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Common Fixed Points for a Countable Family of Set-Valued Mappings with Quasi-Contractive Conditions on Metrically Convex Spaces

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ABSTRACT

In this paper, we consider a countable family of set-valued mappings satisfying some quasi-contractive conditions. We also construct a sequence by the quasi-contractive conditions of mappings and the boundary condition of a closed subset of a metrically convex space, and then prove that the unique limit of the sequence is the unique common fixed point of the mappings. Finally, we give more generalized common fixed point theorems for a countable family of single-valued mappings. The main results generalize and improve many common fixed point theorems for a finite or countable family of single valued or set-valued mappings with quasi-contractive conditions.

KEYWORDS

1. Introduction

There have appeared many fixed point theorems for a single-valued self map of a closed subset of a Banach space. However, in many applications, the mapping under considerations is not a self-mapping on a closed sub-set. In 1976, Assad [1] gave sufficient condition for such single valued mapping to obtain a fixed point by prov-ing a fixed point theorem for Kannan mappings on a Banach space and putting certain boundary conditions on the mapping. Similar results for multi-valued mappings were respectively given by Assad [2] and Assad and Kirk [3]. On the other hand, many authors discussed common fixed point problems [4-7] for finite single or multi-valued mappings on a complete 2-metric convex space or a complete cone metric space respectively. And some authors also discussed common fixed point problems [8-13] for a countable family of self-single-valued mappings with contractive or quasi-contractive conditions on a metric space or a metrically convex space re-spectively. These results improved and generalized many previous works.

In this paper, we will discuss the existent problems of common fixed points for a countable family of surjec-tive set-valued mappings, which satisfy certain quasi-contractive condition, defined on a complete metrically convex space and obtain some important theorems. The main results in this paper further generalize and improve many common fixed point theorems for single valued or multi-valued mappings with quasi-contractive type conditions.

Through this paper, (X, d) (or X) is a metric space. Let $bc(X)$ denote the families of all bounded closed subset of X.

Let $d(x, A) = \inf_{y \in A} d(x, y)$, the distance between x and $A \in bc(X)$.

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Definition 1.1. ([8-10]) A metric space (X,d) is said to be metrically convex, if any $x, y \in X$ with $x \neq y$, there exists $z \in X$ such that $z \neq x$, $z \neq y$ and $d(x,z) + d(z,y) = d(x,y)$.

Lemma 1.1. ([3,8]) If K is a nonempty closed subset of a complete metrically convex space (X,d), then for any $x \in K$ and $y \notin K$, there exists $z \in \partial K$ which satisfies $d(x,z) + d(z,y) = d(x,y)$.

Lemma 1.2. ([13]) If X is a complete metric space and $A \in bc(X)$, then $d(\cdot, A)$ is continuous on X. Moreover, we have :

1) $A = \{x \in X : d(x,A) = 0\}$;
2) $A \subset B$ if and only if $x \in X$, $d(x,B) \leq d(x,A)$;
3) for any $x, y \in X$, $d(x,A) \leq d(x,y) + d(y,A)$.

2. Main Results

Theorem 2.1. Let K be a nonempty closed subset of a complete metrically convex space (X,d) with $\text{int} K \neq \varnothing$, $\{T_i : X \to bc(X)\}_{i \in \mathbb{N}}$ a countable family of surjective set-valued mappings with nonempty values such that for any $i, j \in \mathbb{N}$ with $i < j$, any $x, y \in X$,

$$d(x,y) \leq \lambda u_{i,j}(x,y), \tag{1}$$

where $u_{i,j}(x,y) \in \left\{ \dfrac{d(x,T_i x) + d(y,T_j y)}{2}, \dfrac{d(x,T_j y) + d(y,T_i x)}{2} \right\}$ and $\lambda \in \left(0, \dfrac{-3 + \sqrt{17}}{2}\right)$ is a constant number.

Furthermore, if $T_i(K^C) \cap \partial K = \varnothing$ for all $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$ and $p \in K$ and any $z \in T_i p$, there exists $j \in \mathbb{N}$ such that $p \in T_j z$, then $\{T_i\}_{i \in \mathbb{N}}$ has a unique common fixed point in K.

Proof Take $x_0 \in K$. We will construct two sequences $\{x_n\}$ and $\{x_n'\}$ in the following manner. Since T_1 is on-to, there exists $x_1' \in X$ such that $x_0 \in T_1 x_1'$. If $x_1' \in K$, then put $x_1 = x_1'$; if $x_1' \notin K$, then by Lemma 1.1 there exits $x_1 \in \partial K$ such that $d(x_0, x_1) + d(x_1, x_1') = d(x_0, x_1')$. For x_1, since T_2 is on-to, there exists $x_2' \in X$ such that $x_1 \in T_2 x_2'$. If $x_2' \in K$, then put $x_2 = x_2'$; if $x_2' \notin K$, then by Lemma 1.1 there exists $x_2 \in \partial K$ such that $d(x_1, x_2) + d(x_2, x_2') = d(x_1, x_2')$. Continuing this way, we obtain $\{x_n\}$ and $\{x_n'\}$:

1) $x_{n-1} \in T_n x_n'$;
2) if $x_n' \in K$, then put $x_n = x_n'$;
3) if $x_n' \notin K$, then by Lemma 1.1 there exists $x_n \in \partial K$ such that $d(x_{n-1}, x_n) + d(x_n, x_n') = d(x_{n-1}, x_n')$;
4) $x_n \in K$ for all $n = 0, 1, 2, \cdots$.

Let $P = \{x_i \in \{x_n\} : x_i = x_i'\}$ and $Q = \{x_i \in \{x_n\} : x_i \neq x_i'\}$. If there exists $n \in \mathbb{N}$ such that $x_n \in Q$, then $x_{n-1}, x_{n+1} \in P$. In fact, By 3) and the definition of Q, we have that $x_n \neq x_n'$, $x_n \in \partial K$, $x_n' \notin K$. If $x_{n-1} \in Q$, then $x_{n-1} \in \partial K$. On the other hand, since $T_n(K^C) \cap \partial K = \varnothing$ and $x_{n-1} \in T_n x_n' \subset T_n(K^C)$, hence $x_{n-1} \notin \partial K$, which is a contradiction. If $x_{n+1} \in Q$, then $x_{n+1} \in \partial K$ and $x_{n+1}' \notin K$, hence $x_n \in T_{n+1} x_{n+1}' \subset T_{n+1}(K^C)$, so $x_n \notin \partial K$, which is another contradiction.

By the definitions and properties of P and Q, we can estimate $d(x_n, x_{n+1})$ into three cases:

Case I. $x_n, x_{n+1} \in P$. In this case, $x_n = x_n'$, $x_{n+1} = x_{n+1}'$, $x_{n-1} \in T_n x_n'$ and $x_n \in T_{n+1} x_{n+1}'$. And we have

$$d(x_n, x_{n+1}) = d(x_n', x_{n+1}') \leq \lambda u_{n,n+1}(x_n', x_{n+1}'),$$

where

$$u_{n,n+1}(x_n', x_{n+1}') \in \left\{ \frac{d(x_n', T_n x_n') + d(x_{n+1}', T_{n+1} x_{n+1}')}{2}, \frac{d(x_n', T_{n+1} x_{n+1}') + d(x_{n+1}', T_n x_n')}{2} \right\}.$$

If $u_{n,n+1}(x_n', x_{n+1}') = \dfrac{d(x_n', T_n x_n') + d(x_{n+1}', T_{n+1} x_{n+1}')}{2}$, then

$$d(x_n, x_{n+1}) = d(x_n', x_{n+1}') \leq \frac{\lambda}{2} \left[d(x_n', T_n x_{n'}) + d(x_{n+1}', T_{n+1} x_{n+1}') \right]$$

$$\leq \frac{\lambda}{2} \left[d(x_n, x_{n-1}) + d(x_{n+1}, x_n) \right]$$

$$\leq \frac{\lambda}{2} d(x_n, x_{n-1}) + \frac{1}{2} d(x_{n+1}, x_n),$$

hence

$$d(x_n, x_{n+1}) = d(x'_n, x'_{n+1}) \le \lambda d(x_{n-1}, x_n);$$

If $u_{n,n+1}(x'_n, x'_{n+1}) = \dfrac{d(x'_n, T_{n+1}x'_{n+1}) + d(x'_{n+1}, T_n x'_n)}{2}$, then

$$d(x_n, x_{n+1}) = d(x'_n, x'_{n+1}) \le \frac{\lambda}{2}\left[d(x'_n, T_{n+1}x'_{n+1}) + d(x'_{n+1}, T_n x'_n)\right]$$

$$\le \frac{\lambda}{2}\left[d(x_n, x_n) + d(x_{n+1}, x_{n-1})\right] \le \frac{\lambda}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]$$

$$\le \frac{\lambda}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right],$$

hence

$$d(x_n, x_{n+1}) = d(x'_n, x'_{n+1}) \le \lambda d(x_{n-1}, x_n).$$

Therefore, in any situation, we have

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Case II. $x_n \in P$ and $x_{n+1} \in Q$. In this case, $x_n = x'_n$, $x_{n+1} \ne x'_{n+1}$ and $x_{n-1} \in T_n x'_n$ and $x_n \in T_{n+1}x'_{n+1}$. And we have

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \le \lambda u_{n,n+1}(x'_n, x'_{n+1}),$$

where

$$u_{n,n+1}(x'_n, x'_{n+1}) \in \left\{ \frac{d(x'_n, T_n x_{n'}) + d(x'_{n+1}, T_{n+1}x'_{n+1})}{2}, \frac{d(x'_n, T_{n+1}x'_{n+1}) + d(x'_{n+1}, T_n x'_n)}{2} \right\}$$

If $u_{n,n+1}(x'_n, x'_{n+1}) = \dfrac{d(x'_n, T_n x'_n) + d(x'_{n+1}, T_{n+1}x'_{n+1})}{2}$, then

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \le \frac{\lambda}{2}\left[d(x'_n, T_n x'_n) + d(x'_{n+1}, T_{n+1}x'_{n+1})\right]$$

$$\le \frac{\lambda}{2}\left[d(x_n, x_{n-1}) + d(x'_{n+1}, x_n)\right]$$

$$\le \frac{\lambda}{2}d(x_n, x_{n-1}) + \frac{1}{2}d(x'_{n+1}, x_n),$$

hence

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \le \lambda d(x_{n-1}, x_n);$$

If $u_{n,n+1}(x'_n, x'_{n+1}) = \dfrac{d(x'_n, T_{n+1}x'_{n+1}) + d(x'_{n+1}, T_n x'_n)}{2}$, then

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \le \frac{\lambda}{2}\left[d(x'_n, T_{n+1}x'_{n+1}) + d(x'_{n+1}, T_n x'_n)\right]$$

$$\le \frac{\lambda}{2}\left[d(x_n, x_n) + d(x'_{n+1}, x_{n-1})\right] \le \frac{\lambda}{2}\left[d(x_{n-1}, x_n) + d(x_n, x'_{n+1})\right]$$

$$\le \frac{\lambda}{2}d(x_n, x_{n-1}) + \frac{1}{2}d(x'_{n+1}, x_n),$$

hence

$$d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \le \lambda d(x_{n-1}, x_n).$$

Therefore, in any situation, we have

$$d\left(x_n, x'_{n+1}\right) \le \lambda d\left(x_{n-1}, x_n\right) \text{ for all } n \in \mathbb{N}.$$

But $d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x'_{n+1}\right) = d\left(x_n, x'_{n+1}\right)$, hence we obtain

$$d\left(x_n, x_{n+1}\right) \le d\left(x_n, x'_{n+1}\right) \le \lambda d\left(x_{n-1}, x_n\right) \text{ for all } n \in \mathbb{N}.$$

Case III. $x_n \in Q$ and $x_{n+1} \in P$. In this case, $x_{n-1} \in P$ by the property of P and Q, and $x_n \ne x'_n$, $x_{n+1} = x'_{n+1}$, $x_{n-1} \in T_n x'_n$ and $x_n \in T_{n+1} x'_{n+1}$. And we have

$$d\left(x'_n, x_{n+1}\right) = d\left(x'_n, x'_{n+1}\right) \le \lambda u_{n,n+1}\left(x'_n, x'_{n+1}\right),$$

where

$$u_{n,n+1}\left(x'_n, x'_{n+1}\right) \in \left\{ \frac{d\left(x'_n, T_n x'_n\right) + d\left(x'_{n+1}, T_{n+1}x'_{n+1}\right)}{2}, \frac{d\left(x'_n, T_{n+1}x'_{n+1}\right) + d\left(x'_{n+1}, T_n x'_n\right)}{2} \right\}.$$

Here, we give two basic properties:
1) since $x_n \in Q$, so $d\left(x_{n-1}, x_n\right) + d\left(x_n, x'_n\right) = d\left(x_{n-1}, x'_n\right)$, and hence $d\left(x_{n-1}, x_n\right) \le d\left(x_{n-1}, x'_n\right)$;
2) since

$$d\left(x_n, x_{n+1}\right) \le d\left(x_n, x'_n\right) + d\left(x'_n, x_{n+1}\right) \le d\left(x_{n-1}, x_n\right) + d\left(x_n, x'_n\right) + d\left(x'_n, x_{n+1}\right) = d\left(x_{n-1}, x'_n\right) + d\left(x'_n, x_{n+1}\right),$$

hence $d\left(x_n, x_{n+1}\right) - d\left(x_{n-1}, x'_n\right) \le d\left(x'_n, x_{n+1}\right)$.

If $u_{n,n+1}\left(x'_n, x'_{n+1}\right) = \dfrac{d\left(x'_n, T_n x'_n\right) + d\left(x'_{n+1}, T_{n+1}x'_{n+1}\right)}{2}$, then

$$d\left(x'_n, x_{n+1}\right) = d\left(x'_n, x'_{n+1}\right) \le \frac{\lambda}{2}\left[d\left(x'_n, T_n x'_n\right) + d\left(x'_{n+1}, T_{n+1}x'_{n+1}\right)\right]$$

$$\le \frac{\lambda}{2}\left[d\left(x'_n, x_{n-1}\right) + d\left(x_{n+1}, x_n\right)\right],$$

hence by 2),

$$d\left(x_n, x_{n+1}\right) - d\left(x_{n-1}, x'_n\right) \le \frac{\lambda}{2}\left[d\left(x'_n, x_{n-1}\right) + d\left(x_{n+1}, x_n\right)\right]$$

So by Case II, we obtain

$$d\left(x_n, x_{n+1}\right) \le \frac{2+\lambda}{2-\lambda} d\left(x_{n-1}, x'_n\right) \le \frac{\lambda(2+\lambda)}{2-\lambda} d\left(x_{n-2}, x_{n-1}\right) \text{ for all } n \in \mathbb{N}, n \ge 2;$$

If $u_{n,n+1}\left(x'_n, x'_{n+1}\right) = \dfrac{d\left(x'_n, T_{n+1}x'_{n+1}\right) + d\left(x'_{n+1}, T_n x'_n\right)}{2}$, then

$$d\left(x'_n, x_{n+1}\right) = d\left(x'_n, x'_{n+1}\right) \le \frac{\lambda}{2}\left[d\left(x'_n, T_{n+1}x'_{n+1}\right) + d\left(x'_{n+1}, T_n x'_n\right)\right]$$

$$\le \frac{\lambda}{2}\left[d\left(x'_n, x_n\right) + d\left(x_{n+1}, x_{n-1}\right)\right] \le \frac{\lambda}{2}\left[d\left(x'_n, x_n\right) + d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right]$$

$$\le \frac{\lambda}{2}\left[d\left(x_{n-1}, x'_n\right) + d\left(x_n, x_{n+1}\right)\right],$$

hence by 2),

$$d\left(x_n, x_{n+1}\right) - d\left(x_{n-1}, x'_n\right) \le \frac{\lambda}{2}\left[d\left(x'_n, x_{n-1}\right) + d\left(x_{n+1}, x_n\right)\right]$$

So by Case II again, we obtain

$$d\left(x_n, x_{n+1}\right) \le \frac{2+\lambda}{2-\lambda} d\left(x_{n-1}, x'_n\right) \le \frac{\lambda(2+\lambda)}{2-\lambda} d\left(x_{n-2}, x_{n-1}\right) \text{ for all } n \in \mathbb{N}, n \ge 2.$$

Hence in any situation, we have

$$d\left(x_n, x_{n+1}\right) \le \frac{\lambda\left(2+\lambda\right)}{2-\lambda} d\left(x_{n-2}, x_{n-1}\right) \text{ for all } n \in \mathbb{N}, n \ge 2.$$

Therefore, from Case I, Case II and Case III, we obtain

$$d\left(x_n, x_{n+1}\right) \le \frac{\lambda\left(2+\lambda\right)}{2-\lambda} \max\left\{d\left(x_{n-1}, x_n\right) d\left(x_{n-2}, x_{n-1}\right)\right\} \text{ for all } n \in \mathbb{N}, n \ge 2.$$

Let $h = \dfrac{\lambda\left(2+\lambda\right)}{2-\lambda}$, then $0 < h < 1$ since $\lambda \in \left(0, \dfrac{-3+\sqrt{17}}{2}\right)$, hence we have

$$d\left(x_n, x_{n+1}\right) \le h \max\left\{d\left(x_{n-1}, x_n\right) d\left(x_{n-2}, x_{n-1}\right)\right\} \text{ for all } n \in \mathbb{N}, n \ge 2,$$

so

$$d\left(x_n, x_{n+1}\right) \le h^{\frac{n}{2}-1} \max\left\{d\left(x_2, x_1\right) d\left(x_1, x_0\right)\right\} \text{ for all } n \in \mathbb{N}, n \ge 2.$$

Let $\delta = h^{-1} \max\left\{d\left(x_2, x_1\right), d\left(x_1, x_0\right)\right\}$, then for $m > n \ge N \ge 2$,

$d\left(x_n, x_m\right) \le \sum_{i=n}^{m-1} d\left(x_i, x_{i+1}\right) \le \sum_{i=N}^{\infty} d\left(x_i, x_{i+1}\right) \le \sum_{i=N}^{\infty} h^{\frac{i}{2}} \delta \to 0$ as $N \to \infty$. Hence $\left\{x_n\right\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since X is complete, $\left\{x_n\right\}$ has a limit x^*. But K is closed and $x_n \in K$ for all $n \in \mathbb{N}$, hence $x^* \in K$.

By the property of P and Q, we can see that there exists an infinite subsequence $\left\{x_{n_k+1}\right\}$ of $\left\{x_n\right\}$ such that $x_{n_k+1} \in P$, hence $x_{n_k+1} = x'_{n_k+1}$ and $x_{n_k} \in T_{n_k+1} x'_{n_k+1}$.

Next, we will prove that x^* is a common fixed point of $\left\{T_i\right\}_{i \in \mathbb{N}}$. Fix any $i \in \mathbb{N}$, for each fixed $z \in T_i x^*$, there exists $j \in \mathbb{N}$ such that $x^* \in T_j z$. Take an enough large $k \in \mathbb{N}$ such that $n_k + 1 > i$ and $n_k + 1 > j$. By Lemma 1.2 3) and (1), we have

$$d\left(T_i x^*, x^*\right) \le d\left(T_i x^*, x_{n_k+1}\right) + d\left(x_{n_k+1}, x^*\right) \le d\left(z, x_{n_k+1}\right) + d\left(x_{n_k+1}, x^*\right),$$

and

$$d\left(z, x_{n_k+1}\right) \le \lambda u_{j, n_k+1}\left(z, x_{n_k+1}\right),$$

where

$$u_{j, n_k+1}\left(z, x_{n_k+1}\right) \in \left\{\frac{d\left(z, T_j z\right) + d\left(x_{n_k+1}, T_{n_k+1} x_{n_k+1}\right)}{2}, \frac{d\left(z, T_{n_k+1} x_{n_k+1}\right) + d\left(x_{n_k+1}, T_j z\right)}{2}\right\}.$$

If $u_{j, n_k+1}\left(z, x_{n_k+1}\right) = \dfrac{d\left(z, T_j z\right) + d\left(x_{n_k+1}, T_{n_k+1} x_{n_k+1}\right)}{2}$, then

$$d\left(T_i x^*, x^*\right) \le \frac{\lambda}{2}\left[d\left(z, T_j z\right) + d\left(x_{n_k+1}, T_{n_k+1} x_{n_k+1}\right)\right] + d\left(x_{n_k+1}, x^*\right) \le \frac{\lambda}{2}\left[d\left(z, x^*\right) + d\left(x_{n_k+1}, x_{n_k}\right)\right] + d\left(x_{n_k+1}, x^*\right).$$

Let $k \to \infty$, then $d\left(T_i x^*, x^*\right) \le \dfrac{\lambda}{2} d\left(z, x^*\right)$ since $x_n \to x^*$, hence

$d\left(T_i x^*, x^*\right) \le \dfrac{\lambda}{2} \inf_{z \in T_i x^*} d\left(z, x^*\right) = \dfrac{\lambda}{2} d\left(T_i x^*, x^*\right)$. So $d\left(T_i x^*, x^*\right) = 0$ since $\lambda < 1$, therefore $x^* \in T_i x^*$ by Lemma 1.2 1).

If $u_{j, n_k+1}\left(z, x_{n_k+1}\right) = \dfrac{d\left(z, T_{n_k+1} x_{n_k+1}\right) + d\left(x_{n_k+1}, T_j z\right)}{2}$, then

$$d\left(T_i x^*, x^*\right) \le \frac{\lambda}{2}\left[d\left(z, T_{n_k+1} x_{n_k+1}\right) + d\left(x_{n_k+1}, T_j z\right)\right] + d\left(x_{n_k+1}, x^*\right)$$

$$\le \frac{\lambda}{2}\left[d\left(z, x_{n_k}\right) + d\left(x_{n_k+1}, x^*\right)\right] + d\left(x_{n_k+1}, x^*\right).$$

Let $k \to \infty$, then $d\left(T_i x^*, x^*\right) \leq \dfrac{\lambda}{2} d\left(z, x^*\right)$ since $x_n \to x^*$, hence similarly, $x^* \in T_i x^*$.

So in any situation, $x^* \in T_i x^*$ for all $i \in \mathbb{N}$, so x^* is a common fixed point of $\left\{T_i\right\}_{i \in \mathbb{N}}$.

If x^* and y^* are all common fixed points of $\left\{T_i\right\}_{i \in \mathbb{N}}$, then we will have

$$d\left(x^*, y^*\right) \leq \lambda u_{1,2}\left(x^*, y^*\right),$$

where

$$u_{1,2}\left(x^*, y^*\right) \in \left\{\frac{d\left(x^*, T_1 x^*\right) + d\left(y^*, T_2 y^*\right)}{2}, \frac{d\left(x^*, T_2 y^*\right) + d\left(y^*, T_1 x^*\right)}{2}\right\}.$$

If $u_{1,2}\left(x^*, y^*\right) = \dfrac{d\left(x^*, T_1 x^*\right) + d\left(y^*, T_2 y^*\right)}{2}$, then $d\left(x^*, y^*\right) = 0$, hence $x^* = y^*$;

If $u_{1,2}\left(x^*, y^*\right) = \dfrac{d\left(x^*, T_2 y^*\right) + d\left(y^*, T_1 x^*\right)}{2}$, then $d\left(x^*, y^*\right) \leq \dfrac{\lambda}{2}\left[d\left(x^*, T_2 y^*\right) + d\left(y^*, T_1 x^*\right)\right] = \lambda d\left(x^*, y^*\right)$,

hence $d\left(x^*, y^*\right) = 0$ since $\lambda < 1$, so $x^* = y^*$.

Hence in any situation, $x^* = y^*$. So x^* is the unique common fixed points of $\left\{T_i\right\}_{i \in \mathbb{N}}$.

If the mappings in Theorem 2.1 are all single-valued, then Theorem 2.1 becomes the next form.

Theorem 2.2. Let K be a nonempty closed subset of a complete metrically convex space (X, d) with $\operatorname{int} K \neq \varnothing$, $\left\{T_i : X \to X\right\}_{i \in \mathbb{N}}$ a countable family of surjective single-valued mappings such that for any $i, j \in \mathbb{N}$ with $i < j$, any $x, y \in X$,

$$d(x, y) \leq \lambda u_{i,j}(x, y), \tag{2}$$

where $u_{i,j}(x, y) \in \left\{\dfrac{d\left(x, T_i x\right) + d\left(y, T_j y\right)}{2}, \dfrac{d\left(x, T_j y\right) + d\left(y, T_i x\right)}{2}\right\}$ and $\lambda \in \left(0, \dfrac{-3 + \sqrt{17}}{2}\right)$ is a constant number.

Furthermore, if $T_i\left(K^C\right) \bigcap \partial K = \varnothing$ for all $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$ and $p \in K$, there exists $j \in \mathbb{N}$ such that $p = T_j T_i p$, then $\left\{T_i\right\}_{i \in \mathbb{N}}$ has a unique common fixed point in K.

From Theorem 2.2, we can obtain the following more generalized common fixed point theorem.

Theorem 2.3. Let K be a nonempty closed subset of a complete metrically convex space (X, d) with $\operatorname{int} K \neq \varnothing$, $\left\{T_{i,j} : X \to X\right\}_{i, j \in \mathbb{N}}$ a family of subjective single-valued mappings, $\left\{m_{i,j}\right\}_{i, j \in \mathbb{N}}$ a family of positive integral numbers such that for any $i_1, i_2, j \in \mathbb{N}$, $x, y \in X$,

$$d(x, y) \leq \lambda u_{i_1, i_2, j}(x, y), \tag{3}$$

where $u_{i_1, i_2, j}(x, y) \in \left\{\dfrac{d\left(x, T_{i_1, j}^{m_{i_1, j}} x\right) + d\left(y, T_{i_2, j}^{m_{i_2, j}} y\right)}{2}, \dfrac{d\left(x, T_{i_2, j}^{m_{i_2, j}} y\right) + d\left(y, T_{i_1, j}^{m_{i_1, j}} x\right)}{2}\right\}$ and $\lambda \in \left(0, \dfrac{-3 + \sqrt{17}}{2}\right)$ is a

constant number. Furthermore, if 1) $T_{i,j}^{m_{i,j}}\left(K^C\right) \bigcap \partial K = \varnothing$ for all $i \in \mathbb{N}$, 2) for each $i_1, j \in \mathbb{N}$ and $p \in X$, there exists $i_2 \in \mathbb{N}$ such that $p = T_{i_2, j}^{m_{i_2, j}} T_{i_1, j}^{m_{i_1, j}} p$, 3) for each $i_1, i_2, \mu, \nu \in \mathbb{N}$ with $\mu \neq \nu$, $T_{i_1, \mu} T_{i_2, \nu} = T_{i_2, \nu} T_{i_1, \mu}$. Then $\left\{T_{i,j}\right\}_{i, j \in \mathbb{N}}$ has a unique common fixed point in K.

Proof Fix $j \in \mathbb{N}$, and let $S_{i,j} = T_{i,j}^{m_{i,j}}$, then $\left\{S_{i,j}\right\}_{i \in \mathbb{N}}$ satisfies all of the conditions of Theorem 2.2, hence $\left\{S_{i,j}\right\}_{i \in \mathbb{N}}$ has an unique common fixed point p_j in K. Now, we will prove that p_j is also unique common fixed point of $\left\{T_{i,j}\right\}_{i \in \mathbb{N}}$. In fact, for any fixed $i \in \mathbb{N}$,

$S_{i,j}\left(T_{i,j}\left(p_j\right)\right) = T_{i,j}^{m_{i,j}}\left(T_{i,j}\left(p_j\right)\right) = T_{i,j}\left(T_{i,j}^{m_{i,j}}\left(p_j\right)\right) = T_{i,j}\left(S_{i,j}\left(p_j\right)\right) = T_{i,j}\left(p_j\right)$. This means that $T_{i,j}\left(p_j\right)$ is a

fixed point of $S_{i,j}$. For any $k \in \mathbb{N}$ with $k \neq i$, there exists $k' \in \mathbb{N}$ such that $S_{k', j} S_{k, j}\left(T_{i,j}\left(p_j\right)\right) = T_{i,j}\left(p_j\right)$ by 2), and by (3) we have that

$$d\left(T_{i,j}\left(p_j\right), S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right) \leq \lambda u_{k, k'j}\left(T_{i,j}\left(p_j\right), S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right),$$

where

$$u_{k,k'j}\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)$$

$$\in\left\{\frac{d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)+d\left(S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right),S_{k',j}\left(S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)\right)}{2},\right.$$

$$\left.\frac{d\left(T_{i,j}\left(p_j\right),S_{k',j}\left(S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)\right)+d\left(S_{k,j}\left(T_{i,j}\left(p_j\right)\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)}{2}\right\}$$

$$=\left\{0,d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)\right\}.$$

If $u_{k,k'j}\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)=0$, then $d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)=0$, hence $T_{i,j}\left(p_j\right)=S_{k,j}\left(T_{i,j}\left(p_j\right)\right)$;

If $u_{k,k'j}\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)=d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)$, then

$d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)\le\lambda d\left(T_{i,j}\left(p_j\right),S_{k,j}\left(T_{i,j}\left(p_j\right)\right)\right)$, hence $T_{i,j}\left(p_j\right)=S_{k,j}\left(T_{i,j}\left(p_j\right)\right)$.

Hence in any situation, we have that $T_{i,j}\left(p_j\right)$ is a fixed point of $S_{k,j}$ for each k with $k\ne i$. So $T_{i,j}\left(p_j\right)$ is a common fixed point of $\left\{S_{i,j}\right\}_{i\in\mathbb{N}}$. By uniqueness of common fixed points of $\left\{S_{i,j}\right\}_{i\in\mathbb{N}}$, we have $T_{i,j}\left(p_j\right)=p_j$ for each $i\in\mathbb{N}$. Hence p_j is a common fixed point of $\left\{T_{i,j}\right\}_{i\in\mathbb{N}}$.

If u_j and v_j are all common fixed points of $\left\{T_{i,j}\right\}_{i\in\mathbb{N}}$, then they are also common fixed points of $\left\{S_{i,j}\right\}_{i\in\mathbb{N}}$, hence by the uniqueness of common fixed points of $\left\{S_{i,j}\right\}_{i\in\mathbb{N}}$, we obtain $u_i=p_j=v_j$. This means that for each $j\in\mathbb{N},\left\{T_{i,j}\right\}_{i\in\mathbb{N}}$ has a unique common fixed point p_j.

Now, we prove $p_\mu=p_\nu$ for each $\mu,\nu\in\mathbb{N}$. In fact, for any $i_1,i_2,\mu,\nu\in\mathbb{N}$ with $\mu\ne\nu$, since $T_{i_1,\mu}\left(p_\mu\right)=p_\mu$ and $T_{i_2,\nu}\left(p_\nu\right)=p_\nu$, so $T_{i_1,\mu}\left(T_{i_2,\nu}\left(p_\nu\right)\right)=T_{i_1,\mu}\left(p_\nu\right)$, hence $T_{i_2,\nu}\left(T_{i_1,\mu}\left(p_\nu\right)\right)=T_{i_1,\mu}\left(T_{i_2,\nu}\left(p_\nu\right)\right)=T_{i_1,\mu}\left(p_\nu\right)$ by 3). Therefore, $T_{i_1,\mu}\left(p_\nu\right)$ is a fixed point of $T_{i_2,\nu}$ for each i_2, i.e., $T_{i_1,\mu}\left(p_\nu\right)$ is a common fixed point of $\left\{T_{i_2,\nu}\right\}_{i_2\in\mathbb{N}}$. But $\left\{T_{i_2,\nu}\right\}_{i_2\in\mathbb{N}}$ has a unique common fixe point p_ν, hence $T_{i_1,\mu}\left(p_\nu\right)=p_\nu$ for each i_1, and therefore p_ν is a common fixed point of $\left\{T_{i_1,\mu}\right\}_{i_1\in\mathbb{N}}$. But $\left\{T_{i_1,\mu}\right\}_{i_1\in\mathbb{N}}$ has a unique common fixed point p_μ, hence $p_\mu=p_\nu$. Let $p^*=p_j$, then p^* is the common fixed point of $\left\{T_{i,j}\right\}_{i,j\in\mathbb{N}}$. The uniqueness of common fixed points of $\left\{T_{i,j}\right\}_{i,j\in\mathbb{N}}$ is obvious.

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