Uniqueness of Meromorphic Functions of Differential Polynomials Sharing Two Values IM

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ABSTRACT

In this paper, we shall study the uniqueness problems of meromorphic functions of differential polynomials sharing two values IM. Our results improve or generalize many previous results on value sharing of meromorphic functions.

KEYWORDS

Differential Polynomial; Uniqueness; Meromorphic Function; Shared Value

1. Introduction

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions defined in the open complex plane $\mathbf{C}$. Let $a \in \mathbf{C} \cup \{\infty\}$, we say that $f(z)$ and $g(z)$ share a CM (counting multiplicities) if $f(z) - a$, $g(z) - a$ have the same zeros with the same multiplicities and we say that $f(z)$ and $g(z)$ share a (ignoring multiplicities) if we do not consider the multiplicities. We denote by $T(r, f)$ the Nevanlinna characteristic function of the meromorphic function f and by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measure. $N_k(r, f)$ denotes the truncated counting function bounded by k. Moreover, $GCD(n_1, n_2, \cdots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \cdots, n_k$.

For the sake of simplicity, let m be a nonnegative integer, $a_0 \neq 0, a_1, \cdots, a_{m-1}, a_m \neq 0$ be complex constants. Define

$$P(w) = \begin{cases} a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0 & (m > 0) \\ a_0 & (m = 0) \end{cases} \tag{1.1}$$

In 1929, Nevanlinna [1] proved the following well-know result which is the so called Nevanlinna five values theorem.

Theorem A *Let f and g be two non-constant meromorphic functions. If f and g share five distinct values IM, then $f \equiv g$.*

Moreover, he got.

Theorem B *Let f and g be two distinct non-constant meromorphic functions and $a_j (j = 1, 2, 3, 4)$ be four distinct values. If f and g share $a_j (j = 1, 2, 3, 4)$ CM, then f is a Mobius transformation of g.*

In 1976, L. Rubel asked the following question:

Whether CM can be replaced by IM in the hypothesis of Theorem A with the same conclusion or not?

In 1979, G. G. Gundersen [2] gave a negative answer for this question by the following counterexample:

$$f = (e^h + 1) / (e^h - 1)^2, \quad g = (e^h + 1)^2 / [8(e^h - 1)],$$

where h is a non-constant entire function. It is easy to verify that f *and* g share the four values $0, 1, \infty, -1/8$, where none of the four values are shared CM, and f is not a Mobius transformation of g.

On the other hand, G. G. Gundersen [3] proved the following result which is an improvement of Theorem B.

Theorem C. If two distinct non-constant meromorphic functions share two values CM and share two other values IM, then the functions share all four values CM (hence the conclusions of Theorem B hold).

In this paper, we shall show that similar conclusions hold for certain types of differential polynomials when they share two values IM.

Theorem 1.1 Let f and g be two non-constant meromorphic functions, $n > 0, k > 0$, and $m \geq 0$ be three integers with $n > 4m + 9k + 10$ and $P(w)$ be defined as in (1.1). If $\left[f^n P(f) \right]^{(k)}$ and $\left[g^n P(g) \right]^{(k)}$ share 1 and ∞ IM, then

1) when $m > 0$, $f^n P(f) \equiv g^n P(g)$;

2) when $m = 0$, one of the following two cases holds:

3) $f \equiv tg$ for a constant t such that $t^n = 1$,

4) $f(z) = c_1 / \sqrt[n]{a_0} \, \mathrm{e}^{cz}$, $g(z) = c_2 / \sqrt[n]{a_0} \, \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

Remark 1.1 "$\left[f^n P(f) \right]^{(k)}$ and $\left[g^n P(g) \right]^{(k)}$ share ∞ IM" $\Leftrightarrow$ f and g share ∞ IM". Moreover, from $f^n P(f) \equiv g^n P(g)$, one cannot get $f \equiv tg$ for some constant t. For example, let $P(f) = f - 1$, $P(g) = g - 1$, then $f = (1 - h^n) h / (1 - h^{n+1})$, $g = (1 - h^n) / (1 - h^{n+1})$, where h is a non-constant meromorphic function. Obviously, $f \not\equiv tg$ for some canstant t but $f^n (f - 1) \equiv g^n (g - 1)$.

Now we give some corollaries of Theorem 1.1. Corollary 1.2 and Corollary 1.3 improve Theorems D and E, respectively.

Corollary 1.2 Let f and g be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 9k + 10$. If $\left[f^n \right]^{(k)}$ and $\left[g^n \right]^{(k)}$ share 1 IM, f and g share ∞ IM, then either $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

Corollary 1.3 Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > 2/n$, and let n, k be two positive integers with $n > 9k + 14$. If $\left[f^n (f - 1) \right]^{(k)}$ and $\left[g^n (g - 1) \right]^{(k)}$ share 1 IM, f and g share ∞ IM, then $f \equiv g$.

Corollary 1.4 Let f and g be two non-constant meromorphic functions, and let $n, k, m (\geq 2)$ be two positive integers with $n > 4m + 9k + 10$, a be a nonzero constant. If $\left[f^n (f^m + a) \right]^{(k)}$ and $\left[g^n (g^m + a) \right]^{(k)}$ share 1 IM, f and g share ∞ IM, then $f \equiv tg$ for some constant t such that $t^d = 1$, where $d = GCD(n + m, n)$.

Theorem 1.1 generalizes the following result that was obtained by Zhang, Chen and Lin [4].

Theorem D Let f and g be two non-constant entire functions. Let n, k, and m be three positive integers with $n \geq 3m + 2k + 5$ and let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ or $P(w) \equiv c_0$, where $a_0 \neq 0, a_1, \cdots, a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $\left[f^n P(f) \right]^{(k)}$ and $\left[g^n P(g) \right]^{(k)}$ share 1 CM, then

1) when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$, either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \cdots, n + m - i, \cdots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \cdots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$,

where $R(\omega_1, \omega_2) = \omega_1^n \left(a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_0 \right) - \omega_2^n \left(a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_0 \right)$;

2) when $P(w) \equiv c_0$, either $f(z) = c_1 / \sqrt[n]{c_0} \, \mathrm{e}^{cz}$, $g(z) = c_2 / \sqrt[n]{c_0} \, \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants

satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

Corollaries 1.2-1.4 greatly improve the following result that was obtained by Liu [5] by reducing the lower bound of n. Moreover, the proofs of Corollaries 1.2 - 1.4 fill some gaps appeared in the proof of Theorem E.

Theorem E Let f and g be two non-constant meromorphic functions, and let n, k, and m be three positive integers with $n > 6m^* + 9k + 14$, and λ, μ be two constants such that $|\lambda| + |\mu| \neq 0$. If

$$\left[f^n \left(\mu f^m + \lambda \right) \right]^{(k)} \text{ and } \left[g^n \left(\mu g^m + \lambda \right) \right]^{(k)} \text{ share 1 IM, } f \text{ and } g \text{ share } \infty \text{ IM, then} \backslash\backslash$$

    1) when $\lambda \mu \neq 0$, If $m \geq 2$ and $\delta(\infty, f) > 3/(m+n)$, then $f \equiv g$.

If $m = 1$ and $\delta(\infty, f) > 3/(n+1)$, then $f \equiv g$;

    2) when $\lambda \mu = 0$, if $f \neq \infty$ and $g \neq \infty$, then either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} \left((n+m^*)c \right)^{2k} = 1 \text{ or } (-1)^k \mu^2 (c_1 c_2)^{n+m^*} \left((n+m^*)c \right)^{2k} = 1 \text{ Here, } m^* := \chi_\mu m \text{, where}$$

$\chi_\mu = 0$ if $\mu = 0$, $\chi_\mu = 1$ if $\mu \neq 0$.

2. Preliminary Lemmas

Let

$$H = F''/F' - 2F'/(F-1) - G''/G' + 2G'/(G-1), \tag{2.1}$$

$$V = F'/(F-1) - F'/F - G'/(G-1) + G'/G, \tag{2.2}$$

where F and G are meromorphic functions.

Lemma 2.1 [6] Let f be a non-constant meromorphic function and let $a_0(z), a_1(z), \cdots, a_n(z)(\not\equiv 0)$ be small functions with respect to f. Then

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0 \right) = nT(r, f) + S(r, f).$$

Lemma 2.2 [7] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then

$$N_s \left(r, 1/f^{(k)} \right) \leq T\left(r, f^{(k)} \right) - T(r, f) + N_{s+k}(r, 1/f) + S(r, f),$$

$$N_s \left(r, 1/f^{(k)} \right) \leq k\bar{N}(r, f) + N_{s+k}(r, 1/f) + S(r, f).$$

Lemma 2.3 [8-10] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then

$$N\left(r, 1/f^{(k)} \right) \leq N(r, 1/f) + k\bar{N}(r, f) + S(r, f).$$

By using the similar method to Banerjee [11, Lemma 2.14], we can prove the following Lemma.

Lemma 2.4 Let F, G and H be defined as in (2.1). If F and G share 1 CM and ∞ IM, and $H \not\equiv 0$, then $F \not\equiv G$, and

$$T(r, F) \leq N_2(r, 1/F) + N_2(r, 1/G) + 2\bar{N}(r, 1/F) + \bar{N}(r, 1/G) + 6\bar{N}(r, F) + S(r, F) + S(r, G),$$

the same inequality holding for $T(r, G)$.

Lemma 2.5 [12] Let F, G and V be defined as in (2.2). If F and G share ∞ IM, and $V \equiv 0$, then $F \equiv G$.

Lemma 2.6 [13] If F and G share 1 IM, then

$$\bar{N}_L \left(r, 1/(F-1) \right) \leq \bar{N}(r, 1/F) + \bar{N}(r, F) + S(r, F) + S(r, G).$$

Lemma 2.7 Let f, g be two non-constant meromorphic functions, V be defined as in (2.2), where $F = \left[f^n P(f) \right]^{(k)}$, $G = \left[f^n P(g) \right]^{(k)}$, $P(w)$ is defined as in (1.1), $n > 0$, $k > 0$ and $m \geq 0$ are three integers. If $V \not\equiv 0$, F and G share 1 CM and ∞ IM, then

$$(n+m+k-3)\bar{N}(r,f)=(n+m+k-3)\bar{N}(r,g)\le 2\bar{N}(r,1/F)+2\bar{N}(r,1/G)+S(r,f)+S(r,g). \quad (2.3)$$

Proof Since $V\not\equiv 0$, f and g share ∞ IM, suppose that z_0 is a pole of f with multiplicity p, a pole of g with multiplicity q, then z_0 is a pole of F with multiplicity $(n+m)p+k$, a pole of G with multiplicity $(n+m)q+k$, thus z_0 is a zero of $F'/(F-1)-F'/F=F'/[F(F-1)]$ with multiplicity $(n+m)p+k-1(\ge n+m+k-1)$, and z_0 is a zero of $G'/(G-1)-G'/G=G'/[G(G-1)]$ with multiplicity $(n+m)q+k-1(\ge n+m+k-1)$, hence z_0 is a zero of V with multiplicity at least $n+m+k-1$. So

$$(n+m+k-1)\bar{N}(r,f)=(n+m+k-1)\bar{N}(r,g)\le N(r,1/V). \quad (2.4)$$

By the logarithmic derivative lemma, we have $m(r,V)=S(r,f)+S(r,g)$. Note that F and G share 1 IM, by Lemma 2.6, so we have

$$N(r,1/V)\le T(r,V)=m(r,V)+N(r,V)$$
$$\le \bar{N}(r,1/F)+\bar{N}(r,1/G)+\bar{N}_L\left(r,\frac{1}{F-1}\right)+\bar{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g) \quad (2.5)$$
$$\le 2\bar{N}(r,1/F)+2\bar{N}(r,1/G)+2\bar{N}(r,f)+S(r,f)+S(r,g).$$

From (2.4) and (2.5) we get (2.3). This proves Lemma 2.7.

Lemma 2.8 [14] Let f and g be two non-constant meromorphic functions, and $k,n>2k+1$ be two positive integers. If $\left[f^n\right]^{(k)}=\left[g^n\right]^{(k)}$, then $f\equiv tg$ for a constant t such that $t^n=1$.

By the same reason as in Lemma 5 of [8], we obtain the following lemma.

Lemma 2.9 Let f and g be two non-constant meromorphic functions. Let $P(w)$ be defined as in (1.1), and $n>0,k>0$, and $m\ge 0$ be three integers with $n>2k+m+1$. If $\left[f^nP(f)\right]^{(k)}=\left[g^nP(g)\right]^{(k)}$, then $f^nP(f)=g^nP(g)$.

Lemma 2.10 [15] Let f and g be non-constant meromorphic functions, n,k be two positive integers with $n>k+2$, and let $P(w)$ be defined as in (1.1), $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. If $\left[f^nP(f)\right]^{(k)}\left[g^nP(g)\right]^{(k)}=a^2$, f and g share ∞ IM, then $P(w)$ is reduced to a nonzero monomial.

Use the proof of Theorem 3 in [15] and we obtain.

Lemma 2.11 Let f and g be non-constant meromorphic functions, n,k be two positive integers with $n>k$. If $\left[f^n\right]^{(k)}\left[g^n\right]^{(k)}=1$, f and g share ∞ IM, then $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$.

Lemma 2.12 [16] Let $s>0$ and t are relatively prime integers, and let c be a complex number such that $c^s=1$. Then there exists one and only one common zero of w^s-1 and w^t-c.

3. Proof of Theorem 1.1

Let $F=\left[f^nP(f)\right]^{(k)}$, $G=\left[g^nP(g)\right]^{(k)}$, $F_1=f^nP(f)$, $G_1=g^nP(g)$, then F and G share 1 IM and ∞ IM. Suppose that $H\not\equiv 0$, then $F\not\equiv G$, and $V\not\equiv 0$.

Case 1. $m>0$. By Lemma 2.4 we have

$$T(r,F)\le N_2(r,1/F)+N_2(r,1/G)+2\bar{N}(r,1/F)+\bar{N}(r,1/G)+6\bar{N}(r,F)+S(r,F)+S(r,G), \quad (3.1)$$

By Lemma 2.2 with $s=2$, we obtain

$$T(r,F_1)\le T(r,F)-N_2(r,1/F)+N_{k+2}(r,1/F_1)+S(r,F), \quad (3.2)$$

and

$$N_2\left(r,1/G\right) \le N_{k+2}\left(r,1/G_1\right) + k\overline{N}\left(r,G\right) + S\left(r,G\right). \tag{3.3}$$

Combining (3.1) - (3.3) gives

$$\begin{aligned}
T\left(r,F_1\right) &\le N_{k+2}\left(r,1/F_1\right) + N_{k+2}\left(r,1/G_1\right) + \left(4k+6\right)\overline{N}\left(r,f\right) \\
&\quad + 2N_{k+1}\left(r,1/F_1\right) + N_{k+1}\left(r,1/G_1\right) + S\left(r,f\right) + S\left(r,g\right) \\
&\le \left(3k+4\right)\overline{N}\left(r,1/f\right) + 3N\left(r,1/P\left(f\right)\right) + \left(2k+3\right)\overline{N}\left(r,1/g\right) \\
&\quad + 2N\left(r,1/P\left(g\right)\right) + \left(4k+6\right)\overline{N}\left(r,f\right) + S\left(r,f\right) + S\left(r,g\right).
\end{aligned}$$

It follows from Lemma 2.1 and the above inequality that

$$\begin{aligned}
\left(n+m\right)T\left(r,f\right) &\le \left(3k+4\right)\overline{N}\left(r,1/f\right) + 3N\left(r,1/P\left(f\right)\right) + \left(2k+3\right)\overline{N}\left(r,1/g\right) \\
&\quad + 2N\left(r,1/P\left(g\right)\right) + \left(4k+6\right)\overline{N}\left(r,f\right) + S\left(r,f\right) + S\left(r,g\right) \\
&\le \left(3k+3m+4\right)T\left(r,f\right) + \left(2k+2m+3\right)T\left(r,g\right) \\
&\quad + \left(4k+6\right)\overline{N}\left(r,f\right) + S\left(r,f\right) + S\left(r,g\right).
\end{aligned} \tag{3.4}$$

Similarly we have

$$\begin{aligned}
\left(n+m\right)T\left(r,g\right) &\le \left(3k+3m+4\right)T\left(r,g\right) + \left(2k+2m+3\right)T\left(r,f\right) \\
&\quad + \left(4k+6\right)\overline{N}\left(r,g\right) + S\left(r,f\right) + S\left(r,g\right).
\end{aligned} \tag{3.5}$$

Note that . $\overline{N}\left(r,f\right) = \overline{N}\left(r,g\right)$. From (3.4) and (3.5) we deduce that

$$\left(n-5k-4m-7\right)\left(T\left(r,f\right)+T\left(r,g\right)\right) \le 2\left(4k+6\right)\overline{N}\left(r,f\right) + S\left(r,f\right) + S\left(r,g\right). \tag{3.6}$$

Note that $V \not\equiv 0$ and we get (2.3). By Lemma 2.2 with $s=1$, we obtain

$$\begin{aligned}
N_1\left(r,1/F\right) = \overline{N}\left(r,1/F\right) &\le N_{k+1}\left(r,1/F_1\right) + k\overline{N}\left(r,f\right) + S\left(r,f\right) \\
&\le \left(k+1\right)\overline{N}\left(r,1/f\right) + N\left(r,1/P\left(f\right)\right) + k\overline{N}\left(r,f\right) + S\left(r,f\right),
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
N_1\left(r,1/G\right) = \overline{N}\left(r,1/G\right) &\le N_{k+1}\left(r,1/G_1\right) + k\overline{N}\left(r,g\right) + S\left(r,g\right) \\
&\le \left(k+1\right)\overline{N}\left(r,1/g\right) + N\left(r,1/P\left(g\right)\right) + k\overline{N}\left(r,g\right) + S\left(r,g\right).
\end{aligned} \tag{3.8}$$

From (2.3), (3.7) and (3.8) we get

$$\left(n+m-3k-3\right)\overline{N}\left(r,f\right) \le 2\left(k+m+1\right)\left(T\left(r,f\right)+T\left(r,g\right)\right) + S\left(r,f\right) + S\left(r,g\right). \tag{3.9}$$

Combining (3.6) - (3.9) gives

$$\left[\left(n-5k-4m-7\right)\left(n+m-3k-3\right) - 4\left(4k+6\right)\left(k+m+1\right)\right]\left(T\left(r,f\right)+T\left(r,g\right)\right) \le S\left(r,f\right) + S\left(r,g\right), \tag{3.10}$$

which is a contradiction since $n > 4m+9k+10$. Thus $H \equiv 0$. Similar to the proof of [17, Lemma 3], we obtain

1) $\left[f^n P\left(f\right)\right]^{(k)}\left[g^n P\left(g\right)\right]^{(k)} = 1$, or

2) $\left[f^n P\left(f\right)\right]^{(k)} \equiv \left[g^n P\left(g\right)\right]^{(k)}$.

By Lemma 2.10, the case of 1) is impossible. By Lemma 2.9, we get $f^n P\left(f\right) \equiv g^n P\left(g\right)$ from 2).

Case 2. $m=0$. Similar to the proof of Case 1, we get

$$\left[\left(n-5k-7\right)\left(n-3k-3\right) - 4\left(4k+6\right)\left(k+1\right)\right]\left(T\left(r,f\right)+T\left(r,g\right)\right) \le S\left(r,f\right) + S\left(r,g\right), \tag{3.11}$$

which is a contradiction since $n > 9k+10$. Thus $H \equiv 0$. and we have

3) $\left[f^n\right]^{(k)}\left[g^n\right]^{(k)} = 1$, or

4) $\left[f^n\right]^{(k)} \equiv \left[g^n\right]^{(k)}$.

For 3), by Lemma 2.11, we get $f(z) = c_1 / \sqrt[n]{a_0}\, e^{cz}$, $g(z) = c_2 / \sqrt[n]{a_0}\, e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

For 4), By Lemma 2.8, we get $f \equiv tg$ *for a constant* t *such that* $t^n = 1$. This completes the proof of Theorem 1.1.

4. Proof of Corollaries 1.2 - 1.4

The proof of Corollary 1.2 is the same to the proof of Case 2 of Theorem 1.1, we only need to let $a_0 = 1$. Thus we omit the proof here.

Now we prove Corollary 1.3, Let $m = 1$, similar to (3.10), we get

$$\left[(n-5k-11)(n-3k-2) - 4(4k+6)(k+2)\right]\left(T(r,f) + T(r,g)\right) \leq S(r,f) + S(r,g), \tag{3.12}$$

which is a contradiction since $n > 9k + 14$. Thus $H \equiv 0$ and we have

1) $\left[f^n (f-1)\right]^{(k)} \left[g^n (g-1)\right]^{(k)} = 1$, or

2) $\left[f^n (f-1)\right]^{(k)} \equiv \left[g^n (g-1)\right]^{(k)}$.

By Lemma 2.10, the case of (i) is impossible. By Lemma 2.9, we get $f^n (f-1) \equiv g^n (g-1)$ from 2).

Similar to the proof of Theorem 2 in [14], we get $f \equiv g$. This proves Corollary 1.3.

Next we prove Corollary 1.4.

According to the proof of Case 1 in Theorem 1.1, we have

1) $\left[f^n (f^m-1)\right]^{(k)} \left[g^n (g^m-1)\right]^{(k)} = 1$, or

2) $\left[f^n (f^m-1)\right]^{(k)} \equiv \left[g^n (g^m-1)\right]^{(k)}$.

By Lemma 2.10, the case of 1) is impossible. By Lemma 2.9, we get $f^n (f^m-1) \equiv g^n (g^m-1)$ from 2).

Let $h = f/g$. If h is not a constant, then substitute $f = gh$ into $f^n (f^m-1) \equiv g^n (g^m-1)$ and we get

$$g^m = \frac{1-h^n}{1-h^{n+m}} = \frac{(h-a_1)\cdots(h-a_n)}{(h-b_1)\cdots(h-b_{n+m})},$$

where $a_j \; (j=1,2,\cdots,n)$ are distinct roots of the algebraic equation $l^n = 1$, $b_j \; (j=1,2,\cdots,n+m)$ are distinct roots of the algebraic equation $l^{n+m} = 1$.

Suppose that $d = GCD(n+m,n)$, then $n+m = sd$, $n = td$, where s, t are co-prime integers and $s > t$, thus $m = (s-t)d$, which implies $d \leq m$. By Lemma 2.12, there exists one and only one common zero of $w^s - 1$ and $w^t - 1$, namely $w = 1$. Therefore, there exists at least n of $b_j \; (j=1,2,\cdots,n+m)$ different from $a_j \; (j=1,2,\cdots,n)$. Suppose that $b_j \; (j=1,2,\cdots,n)$ are different from $a_j \; (j=1,2,\cdots,n)$, then all zeros of $h - b_j \; (j=1,2,\cdots,n)$ have order of at least m. Applying the second fundamental theorem to h gives

$$(n-2)T(r,h) \leq \sum_{j=1}^{n} \bar{N}\left(r, \frac{1}{h-b_j}\right) + S(r,h) \leq \frac{1}{2}\sum_{j=1}^{n} N\left(r, \frac{1}{h-b_j}\right) + S(r,h) \leq \frac{n+1}{2}T(r,h) + S(r,h).$$

Note that $n > 4m + 9k + 10$ and we get a contradiction. Thus h is a constant. From (4.2) we have $h^{n+m} = 1$ and $h^n = 1$, thus $f \equiv tg$ for some constant t such that $t^d = 1$, where $d = GCD(n+m,n)$. This proves Corollary 1.4.

5. Open Problem

For further study, we pose the following. **Problem:** What form of $f^n P(f) \equiv g^n P(g)$ implies $f \equiv tg$ for some constant t?

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