \cdots + \epsilon\left(\frac{mh}{p_n}\right)\pi\left([v],p_n,m\right).$$

But consider

$$\epsilon\left(\frac{h}{p_n}\right)\pi\left([v],p_v,1\right) - \epsilon\left(\frac{h}{p_v}\right)\frac{ls[v]}{\phi(p_n)}$$

$$\vdots$$

$$\epsilon\left(\frac{mh}{p_n}\right)\pi\left([v],p_v,m\right) - \epsilon\left(\frac{mh}{p_v}\right)\frac{ls[v]}{\phi(p_v)};$$

so that uniformly $0 \le v \le n$

$$\left|\epsilon\left(\frac{h}{p_n}\right)\pi\left([v],p_v,1\right) - \epsilon\left(\frac{h}{p_v}\right)\frac{ls[v]}{\phi(p_v)}\right| \le \left|\pi\left([v],p_v,1\right) - \frac{ls[v]}{\phi(p_v)}\right| \le n\log^{-100}n.$$

$$\vdots$$

$$\left|\epsilon\left(\frac{mh}{p_n}\right)\pi\left([v],p_v,m\right) - \epsilon\left(\frac{mh}{p_n}\right)\frac{ls[v]}{\phi(p_v)}\right| \le \left|\pi\left([v],p_v,m\right) - \frac{ls[v]}{\phi(p_v)}\right| \le n\log^{-100}n.$$

Hence for $0 < h < p_n$ uniformly $0 \le v \le n$

$$\left|f\left(\frac{h}{p_n},v\right) - \left(\sum_{l=1}^{m}\epsilon\left(\frac{lh}{p_n}\right)\right)\frac{ls[v]}{\phi(p_v)}\right| \le C_{10}mn\log^{-100}n.$$

165

So by (100) and (101) for $|y| \leq x_0, (0 < h < p_n)$

$$\left| f\left(\frac{h}{p_n} + y, n\right) - \left(\sum_{l=1}^{m} \epsilon\left(\frac{lh}{p_n}\right)\right) \frac{g(y,n)}{\phi(p_n)} \right| \leq C_{11} mn \log^{-98+\epsilon} n.$$

But since $m < (p_n - 1) = \phi(p_n)$, $\left| \dfrac{\sum_{l=1}^{m} \epsilon\left(\frac{lh}{p_n}\right) g(y,n)}{\phi(p_n)} \right| \leq n$ and since $\left| f\left(\dfrac{h}{p_n} + y, n\right) \right| \leq n$, we have by the in-

equalities immediately below (3.2) for $|y| \leq x_0, (0 < h < p_n)$

$$\left| f^2\left(\frac{h}{p_n} + y, n\right) - \left(\sum_{l=1}^{m} \epsilon\left(\frac{lh}{p_n}\right)\right)^2 \frac{g^2(y,n)}{\phi^2(p_n)} \right| \leq C_{12} mn^2 \log^{-98+\epsilon} n.$$

But

$$\left| \left(\sum_{l=1}^{m} \epsilon\left(\frac{lh}{p_n}\right)\right)^2 \left(\frac{g^2(y,n)}{\phi^2(p_n)}\right) - \left(\frac{g^2(y,n)}{\phi^2(p_n)}\right) \right| \leq C_{13} m^2 n^2 \log^{-2\theta-2} n.$$

Hence for $|y| \leq x_0, (0 < h < p_n)$

$$\left| f^2\left(\frac{h}{p_n} + y, n\right) - \frac{g^2(y,n)}{\phi^2(p_n)} \right| \leq C_{12} mn^2 \log^{-98+\epsilon} n + C_{13} m^2 n^2 \log^{-2\theta-2} n. \tag{3.10}$$

By a change of variable $y = \left(x - \dfrac{h}{p_n}\right)$ we have

$$T(h, p_n) = \epsilon\left(-\frac{nh}{p_n}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{p_n} + y, n\right) \epsilon(-ny) \, dy.$$

However,

$$\left| \epsilon\left(-\frac{nh}{p_n}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{p_n} + y, n\right) \epsilon(-ny) \, dy - \frac{1}{\phi^2(p_n)} \epsilon\left(-\frac{nh}{p_n}\right) \int_{-x_0}^{x_0} g^2(y,n) \epsilon(-ny) \, dy \right|$$

$$\leq \int_{-x_0}^{x_0} \left| f^2\left(\frac{h}{p_n} + y, n\right) - \frac{g^2(y,n)}{\phi^2(p_n)} \right| dy \leq C_{14} mn \log^{-96+2\epsilon} n + C_{15} m^2 n \log^{-2\theta+\epsilon} n.$$

Let

$$T_1(n) = \int_{-x_0}^{x_0} g^2(y,n) \epsilon(-ny) \, dy.$$

Hence, if $0 < h < p_n$,

$$\left| T(h, p_n) - \frac{T_1(n)}{\phi^2(p_n)} \epsilon\left(-\frac{mh}{p_n}\right) \right| \leq C_{16} m^2 n \log^{-2\theta+\epsilon} n.$$

Let $T(n) = \sum_{\substack{m_1, m_2 \\ m_1 \geq m_2, m_2 \geq 2 \text{ and } m_1 + m_2 = n}} \log^{-1} m_1 \log^{-1} m_2.$

By (3.5) we have

$$\left| T(n) - T_1(n) \right| \leq C_4 n \log^{-(2+\epsilon)} n,$$

and

$$\frac{1}{3}n\log^{-2}n < T(n) < n.$$

But

$$\left|\epsilon\left(-\frac{nh}{p_n}\right)\left(\frac{1}{\phi^2(p_n)}\right)\right|\left|T(n)-T_1(n)\right| \le C_4\frac{n\log^{-(2+\epsilon)}n}{\phi^2(p_n)}.$$

Hence, if $0 < h < p_n$,

$$\left|T(h,p_n)-\frac{T(n)}{\phi^2(p_n)}\epsilon\left(-\frac{nh}{p_n}\right)\right| \le C_{16}m^2 n\log^{-2\theta+\epsilon}n + C_4\frac{n\log^{-(2+\epsilon)}n}{\phi^2(p_n)};$$

so that

$$\left|\sum_{0<h<p_n}T(h,p_n)-\frac{T(n)}{\phi^2(p_n)}C_{p_n}(n)\right| \le C_{17}m^2 n\log^{-\theta+\epsilon}n + C_4\frac{n\log^{-(2+\epsilon)}n}{\phi(p_n)}$$

where

$$C_{p_n}(n) = \sum_{0<h<p_n}\epsilon\left(-\frac{nh}{p_n}\right).$$

But by (3.6) we have

$$\left|T(1,1)-T(n)\right| \le C_{18}n\log^{-(2+\epsilon)}n;$$

so that

$$\left|\sum_{0<h<p_n}T(h,p_n)+T(1,1)-T(n)\left(1+\frac{C_{p_n}(n)}{\phi^2(p_n)}\right)\right| \le C_{19}n\log^{-\theta+2\theta_1+\epsilon}n;$$

so that

$$\left|r(n)-T(n)\left(\frac{1+C_{p_n}(n)}{\phi^2(p_n)}\right)\right| \le C_{19}n\log^{-\theta+2\theta_1+\epsilon}n + \left|I(n)\right| + \left|m(n)\right|.$$

Let

$$S(n) = \left(1+\frac{C_{p_n}(n)}{\phi^2(p_n)}\right).$$

By Theorem 272 in [5]

$$C_{p_n}(n) = \begin{cases} -1 & \text{if } (n,p_n)=1 \\ (p_n-1) & \text{if } (n,p_n)>1 \end{cases}$$

so that $S(n) > \dfrac{1}{2}$, since we assume $n > C_0$.

Hence

$$\left|r(n)-T(n)S(n)\right| \le C_{19}n\log^{-\theta+2\theta_1+\epsilon}n + \left|I(n)\right| + \left|m(n)\right|$$

so that $r(n) > 0$ if Conjecture 1 is true.

Let $r(n)$ be the number of twin primes, each of which is in one of the arithmetic progressions mod p_n defined above.

Clearly,

$$r(n) = \int_{X_0}^{1+x_0}\left|f_1(x,n)\right|^2\epsilon(-2x)\,dx.$$

We decompose the integral

$$r(n) = m(n) + T^*(1,1) + \sum_{0 < h < p_n} T(h, p_n), \text{ where}$$

$$m(n) = \int_{E_n} |f_1(x,n)|^2 \, \epsilon(-2x) \, dx,$$

$$T^*(1,1) = \int_{1-x_0}^{1+x_0} |f_1(x,n)|^2 \, \epsilon(-2x) \, dx,$$

$$T(h, p_n) = \int_{\frac{h}{p_n}-x_0}^{\frac{h}{p_n}+x_0} |f_1(x,n)|^2 \, \epsilon(-2x) \, dx,$$

$$T^*(1,1) = T(1,1) + \int_{1-x_0}^{1+x_0} \left(|f_1(x,n)|^2 - |f(x,n)|^2 \right) \epsilon(-2x) \, dx.$$

Conjecture 2.

$$I(n) = \int_{1-x_0}^{1+x_0} \left(|f_1(x,n)|^2 - |f(x,n)|^2 \right) \epsilon(-2x) \, dx = o\left(\frac{n}{\log^2 n} \right), \text{ as } n \text{ goes to}$$

infinity in some suitable sequence.

From (3.10) we have for $|y| \le x_0, (0 < h < p_n)$

$$\left| \left| f\left(\frac{h}{p_n} + y, n \right) \right|^2 - \frac{|g(y,n)|^2}{\phi^2(p_n)} \right| \le C_{20} m^2 n^2 \log^{-2\theta-2} n.$$

By change of variable $y = \left(x - \frac{h}{p_n} \right)$ we have

$$T(h,q) = \epsilon\left(\frac{-2h}{p_n} \right) \int_{-x_0}^{x_0} \left| f^2\left(\frac{h}{p_n} + y, n \right) \right|^2 \epsilon(-2y) \, dy.$$

However,

$$\left| \epsilon\left(\frac{-2h}{n} \right) \int_{-x_0}^{x_0} \left| f\left(\frac{h}{p_n} + y, n \right) \right|^2 \in(-2y) - \frac{1}{\phi^2(p_n)} \epsilon\left(\frac{-2h}{n} \right) \int_{-x_0}^{x_0} |g(y,n)|^2 \, \epsilon(-2y) \, dy \right|$$

$$\le \int_{-x_0}^{x_0} \left| \left| f\left(\frac{h}{p_n} + y, n \right) \right|^2 - \frac{|g(y,n)|^2}{\phi^2(p_n)} \right| dy \le C_{21} m^2 n \log^{-2\theta+\epsilon} n.$$

Let

$$T_1(n) = \int_{-x_0}^{x_0} |g(y,n)|^2 \, \epsilon(-2y) \, dy.$$

Hence, if $0 < h < p_n$

$$\left| T(h, p_n) - \frac{T_1(n)}{\phi^2(p_n)} \epsilon\left(-\frac{2h}{p_n} \right) \right| \le C_{21} m^2 n \log^{-2\theta+\epsilon} n.$$

Let $T(n) = \sum_{\substack{m_1, m_2 \\ m_1 \ge 2, m_2 \ge 2 \text{ and } m_2 - m_1 = 2}} \log^{-1} m_1, \ \log^{-1} m_2.$

Clearly,

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(y,n)|^2 \, \epsilon(-2y) \, \mathrm{d}y.$$

So that by (3.8)

$$|T(n) - T_1(n)| \le C_4 n \log^{-(2+\epsilon)} n$$

and $\dfrac{1}{3} n \log^{-2} n < T(n) < n$.

$$\left| \epsilon\left(-\frac{2h}{p_n} \right) \left(\frac{1}{\phi^2(p_n)} \right) \|T(n) - T_1(n)\| \right| \le C_4 \frac{n \log^{-(2+\epsilon)} n}{\phi^2(p_n)}.$$

Hence, if $0 < h < p_n$,

$$\left| T(h, p_n) - \frac{T(n)}{\phi^2(p_n)} \epsilon\left(-\frac{2h}{p_n} \right) \right| \le C_{21} m^2 n \log^{-2\theta+\epsilon} n + C_4 \frac{n \log^{-(2+\epsilon)} n}{\phi^2(p_n)};$$

so that

$$\left| \sum_{0 < h < p_n} T(h, p_n) - \frac{T(n)}{\phi^2(p_n)} C_{p_n}(2) \right| \le C_{22} \log^{-\theta+2\theta_1+\epsilon} n + C_4 \frac{n \log^{-(2+\epsilon)} n}{\phi(p_n)}$$

where

$$C_{p_n}(2) = \sum_{0 < h < p_n} \epsilon\left(-\frac{2h}{p_n} \right).$$

But by (3.6)

$$|T(1,1) - T(n)| \le C_4 n \log^{-(2+\epsilon)} n;$$

so that

$$\left| \sum_{0 < h < p_n} T(h, p_n) + T(1,1) - T(n)\left(1 + \frac{C_{p_n}(2)}{\phi^2(p_n)} \right) \right| \le C_{22} n \log^{-\theta+2\theta_1+\epsilon} n.$$

Hence

$$\left| r(n) - T(n)\left(1 + \frac{C_{p_n}(2)}{\phi^2(p_n)} \right) \right| \le C_{22} n \log^{-\theta+2\theta_1+\epsilon} n + |I(n)| + |m(n)|.$$

Let $S(n) = \left(1 + \dfrac{C_{p_n}(2)}{\phi^2(p_n)} \right)$. $\quad S(n) > \dfrac{1}{2}$ if $n > C_0$.

Hence

$$|r(n) - T(n)S(n)| \le C_{22} n \log^{-\theta+2\theta_1+\epsilon} n + |I(n)| + |m(n)|$$

so that $r(n) \to \infty$; if Conjecture 2 is true, as n goes to infinity in a sequence that satisfies the conjecture.

4. Some Heuristics

<u>Theorem 4.1</u> If

$$\int_{[-(2+x_0),(2+x_0)]-M(n)} f^2(x,n) \epsilon(-nx) \, \mathrm{d}x = o\left(\frac{n}{\log^2 n} \right),$$

then every sufficiently large even integer is the sum of two primes, where $M(n)$ is an exceptional set, whose measure goes to 0 with n.

Proof. This is established in [1].

<u>Theorem 4.2</u> Let m be an arbitrary fixed integer. Then

$$\int_{-2}^{2} \frac{f^2(x,m)x\epsilon(-nx)}{x-t}\,\mathrm{d}x = o(\log\log n)$$

uniformly for almost all $t \in (-1/2, 1/2)$.

Proof. This deep result is immediate by (5-2) in [6].

There is no compelling reason to assume Theorem 4.2 is not true for $t = 0$.

It is worthwile to investigate if Carleson's proof can be modified to establish Theorem 4.2 with m replaced with n and $o(\log\log n)$ replaced with $o\left(\dfrac{n}{\log^2 n}\right)$.

In [7] Tao presents a heuristic argument to establish that the major arc contribution in the circle method is $o\left(\dfrac{n}{\log^2 n}\right)$. He states that his argument can be made rigorous.

However, it follows from the proof of Theorem 1.5 that the major arc contribution is not $o\left(\dfrac{n}{\log^2 n}\right)$ in any sequence of n.

But it is well known that $r'(n) = 0\left(\dfrac{n}{\log^2 n}\right)$ so that the contribution of the minor arc in the circle method approach to the twin primes conjecture (Theorem 1.5) is $o\left(\dfrac{n}{\log^2 n}\right)$, which makes plausible that the required estimate of $o\left(\dfrac{n}{\log^2 n}\right)$ might be true.

It is plausible that in Theorem 1.3

$$\int_{m(n)} f^2(x,m)\epsilon(-mx)\,\mathrm{d}x = 0\left(\int_{m(n)} |f(x,m)|^2 \epsilon(-2x)\,\mathrm{d}x\right)$$

where the latter integral is that of Theorem 1.5, which makes plausible that the required estimate of $o\left(\dfrac{n}{\log^2 n}\right)$ might be true.

Those, who seriously attempt Conjecture 2 have the advantage that there is some degree of freedom in the choice of $m = \left\lceil \log^{\theta_1} n \right\rceil$ and in the choice of p_n for each n; and the $o(n\log^{-2} n)$ estimate is required only as n goes to infinity in some suitable sequence.

Acknowledgements

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