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Some Mappings on Operator Spaces

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Abstract

We discuss two types of maps on operator spaces. Firstly, through example we show that there is an isometry on unit sphere of an operator space cannot be extended to be a complete isometry on the whole operator space. Secondly, we give a new characterization for complete isometry by the concept of approximate isometry.

Keywords

Unit Sphere, Approximate Isometry, Complete Isometry

1. Introduction

The theory of operator spaces is very recent. It was developed after Ruan's thesis (1988) by Effros and Ruan and Blecher and Paulsen and Pisier.

Theorem (Ruan) in [1] suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$. Then X is completely isometrically linearly isomorphic to a subspace of $B(H)$, for some Hilbert space H, if and only if conditions (R1) and (R2) above hold.

R1: $\|\alpha x \beta\|_n \le \|\alpha\| \|x\|_n \|\beta\|$ for $x \in M_n(V)$ and $\alpha, \beta \in M_n$,

R2: $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$ for all $x \in M_n(V)$ and $y \in M_m(V)$.

In this new category, the objects remain Banach spaces but the morphisms become the completely bounded maps. In fact, the completely bounded maps appeared in the early 1980s following Stinespring's pioneering work (1955) and Arveson's fundamental results (1969) on completely positive maps see [1]-[4].

Definition 1 Let X and Y be operator spaces and that $u: X \to Y$ be a linear map. For a positive integer n, we write u_n for the associated map $[x_{ij}] \mapsto [u(x_{ij})]$ from $M_n(X)$ to $M_n(Y)$. This is often called the (nth) amplification of u, and may also be thought of as the map $I_{M_n} \otimes u$ on $M_n \otimes X$. A map u is completely bounded (in short c.b.) if

$$\|u\|_{cb} := \sup\left\{\left\|\left[u\left(x_{ij}\right)\right]\right\|_n : \left\|\left[x_{ij}\right]\right\|_n \le 1, \text{ all } n \in \mathbb{N}\right\} < \infty.$$

We denote by $CB(X,Y)$ the Banach space of all c.b. maps from X into Y equipped with the c.b. norm. If $u: X \to Y$ is c.b. map, we have

$$\|u\| \le \|u\|_{cb}$$

and

$$CB(X,Y) \subset B(X,Y).$$

The notion of isomorphism is replaced by that of "complete isomorphism". If $u: X \to Y$ is a completely bounded linear bijection, and if its inverse is completely bounded too, then we say that u is complete isomorphism. In this case, we say that X and Y are completely isomorphic and we write "$X \approx Y$ as operator spaces". Similarly, if u_n is an isometry for any $n \in \mathbb{N}$, then we say that u is complete isometry. We often identify two operator spaces X and Y if they are completely isometrically isomorphic. In this case we often write "$X \cong Y$ as operator spaces".

In 1932 [5], Mazur and Ulam got the famous theorem that *a surjective isometry between two real normed spaces must be linear up to translation*, which started the study of the theory of isometry. Many researchers have tried to generalize it. In 1972 Mankiewicz [6] considered the extension problem for isometries and every surjective isometry between the open connected subsets of two normed spaces can be extended to a surjective affine isometry on the whole space. This result can be seen as local Mazur-Ulam Theorem. From this result, we get easily that a map's property on unit ball of a normed space determines the relationship between distance preserving and linearity. In 1987, D.Tingley raised the following problem (the isometric extension problem) in [7]:

Problem. Suppose that $V_0: S_1(X) \to S_1(F)$ *is a surjective isometric mapping, does there exist a linear isometric mapping* $V: X \to Y$ *such that* $V\big|_{S_1(X)} = V_0$?

D. Tingley, who is the first one to study the problem, gave some results in [7] under the condition that X and Y are finite dimensional spaces. More precisely, the answer can be formulated as follows: Suppose that $V_0: S_1(X) \to S_1(Y)$ is a surjective isometric mapping, then $V_0(-x) = -V_0(x)$, *i.e.* V_0 is an odd mapping. In the complex spaces, the answer to Tingleys problem is negative. For example, we take $X = Y = \mathbb{C}$ (complex plane) and $V(x) = \tilde{x}$. Some affirmative results have been obtained between classical real Banach spaces, which had been shown in [8]-[12]. The recent development on this problem you can find in [13]. We ask can we discuss the isometric extension problem in operator space category? Through example, we give some results on it.

2. Some Examples about Isometric Extension Problem on Operator Spaces

Theorem Let X be R (resp. C, OH, $MIN(H)$, $MAX(H)$, $(MIN(H), MAX(H))_\theta$, etc.). Let $V_0: S_1(X) \to S_1(X)$ be a 1-Lip operator. Then V_0 can be extended to a linearly complete isometry on X.

Proof. By [11], V_0 can be linearly extended to the whole Hilbert space. Since X are homogeneous operator space, the the result can be got directly.

How about the result on non-homogeneous operator spaces?

Theorem There exists an isometry $V_0: S_1(H_c \otimes H_r) \to S_1(H_c \otimes H_r)$ such that there is no complete isometry V on the whole space $H_c \otimes H_r$ satisfying $V\big|_{S_1(H_c \otimes H_r)} = V_0$.

Proof. Let $\left\{e_i = (0,\cdots,0,1,0\cdots,0)\right\}_{i=1}^n$ be the basis of $\mathbb{C}^n$. To express conveniently, we denote $\{e_i\}$ $(1 \le i \le n)$ to be the basis of the *n*-dimensional column Hilbert operator space H_c and $\{f_k\}$ $(1 \le k \le n)$ be the basis of the *n*-dimensional row Hilbert operator space H_r. Then $\{e_i \otimes f_k\}$ constructs a basis of $H_c \otimes H_r$. Give the n^2 elements an order defined as

$$h_i = \begin{cases} e_1 \otimes f_i & (1 \le i \le n); \\ e_{1+k} \otimes f_1 & (i = n+k, 1 \le k \le n-1); \\ \text{others} & (i \ge 2n). \end{cases}$$

Let $V(h_1)=h_1$, $V(h_i=h_{n+i-1}),(2\le i\le n)$ and V maps other elements $\{e_i\otimes f_j\}_{i,j=1}^{n}-\{e_1\otimes f_i\}_{i=1}^{n}$ of the basis to $\{e_i\otimes f_j\}_{i,j=1}^{n}-\{e_i\otimes f_1\}_{i=1}^{n}$. So V forms a permutation among the basis of $H_c\otimes H_r$. By the isometric theory on Hilbert space, V is an isometry on the Banach space $H_c\otimes H_r$. Of course let $V_0=V\big|_{H_c\otimes H_r}$ is an isometry between unit sphere of $H_c\otimes H_r$. But V is not complete isometry on $H_c\otimes H_r$.

Indeed, for $x=(x_{ij})\in M_n(H_c\otimes H_r)$ where

$$x=\begin{pmatrix} e_1\otimes f_1 & 0 & \cdots & 0\\ e_1\otimes f_2 & 0 & \cdots & 0\\ \cdots & 0 & \cdots & 0\\ e_1\otimes f_i & 0 & \cdots & 0\\ \cdots & 0 & \cdots & 0\\ e_1\otimes f_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$V_n(x)=\begin{pmatrix} e_1\otimes f_1 & 0 & \cdots & 0\\ e_2\otimes f_1 & 0 & \cdots & 0\\ \cdots & 0 & \cdots & 0\\ e_i\otimes f_1 & 0 & \cdots & 0\\ \cdots & 0 & \cdots & 0\\ e_n\otimes f_1 & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to calculate that $\|x\|_c=1$ but $\|V_n(x)\|_r=\sqrt{n}$. So V is not complete isometry.

3. On Stability of ϵ-Isometries of Operator Spaces

The study on the stability of ϵ-isometry is from Banach Space Theory.

Let X,Y be two Banach spaces and $\epsilon\ge 0$. A mapping $f:X\to Y$ is said to be an ϵ-isometry provided

$$\big|\|f(x)-f(y)\|-\|x-y\|\big|<\epsilon,\quad \text{for all } x,y\in X.$$

In 1945, Hyers and Ulam proposed the following question: whether for every surjective ϵ-isometry $f:X\to Y$ with $f(0)=0$ there exist a surjective linear isometry $U:X\to Y$ and $\gamma>0$ such that $\|f(x)-Ux\|\le\gamma\epsilon$ for all $x\in X$. After many efforts of a number of mathematicians (see, for instance [14]-[19]) Omladi $\breve{c}$ and $\breve{S}$ emrl gave the sharp estimate that $\gamma=2$ in 1995 (see [20]).

In this section, we ask whether we can discuss the the stability of ϵ-isometry in operator space category? We try to propose a question and give a little answer.

Firstly, we get the following result.

Theorem Let X and Y be operator spaces. If a mapping $T:X\to Y$ satisfies that for any $n\in\mathbb{N}$,

$$\big|\|T_n(x)-T_n(y)\|_n-\|x-y\|_n\big|\le\epsilon,\quad (\forall x,y\in M_n(X)),$$

then T is an isometry.

Proof. For any $n\in\mathbb{N}$, $T_n:M_n(X)\to M_n(Y)$. Let $x,y\in X$. Specially, we put $x_{ij}\equiv x, y_{ij}\equiv y,(1\le i,j\le n)$, then $[x_{ij}],[y_{ij}]\in M_n(X)$ and we have

$$T_n[x_{ij}]-T_n[y_{ij}]=(Tx-Ty)\otimes\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix},$$

Since

$$\lim_{n\to\infty}\left\|\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}\right\|_n = \infty,$$

we get $\left|\|Tx-Ty\|-\|x-y\|\right| \le \epsilon \left\|\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}\right\|^{-1}$.

So $\|Tx-Ty\| = \|x-y\|$ holds for any $x, y \in X$. T is an isometry.

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