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S-Injective Modules and Rings

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ABSTRACT

We introduce and investigate the concept of *s*-injective modules and strongly *s*-injective modules. New characterizations of *SI*-rings, *GV*-rings and pseudo-Frobenius rings are given in terms of *s*-injectivity of their modules.

KEYWORDS

***S*-Injective Module; Kasch Ring; Pseudo Frobenius Ring**

1. Introduction

In this paper we introduce and investigate the notion of *s-injective Modules and Rings*. A right *R*-module *M* is called right *s-N-injective*, where *N* is a right *R*-module, if every *R*-homomorphism $f: K \to M$ extends to N, where *K* is a submodule of the singular submodule $Z(N)$. *M* is called *strongly s-injective* if *M* is *s*-*N*-injective for every right *R*-module *N*. The connection between this new injectivity condition and other injectivity conditions has been established, and examples are provided to distinguish *s*-injectivity from other injectivity concepts such as mininjectivity, *soc*-injectivity. Several properties of this new class of injectivity are highlighted.

Throughout this paper all rings are associative with identity, and all modules are unitary *R*-modules. For a right *R*-module M_R, we denoted $J(M)$, $soc(M)$, and $Z(M)$ by the Jacobson radical, the socle and the singular submodule of M, respectively. S_r, S_l, Z_r, Z_l and J are used to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, and the Jacobson radical of R, respectively. For a submodule N of M, the notations $N \subseteq^{ess} M$, $N \subseteq^{\max} M$ and $N \subseteq^{\oplus} M$ mean, respectively, that *N* is essential, maximal, and direct summand. If X is a subset of a right *R*-module M, right annihilators will be denoted by $r(X) = r_R(X) = \{r \in R : xr = 0 \text{ for all } x \in X\}$, with a similar definition of left annihilators $l(X) = l_R(X)$. Multiplication maps $x \to ax$ and $x \to xa$ will be denoted by $a^\bullet$ and ${}^\bullet a$, respectively. If *M* and *N* are right *R*-modules, then *M* is called *N-injective* if every *R*-homomorphism from a submodule of *N* into *M* can be extended to an *R*-homomorphism from *N* into *M*. Mod-*R* indicates the category of right *R*-modules. We refer to [1-3] for all the undefined notions in this paper.

2. Strongly S-Injective Modules

Definition 1 *A right R-module M is called s-N-injective if every R-homomorphism* $f: K \to M$ *extends to* N, *where* K *is a submodule of the singular submodule* $Z(N)$. *M is called s-injective if* M *is s-R-injective.* M *is called strongly s-injective, if* M *is s-N-injective for all right R-modules* N.

For example every nonsingular *R*-module is strongly *s*-injective. In particular, the ring of integers Z is strongly *s*-injective, but not injective.

Proposition 1

1) Let N be a right *R*-module and $\{M_i : i \in I\}$ a family of right *R*-modules. Then the direct product $\prod_{i\in I} M_i$

is *s*-*N*-injective if and only if each M_i is *s*-*N*-injective, $i \in I$.

2) Let M, N, and K be right *R*-modules with $K \subseteq N$. If M is *s*-*N*-injective, then M is *s*-*K*-injective.

3) Let M, N, and K be right *R*-modules with $K \cong N$. If K is *s*-*M*-injective, then N is *s*-*M*-injective.

4) Let N be a right *R*-module and $\{A_i : i \in I\}$ a family of right *R*-modules. Then N is $s\text{-}\bigoplus_{i\in I} A_i$- injective if and only if N is $s\text{-}A_i$- injective, $\forall i \in I$.

5) A right *R*-module M is *s*-injective if and only if M is *s*-*P*-injective for every projective right *R*-module P.

6) Let M, N, and K be right *R*-modules with $N \subseteq^{\oplus} M$. If M is *s*-*K*-injective, then N is *s*-*K*-injective.

7) If A, B, and M are right *R*-modules, $A_R \cong B_R$, and M is *s*-*A*-injective, then M is *s*-*B*-injective.

Proof. The proofs of 1) through 4) are routine.

5). This follows from 4).

6). Let $f : L \to N$ be an *R*-homomorphism where L is singular submodule of N. Then the map $\iota \circ f : L \to$ can be extended to an *R*-homomorphism $g : K \to M$, where $\iota : N \to M$ the inclusion map. Now, the map $\pi \circ g : K \to N$ is an extension of f, where $\pi : M \to N$ the natural projection map into N.

7). Let $f : A_R \to B_R$ be an *R*-isomorphism, and $g : K \to M_R$ an *R*-homomorphism where K is a singular submodule of B. The restriction of f to $Z(A)$ induces an isomorphism $f/Z(A_R) : Z(A_R) \to Z(B_R)$. By hypothesis, the map $g \circ f : K \to M_R$ can be extended to an *R*-homomorphism $\eta : A_R \to M_R$. Now, the map $\eta \circ f^{-1} : B \to M$ is an extension of g. □

The next two corollaries are immediate consequences of the above proposition.

Corollary 1 *Let N be a right R-module. Then the following statements are true*:

1) A finite direct sum of *s*-*N*-injective modules is again *s*-*N*-injective. In particular a finite direct sum of *s*-injective (strongly *s*-injective) modules is again *s*-injective (strongly *s*-injective).

2) A summand of *s*-*N*-injective (*s*-injective, strongly *s*-injective) module is again *s*-*N*-injective (*s*-injective, strongly *s*-injective) module.

Corollary 2

1) Let M be a right *R*-module and $1 = e_1 + e_2 + \cdots + e_n$ in R, where the e_i are orthogonal idempotents. Then M is *s*-injective if and only if M is $s\text{-}e_i R$- injective for each i, $1 \le i \le n$.

2) Assume that e and f are idempotents of R, $eR \cong fR$, and M is $s - eR -$ injective. Then M is *s*-*fN*-injective.

Proposition 2 *If N is a finitely generated right R-module, then the following conditions are equivalent*:

1) Any direct sum of *s*-*N*-injective modules is *s*-*N*-injective.

2) Any direct sum of injective modules is *s*-*N*-injective.

3) $Z(N)$ is noetherian.

Proof. 1) $\Rightarrow$ 2). Clear.

2) $\Rightarrow$ 3). Consider a chain $U_1 \subseteq U_2 \subseteq \cdots$ of singular submodules of N and $U =_{1\le i} U_i$. Let $E(N/U_i)$ be the injective hull of N/U_i, $i \ge 1$, and $f : U \to \bigoplus_{1\le i} E(N/U_i)$ be a map defined by $f(n) = (n + U_i)$. Since, $\bigoplus_{1\le i} E(N/U_i)$ is *s*-*N*-injective, f can be extended to an *R*-homomorphism $\hat{f} : N \to \bigoplus_{1\le i} E(N/U_i)$. Since N is finitely generated, $\hat{f}(N) \subseteq \bigoplus_{1\le i\le n} E(N/U_i)$ for some n, then $f(U) \subseteq \bigoplus_{1\le i\le n} E(N/U_i)$ and $U = U_{j+n}$ for every $j \ge 1$. Hence $Z(N)$ is noetherian.

3) $\Rightarrow$ 1). Let $E = \bigoplus_{i\in I} E_i$ be a direct sum of *s*-*N*-injective modules, and $f : U \to E_R$ be a homomorphism of right *R*-modules where $U \subseteq Z(N)$. Since $Z(N)$ is noetherian, $f(U) \subseteq \bigoplus_{i\in F} E_i$ for some finite subset $F \subseteq I$. Since finite direct sums of *s*-*N*-injective modules is *s*-*N*-injective, f can be extended to an *R*-homomorphism $\hat{f} : N \to E$. □

The second singular submodule of a right *R*-module M, denoted by $Z_2(M)$, is defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. We see that a $Z_2(M)$ is closed submodule of M and $M/Z_2(M)$ is non-singular. A right *R*-module is Goldie torsion if $Z_2(G) = G$.

Lemma 1 *Let M and N be right R-modules such that $Z_2(M)$ is injective. Then every homomorphism*

$f:K\to M$ where $K\subseteq Z_2(N)$ extends to N.

Proof. Let $M=Z_2(M)\oplus T$ where $Z_2(M)$ is injective and $Z(T)=0$. If $f:K\to M$ is a homomorphism where $K\subseteq Z_2(N)$ such that $f(Z(K))=0$, then $f(K)\cong K/Ker(f)$ is singular. So f extendable to N. Now suppose that $0\neq f(k)\in T$, so $f(kR)\cong kR/Ker(f|_{kR})$ is singular which is a contradiction. Thus $f(K)\cap T=0$. Suppose that $f(k)=a+b$ where $a\in Z_2(M)$ and $b\in T$, $bR\cap f(kR)=0$. So $r(a)\subseteq r(b)$ and the kernal of the map $aR\to bR$ by $ar\mapsto br$ is essential in aR which is a contradiction. Then every homomorphism $f:K\to M$ where $K\subseteq Z_2(N)$ extends to N. □

Proposition 3 *The following statements are equivalent*:

1) M is strongly *s*-injective.
2) M is *s*-$I(M)$-injective, where $I(M)$ is the injective hull of M.
3) $M=E\oplus K$, where K is nonsingular and E is injective with $Z(M)\subseteq^{ess}E$.
4) $Z_2(M)$ is injective.
5) M is *G*-injective for every Goldie torsion module G.
6) M is *I*-injective, where $I=I(Z_2(M))$ is the injective hull of $Z_2(M)$.

Proof. 1) $\Rightarrow$ 2). Clear.

2) $\Rightarrow$ 3). If $Z(M)=0$, we are done. Assume that $Z(M)\neq 0$ and consider the following diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & Z(M) & \xrightarrow{i_1} & D \\ & & i_2\downarrow & & \\ & & M & & \end{array}$$

where ι_1 and i_2 are inclusion maps and D is injective closure of $Z(M)$ in $I(M)$. Since M is *s*-$I(M)$-injective, M is *s*-*D*-injective. So there exists an *R*-homomorphism $\sigma:D\to M$, which extends ι_2. Since $Z(M)\subseteq^{ess}D$, σ is an embedding of D in M. If we write $E=\sigma(D)$, then $M=E\oplus K$ for some submodule K of M because E is injective. Finally K is nonsingular because $Z(M)\subseteq E$.

3) $\Rightarrow$ 4). Since $E/Z(M)$ is singular and $Z_2(M)/Z(M)$ is singular submodule of $M/Z(M)$, so $E\subseteq Z_2(M)$ and $Z_2(M)=E\oplus L$ for some submodule L of M. Then $Z_2(M)=E$ $\left(Z(M)\subseteq^{ess}Z_2(M)\right)$ and $Z_2(M)$ is injective.

4) $\Rightarrow$ 5). Let G be a Goldie torsion right *R*-module and K is a submodule of G. Using the above Lemma, every homomorphism $f:K\to M$ extends to G.

5) $\Rightarrow$ 6). If $I=I(Z_2(M))$ is the injective hull of $Z_2(M)$, then $Z_2(I)=I$ and I is Goldie torsion.

6) $\Rightarrow$ 1). Let N be a right *R*-module and K be a singular submodule of N. Consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{f} & Z_2(M) & \xrightarrow{i_1} & I(Z_2(M)) \\ \downarrow i_3 & & \downarrow i_2 & & \\ N & & M & & \end{array}$$

where ι_1, i_2, and i_3 are the inclusion maps. Since M is *I*-injective and I is injective. So, there exist *R*-homomorphisms $h:I\to M$ and $g:N\to I$ such that $hi_1=i_2$ and $gi_3=i_1f$. Thus $i_2f=hi_1f=hgi_3$. Hence M is strongly *s*-injective. □

Corollary 3 *Let M a Goldie torsion right R-module. Then M is injective if and only if M is strongly s-injective*.

Proposition 4 *For a ring R, the following conditions are equivalent*:

1) R_R is strongly *s*-injective.
2) R_R is *s*-$I(R_R)$-injective, where $I(R_R)$ is the injective hull of R_R.
3) $R=E\oplus T$, where E_R is injective and T is nonsingular. Moreover, if $Z_r\neq 0$, then $Z_r\subseteq^{ess}E$, and in this case E and T are relatively injective.
4) Z_2^r is injective.
5) R_R is *G*-injective for every Goldie torsion right *R*-module G.
6) R_R is *G*-injective, where $I=I(Z_2^r)$ is the injective hull of Z_2^r.
7) Every finitely generated projective right *R*-module is strongly *s*-injective.

Proof. The equivalence between 1), 2), 3), 4), 5) and 6) is from Proposition 3.

1) $\Rightarrow$ 7) Since a finite direct sum of *s*-*N*-injective is *s*-*N*-injective for every right *R*-module N (Corollary 1), so every finitely generated free right *R*-module is strongly *s*-injective. But a direct summand of strongly *s*-injective is strongly *s*-injective (Corollary 1). Therefore every finitely generated projective module is strongly *s*-injective. The converse is clear. □

The following examples show that the two classes of strongly *s*-injective rings and of *soc*-injective rings are different.

Example 1 *Let* $F = Z_2$ *be the field of two elements,* $F_n = F$ *for* $n = 1, 2, \cdots$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \bigoplus_{i=1}^{\infty} F_i$. *If* R *is the subring of* Q *generated by* 1 *and* S, *then* R *is a von Neumann regular ring with* $soc(R) = S$, *and hence* $Z_2^r = 0$ *and* R *is strongly s-injective*. *However*, *the map* $f : S_R \to R_R$, *given by* $(a_1, a_2, a_3, a_4, \cdots) \mapsto (a_{1,} 0, a_3, 0, \cdots)$, *cannot be extended to an R-homomorphism from* R *into* R (*suppose that* $f = c\cdot$ *for some* $c = (c_i) \in R$. *Then for every* $(a_1, a_2, \cdots) \in S_R$, $(a_1, 0, a_3, 0, \cdots) = (c_i)(a_i)$ *which is impossible*), *and so* R *is not a soc-injective ring*.

Example 2 *Let* $R = \mathbb{Z}_2[x_1, x_2, \cdots]$ *where* $x_i^3 = 0$ *for all* i, $x_i x_j = 0$ *for all* $i \neq j$ *and* $x_i^2 = x_j^2 = m \neq 0$ *for all* i *and* j. *Then* R *is a commutative*, *semiprimary*, *local ring with* $J = span\{m, x_1, x_2, \cdots\} = Z_r$, *and* R *has simple essential socle* $J^2 = \mathbb{Z}_2 m$. *It is not difficult to see that* R *is right soc-injective*. *However the R-homomorphism* $\gamma : Z_r \to R$ *defined by* $\gamma(a) = a^2$ *for all* $a \in Z_r$ *can not be extended to an endomorphism of* R, *and so* R *is not s-injective ring*.

Definition 2 *A ring* R *is called a right generalized V-ring* (*right GV-ring*) *if every simple singular right R-module is injective*.

Proposition 5 *A ring* R *is right GV-ring if and only if every simple right R-module is strongly s-injective*.

Proof. Let R be a right *GV*-ring and M be a simple right *R*-module. The module M is either projective or singular, so M is strongly *s*-injective. Conversely, if M is a simple singular right *R*-module, then M is strongly *s*-injective. Thus M is injective by Proposition 3. □

Lemma 2 *For a right R-module* M *the following conditions are equivalent*:

1) M satisfies ACC on essential submodules.

2) $M/Soc(M)$ is noetherian.

Proof. Assume that M has ACC on essential submodules. Then B/A is noetherian for every submodule $A \subseteq^{ess} B$ $B/A \cong (B \oplus L)/(A \oplus L)$ where L is an intersection complement of A and $M/(A \oplus L)$ is noetherian). In particular, every uniform submodule of M is noetherian. Let H be an intersection complement of S $(= Soc(M))$ (see Kasch [2] p.112). Then $M/(H \oplus S)$ is noetherian. So, to prove that M/S is noetherian it is enough to show that H is noetherian. Assume that H contains an infinite direct sum $K = K_1 \oplus K_2 \oplus \cdots$ of nonzero submodules K_i. Since $K_i \cap S = 0$, each K_i contains a proper essential submodule L_i and $L = L_1 \oplus L_2 \oplus \cdots$ is essential in K. But this gives that K/L is noetherian which is impossible because $K/L \cong K_1/L_1 \oplus K_2/L_2 \oplus \cdots$ with each K_i/L_i nonzero. Then H contains k independent uniform submodules U_i such that $U = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ is essential in H. Thus U and H/U are noetherian. Hence H is noetherian. □

It is well-known that, a ring R is right noetherian if and only if all direct sums of injective right *R*-modules are injective. In the next Proposition we obtain a characterization of ring which has ACC on essential right ideals in terms of strongly *s*-injective right *R*-modules.

Proposition 6 *The following conditions on a ring* R *are equivalent*:

1) Every direct sum of strongly *s*-injective right *R*-modules is strongly *s*-injective.

2) Every direct sum of injective right *R*-modules is strongly *s*-injective.

3) Every finitely generated right *R*-module has ACC on essential submodules.

4) R/S_r is noetherian.

Proof. 1) $\Rightarrow$ 2). Clear.

2) $\Rightarrow$ 3). Consider a chain $K_1 \subseteq K_2 \subseteq \cdots$ of essential submodules of a finitely generated right *R*-module M and $K = \bigcup_{1 \le i} K_i$. Let $I(M/K_i)$ be the injective hull of M/K_i, $i \geq 1$, and $f : K \to \bigoplus_{1 \le i} I(M/K_i)$ be a map defined by $f(k) = (k + I_i)$. Since $\bigoplus_{1 \le i} I(M/K_i)$ is strongly *s*-injective and $\bigoplus_{1 \le i} I(M/K_i)$ has an essential singular submodule, so $\bigoplus_{1 \le i} I(M/K_i)$ is injective and f can be extended to an *R*-homomorphism

$\hat{f}: M \to \bigoplus_{1\le i} I(M/K_i)$. Then $\hat{f}(M) \subseteq \bigoplus_{1\le i\le n} I(M/K_i)$ for some n and $f(K) \subseteq \bigoplus_{1\le i\le n} I(M/K_i)$. Thus $K = K_{j+n}$ for every $j \ge 1$. Hence M has ACC on essential submodules.

3) $\Rightarrow$ 4). Above Lemma.

4) $\Rightarrow$ 1). Let $E = \bigoplus_{i\in I} E_i$ be a direct sum of injective Goldie torsion modules. Then E_i, $i \in I$, may be considered as an injective R/S_r-module. Thus E is an injective R/S_r-module. Now, suppose that $f: aR \to E$ is a nonzero map where aR is simple right ideal, so aR is a singular right ideal and $f(aR) \subseteq \bigoplus_{i\in F} E_i$ where $F \subseteq I$ is finite. Thus f extends to $g: R \to \bigoplus_{i\in F} E_i$. Then $0 \ne g(a) = g(1)a = f(a)$ which is a contradiction with $ES_r = 0$. Hence any homomorphism $h: U \to E$ where U is a right ideal of R induces a map $\hat{h}: (U + S_r)/S_r \to E$ with $\hat{h}(u + S_r) = h(u)$. The map $\hat{h}$ extends to a homomorphism $\hat{\alpha}: R/S_r \to E$. Then the map $\alpha = \hat{\alpha}\pi: R \to E$ where π is the natural epimorphism $\pi: R \to R/S_r$, extends h. Hence E is injective. Therefore, every direct sum of strongly *s*-injective right *R*-modules is strongly *s*-injective. □

If I is an ideal of R, R is called *I*-semiperfect if for every right ideal K, there is a decomposition $K = eR \oplus U$ such that $e^2 = e$ and $U = K \cap (1-e)R \subseteq I$ [4].

Lemma 3 *If R is Z_r-semiperfect, then the following statements hold*:

1) A module M is *s*-injective if and only if M is injective.

2) $K = rl(K)$ for all singular right ideals K of R if and only if $K = rl(K)$ for all right ideals K of R.

Proof. 1) Let M be *s*-injective, and $f: T \to M$ be an *R*-homomorphism where T is a right ideal of R. Then $T = eR \oplus U$, where $U = T \cap (1-e)R \subseteq Z_r$. Let $g: R_R \to M$ be an extension of the restriction map f/U. Define $h: R_R \to M$ by $h(x) = h(ex + (1-e)x) = f(ex) + g((1-e)x)$ for all $x \in R$. Clearly, h is an extension of f, and so M is injective by the Baer's Criterion.

2) Let T be a right ideal of R. Since R is right Z_r-semiperfect, then $T = eR \oplus U$, where $e^2 = e \in R$ and $U = T \cap (1-e)R \subseteq Z_r$. So $l(T) = l(e) \cap l(U)$ and $rl(T) = r[R(1-e) \cap l(U)]$. If $x \in r[R(1-e) \cap l(U)]$, then $l((1-e)U) \subseteq l((1-e)x)$ and so $(1-e)x \in rl((1-e)x) \subseteq rl((1-e)U) = (1-e)U$. The last equality is because that $(1-e)U$ is a singular right ideal of R. Write $(1-e_1)x = (1-e_1)u$ where $u \in U$. Then $x = e(x-u) + u \in T$. Therefore, $T = rl(T)$. □

Proposition 7 *Let M be a right R-module. $Z(M)$ is semisimple summand of M if and only if every right R-module is s-M-injective.*

Proof. If $Z(M)$ is semisimple summand of M, then every right *R*-module N is *s*-*M*-injective. Conversely, if every right *R*-module is *s*-*M*-injective, then every identity map $i: K \to K$ where K is singular submodule of M extends to $f: M \to K$. Thus K is a summand of M. Hence $Z(M)$ is a semisimple summand of M. □

Corollary 4 *A ring R is right nonsingular if and only if every right R-module is s-injective*.

A ring R is called a right (left) SI ring if every singular right (left) *R*-module is injective. SI rings were initially introduced and investigated by Goodearl [5]

Theorem 1 *The following statements are equivalent*:

1) R is right SI ring.

2) Every right *R*-module is strongly *s*-injective.

3) Every singular right *R*-module is strongly *s*-injective.

Proof. Clear from Proposition 3. □

A module M is said to satisfy the $C2$-condition, if K and L are submodules of M, $K \cong L$, and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$. We also say M satisfies the $C3$-condition if K and L are submodules of M with $K \cap L = 0$, $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \oplus L$ is a summand of M. It is a well-know fact that the $C2$-condition implies the $C3$-condition. In the next proposition we show that *s*-quasi-injective modules inherit a weaker version of these conditions.

Proposition 8 *Suppose M is a s-quasi-injective right R-module.*

1) $(s\text{-}C2)$ If K and L are singular submodules of M, $K \cong L$, and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$.

2) $(s\text{-}C3)$ Let K and L be singular submodules of M with $K \cap L = 0$. If $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \oplus L$ is a summand of M.

Proof. 1) Since $K \cong L$, and K is *s*-injective, being a summand of the *s*-quasi-injective right *R*-module M, L is *s*-injective. If $\iota : L \to M$ is the inclusion map, the identity map $Id_K : L_R \to L_R$ has an extension $\eta : M \to L$ such that $\iota \circ \eta = Id_K$, and so K is a summand of M.

2) Since K and L are summands of M, and M is *s*-quasi-injective, both K and L are *s*-*M*-injective. Thus the singular module $K \oplus L$ is *s*-*M*-injective, and so is a summand of M. □

It is a well-known fact that the *C*2-condition implies the *C*3-condition.

Proposition 9 *If a module* M *has s-C*2*-condition*, *then* M *has s-C*3*-condition*.

Proof. Consider singular summands M_1 and M_2 of M such that $M_1 \cap M_2 = 0$. Write $M = M_1 \oplus M_1^*$ and let π denote the projection $M_1 \oplus M_1^* \to M_1^*$. Then $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$. If $b \in M_2$, $b = c + d$ where $c \in M_1$ and $d \in M_1^*$ and $\pi|_{M_2}(b) = 0$, $\pi|_{M_2}(c+d) = 0 = d$ and $b \in M_1 \cap M_2 = 0$. Then $\pi|_{M_2}$ is a monomorphism; so $\pi(M_2)$ is a summand of M by $S\text{-}C2$. As $\pi(M_2) \le M_1^*$, $M_1 \oplus \pi(M_2) \subseteq^{\oplus} M$.

Proposition 10 *Let* R *and* S *be Morita-equivalent rings with category equivalence* $F : Mod - R \to Mod - S$. *Let* M, N, *and* K *be right R-modules*. *Then*

1) K_R is singular if and only if $F(K)_S$ is singular.

2) M_R is *s*-*N*-injective if and only if $F(M)_S$ is *s*-injective.

Proof. There is a natural isomorphisms $\eta : GF \to 1_{modR}$ and $\zeta : FG \to 1_{modS}$. This means that for each M_R there is an isomorphism $\eta_M : GF(M) \to M$ in *modR* such that for each M, M' in *modR* and each $f : M \to M'$ in *modR*, the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \uparrow_{\eta_M} & & \uparrow_{\eta_{M'}} \\ GF(M) & \xrightarrow{GF(f)} & GF(M') \end{array}$$

1). The right *R*-module K is singular if and only if there is an exact sequence of right *R*-modules $0 \to A \to B \to K \to 0$ with essential monomorphism $0 \to A \to B$. But using [6, Proposition 21.4 and Proposition 21.6] the sequence $0 \to A \to B \to K \to 0$ is exact with essential monomorphism $0 \to A \to B$ if and only if the sequence $0 \to F(A) \to F(B) \to F(K) \to 0$ of right *S*-modules is exact with essential monomorphism $0 \to F(A) \to F(B)$. So K_R is singular if and only if $F(K)_S$ is singular.

2). Let M be a *s*-*N*-injective and K be a singular submodule of $F(N)$. Let $f : K \to F(M)$ be a homomorphism. Since $G(K)$ is singular, $G(1_K)$ is a monomorphism and the maps η_N and η_M are isomorphisms (we may assume that $G(K)$ is a submodule of N), then we have the commutative diagram

$$\begin{array}{ccc} G(K) & \xrightarrow{\eta_M G(f)} & M \\ & \searrow^{\eta_N G(1_K)} & \uparrow_{\alpha} \\ & & N \end{array}$$

So the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\zeta_K F(\eta_M G(f))} & F(M) \\ & \searrow^{\zeta_K F(\eta_N G(1_K))} & \uparrow_{F(\alpha)} \\ & & F(N) \end{array}$$

is commutative where $F(M)_S$ is $s - F(N) -$ injective. The converse is similarly. □

As for right self-injectivity, right strongly *s*-injectivity turns out to be a Morita invariant.

Theorem 2 *Right strong s-injectivity is a Morita invariant property of rings*.

Proof. Let R and S be Morita-equivalent rings with category equivalences $F : modR \to modS$. and $G : modS \to modR$. Let P, and N be right *R*-modules. P_R is finitely generated projective *R*-module if and only if $F(P)_S$ is finitely generated projective *S*-module [6, Propositions 21.6 and 21.8]. Also P_R is *s*-*N*-injective if and only if $F(P)_S$ is *s*-*F*(*N*)-injective (Proposition 10) and then P_R is strongly *s*-injective if and only if $F(P)_S$ is strongly *s*-injective. Then, every finitely generated projective right *R*-module is strongly *s*-injective if and only if every finitely generated projective right *S*-module is strongly *s*-injective. Therefore right strong *s*-injectivity is a Morita invariant property of rings. □

Proposition 11 *For a projective right R-module* M, *the following conditions are equivalent*:

1) Every homomorphic image of a *s*-*M*-injective right *R*-module is *s*-*M*-injective.
2) Every homomorphic image of an injective right *R*-module is *s*-*M*-injective.
3) Every singular submodule of M is projective.

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Consider the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\eta} & N & \longrightarrow & 0 \\ & & f\uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M \end{array}$$

Where K is a singular submodule of M, E and N are right *R*-modules with E injective, η an *R*-epimorphism, and f an *R*-homomorphism. Since N is *s*-injective, f can be extended to an *R*-homomorphism $g: M \to N$. Since M is projective, g can be lifted to an *R*-homomorphism $\tilde{g}: M \to E$ such that $\eta \circ \tilde{g} = g$. Now, define $\tilde{f}: K \to E$ by $\tilde{f} = \tilde{g}/K$. Clearly, $\eta \circ \tilde{f} = f$, and hence K is projective.

3) $\Rightarrow$ 1) Let N and L be right *R*-modules with $\eta: N \to L$ an *R*-epimorphism, K is a singular submodule of M and N is *s*-*M*-injective. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{inc.} & M \\ & & f\downarrow & & \\ N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Since K is projective, f can be lifted to an *R*-homomorphism $g: K \to N$ such that $\eta \circ g(x) = f(x)$, $\forall x \in K$. Since N is *s*-injective, g can be extended to an *R*-homomorphism $\tilde{g}: M \to N$. Clearly, $\eta \circ \tilde{g}: M \to L$ extends f. □

Corollary 5 *The following conditions are equivalent*:
1) Every quotient of *s*-injective right *R*-module is *s*-injective.
2) Every quotient of an injective right *R*-module is *s*-injective.
3) Every singular right ideal is projective.

Proposition 12 *The following conditions are equivalent*:
1) Every strongly *s*-injective right *R*-module is injective.
2) Every nonsingular right *R*-module is semisimple injective.
3) For every right *R*-module M, $M = E \oplus Z_2(M)$ where E semisimple injective.
4) R is Z_2^r-semiperfect.

Proof. 1) $\Rightarrow$ 2) If M is nonsingular right *R*-module, then M is strongly *s*-injective. Thus M is semisimple injective.

2) $\Rightarrow$ 3) Let M be a right *R*-module. If M is Goldie torsion, we are done. Now suppose that $Z(M)$ is not essential in M and K be a maximal submodule M with respect to $K \cap Z(M) = 0$. Then K is semisimple injective and $M = K \oplus L$. It is clear that $L = Z_2(M)$.

3) $\Rightarrow$ 4). Let K be a right ideal of R. Then $K = E \oplus Z_2(K)$ where E is semisimple injective. We have $Z_2(K) \subseteq Z_r^2$ and E is a summand of R and generated by an idempotent. Hence R is Z_2^r-semiperfect.

4) $\Rightarrow$ 1). Let M be a strongly *s*-injective and $f: T \to M$ be an *R*-homomorphism where T is a right ideal of R. By $T = eR \oplus U$, where $U = T \cap (1-e)R \subseteq Z_2^r$ and $e^2 = e$. Using Proposition 3 let $g: R_R \to M$ be an extension of the restriction map f/U. Define $h: R_R \to M$ by
$h(x) = h(ex + (1-e)x) = f(ex) + g((1-e)x)$ for all $x \in R$. Clearly, h is an extension of f, and so M is injective by the Baer's Criterion. □

Theorem 3 *The following are equivalent for a ring* R:
1) R is a right *PF*-ring.
2) R is Z_r-semiperfect, right strongly *s*-injective ring with essential right socle.
3) R is semiperfect, right min-*C*2, right strongly *s*-injective ring with essential right socle.
4) R is semiperfect with $soc(J) = soc(Z_r)$, right strongly *s*-injective ring with essential right socle.
5) R is right finitely cogenerated, right min-*C*2, right strongly *s*-injective ring.
6) R is a right Kasch, right strongly *s*-injective ring.
7) R is a right strongly *s*-injective ring and the dual of every simple left *R*-module is simple.

Proof. 1) $\Rightarrow$ 2). Clear.

2) $\Rightarrow$ 1). Clear by Lemma 2.

1) $\Rightarrow$ 3). Clear.

3) $\Rightarrow$ 4). Suppose that $0 \neq aR$ is a non singular simple right ideal in J, so $r(a) \cap eR = 0$ for some simple right ideal eR with $e^2 = e$. Thus $eR \cong aR$ and aR is a summand which is a contradiction. Then $soc(J) \subseteq soc(Z_r)$. The other inclusion is clear.

4) $\Rightarrow$ 1). Let R be semiperfect and right strongly *s*-injective ring with essential right socle and $soc(J) = soc(Z_r)$. Then $R = Z_2^r \oplus T$ where Z_2^r is injective and $Z_r \subseteq^{ess} J$. Thus $J \subseteq^{ess} Z_2^r$ and $J(T) = 0$. The right ideal T may be considered as an *R*/*J*-module. Let $f : L \to T$ be a map where L is a right ideal of R, f induces a map $h : (L + J)/J \to T$ given by $h(l + J) = f(l)$. Since T is injective as an *R*/*J*-module so h extends to $g : R/J \to T$. The map $g\pi : R \to T$, where π is the natural epimorphism $\pi : R \to R/J$, extends f and T is injective. Therefore R is right selfinjective and R is right PF.

1) $\Rightarrow$ 5). Clear.

5) $\Rightarrow$ 1). Since R is right strongly *s*-injective ring, it follows from Proposition 3 that $R = Z_2^r \oplus T$, where Z_2^r is injective and T is nonsingular. If aR is simple right ideal in T such that $(aR)^2 = 0$, then, by the proof of 3) $\Rightarrow$ 1) $aR = 0$ and every simple right ideal in T is a summand of T. Since R is right finitely cogenerated, T has a finitely generated essential socle. Thus T is semisimple. Hence R is injective and R is right $PF-$ring.

1) $\Rightarrow$ 6) Proposition 3 and [7, Theorem 5]

1) $\Rightarrow$ 7). Assume that R is right strongly *s*-injective and the dual of every simple left *R*-module is simple. If aR is a nonsingular simple right ideal, then $r(a) \cap eR = 0$ for some simple right ideal eR with $e^2 = e$. But R is $C2$, so $eR \cong aR$ and aR is a summand. Thus R is right min-*CS*, so by [3, Theorem 4.8] R is semiperfect with essential right socle. Hence R is right PF by 3). □

The following is an example of a right perfect, left Kasch ring and right strongly *s*-injective which is not right self-injective ring.

Example 3 *Let K be a field and let R be the ring of all upper triangular, countably infinite square matrices over R with only finitely many off-diagonal entries. Let S be the K-subalgebra of R generated by 1 and $J(R)$. Then S is a right perfect, left Kasch (S has only one simple left R-module M up to isomorphism. So $M \cong S/J(S) \cong K$) such that $(Z_2^r)_R = 0$ whereas S is neither left perfect nor right self-injective because it is not right finite dimensional.*

Remark 1 *Note that the ring of integers $\mathbb{Z}$ is an example of a commutative noetherian strongly s-injective ring which is not quasi-Frobenius.*

Definition 3 *A ring R is called right CF-ring (FGF-ring) if every cyclic (finitely generated) right R-module embeds in a free module. It is not known whether right CF-rings (FGF-rings) are right artinian (quasi-Frobenius). In the next result we provide a positive answer if we assume in addition that the ring R is strongly right s-injective.*

Proposition 13 *Every right CF right strongly s-injective ring is quasi-Frobenius.*

Proof. Theorem 3 and [7, Theorem 5] □

3. S-CS Modules and Rings

A module M is said to satisfy $C1$-condition or called CS module if every submodule of M is essential in a direct summand of M.

Definition 4 *A right R-module M is called s-CS module if every singular submodule of M is essential in a summand of M.*

For example, every nonsingular module is *s*-*CS*. In particular, the ring of integers Z is *s*-*CS* but not CS

Proposition 14 *For a right R-module M, the following statements are equivalent*:

1) The second singular submodule $Z_2(M)$ is CS and a summand of M.

2) M is *s*-*CS*.

Proof. 1) $\Rightarrow$ 2). If the second singular submodule $Z_2(M)$ of M is CS and a summand of M, then every singular submodule of M is a summand of $Z_2(M)$ and a summand of M.

2) $\Rightarrow$ 1). Let M be *s*-*CS* and K is a submodule of $Z_2(M)$. Then $Z(K) \subseteq^{ess} L$ where L is a summand of M and $L \subseteq^{ess} K + L$. But L is closed, so $K \subseteq L$. Since $Z_2(M) \subseteq^{ess} L + Z_2(M)$ and $Z_2(M)$ is closed in M, so $L \subseteq Z_2(M)$ and $Z_2(M)$ is CS. In particular, $Z_2(M)$ is the only closure of $Z(M)$. Thus $Z_2(M)$ is a summand of M. □

A module is called *s*-continuous if it satisfies both the *s*-*C*1- and *s*-*C*2-conditions, and a module is called quasi-*s*-continuous if it satisfies the *s*-*C*1- and *s*-*C*3-conditions, and R is called a right *s*-continuous ring (right quasi-*s*-continuous ring) if R_R has the corresponding property. Clearly every strongly *s*-injective is *s*-continuous.

Proposition 15 *If every singular simple right R-module embeds in* M *and* M *is s-CS, then* $Z_2(M)$ *is finitely cogenerated.*

Proof. Let M be a *s*-*CS* and every singular simple right *R*-module embeds in M. Then $Z_2(M)$ is a CS and summand of M by above Proposition. Also $Z_2(M)$ cogenerats every simple quotient of $Z_2(M)$ then by [3, Theorem 7.29], $Z_2(M)$ is finitely cogenerated.

Proposition 16 *Let* R *be a ring. Then* R *is a right PF-ring if and only if* R_R *is a cogenerator and* $(Z_r^2)_R$ *is* CS.

Proof. Every right *PF*-ring is right self-injective and is a right cogenerator by [3, 1.56]. Conversely, if Z_r^2 is a CS and R is cogenerator then Z_r^2 has finitely generated, essential right socle by Proposition 15. Since Z_r^2 is right finite dimensional and R_R is a cogenerator, let $Soc(Z_r^2) = S_1 \oplus S_2 \oplus \cdots \oplus S_m$ and $I_i = I(S_i)$ be the injective hull of S_i, then there exists an embedding $\sigma : I_i \to R^I$ for some set I. Then $\pi \circ \sigma \neq 0$ for some projection $\pi : R^I \to R$, so $(\pi \circ \sigma)|S_i \neq 0$ and hence is monic. Thus $\pi \circ \sigma : I_i \to R$ is monic, and so $R = E_1 \oplus \cdots \oplus E_m \oplus T$ where T is nonsingular. So R is a right *PF*-ring by Theorem 3. □

Proposition 17 *If every simple singular right R-module embeds in* R *and* $(Z_2^r)_R$ *is continuous, then* R *is semiperfect.*

Proof. Let $(Z_2^r)_R$ be continuous and every simple singular right *R*-module embeds in R. Then $(Z_2^r)_R$ has a finitely generated essential socle by Proposition 15. Thus, by hypothesis, there exist simple submodules $S_1, \cdots, S_n$ of $(Z_2^r)_R$ such that $\{S_1, \cdots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple singular right *R*-modules. Since $(Z_2^r)_R$ is CS, there exist submodules $Q_1, \cdots, Q_n$ of $(Z_2^r)_R$ such that Q_i is a direct summand of $(Z_2^r)_R$ and $(S_i)_R \subseteq^{ess} (Q_i)_R$ for $i = 1, \cdots, n$. Since Q_i is an indecomposable continuous *R*-module, it has a local endomorphism ring; and since Q_i is projective, $J(Q_i)$ is maximal and small in Q_i by [3, 1.54]. Then Q_i is a projective cover of the simple module $Q_i / J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i / J(Q_i) \cong Q_j / J(Q_j)$; and the converse also holds because every module has at most one projective cover up to isomorphism. But it is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$ if and only if $i = j$. Moreover, every $Q_i / J(Q_i)$ is singular. Thus, $\{Q_1 / J(Q_1), \cdots, Q_n / J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple singular right *R*-modules. Hence every simple singular right *R*-module has a projective cover. Since every non-singular simple right *R*-module is projective, we conclude that R is semiperfect. □

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