

Value Distribution of the k th Derivatives of Meromorphic Functions

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ABSTRACT

In the paper, we take up a new method to prove a result of value distribution of meromorphic functions: let f be a meromorphic function in \mathbb{C} , and let $a(z) = P(z)e^z \neq 0$, where P is a polynomial. Suppose that all zeros of f have multiplicity at least $k+1$, except possibly finite many, and $T(r, a) = o(T(r, f))$ as $r \rightarrow \infty$. Then $f^{(k)} - R$ has infinitely many zeros.

KEYWORDS

Meromorphic Function; Spherical Derivative; Quasi-Normality

1. Introduction

The value distribution theory of meromorphic functions occupies one of the central places in Complex Analysis which now has been applied to complex dynamics, complex differential and functional equations, Diophantine equations and others.

In his excellent paper [1], W.K. Hayman studied the value distribution of certain meromorphic functions and their derivatives under various conditions. Among other important results, he proves that if $f(z)$ is a transcendental meromorphic function in the plane, then either $f(z)$ assumes every finite value infinitely often, or every derivative of $f(z)$ assumes every finite nonzero value infinitely often. This result is known as Hayman's alternative. Thereafter, the value distribution of derivatives of transcendental functions continued to be studied.

In this paper, we study the value distribution of transcendental meromorphic functions, all but finitely many of whose zeros have multiplicity at least $k+1$, where k is a positive integer.

In 2008, Liu *et al.* [2] proved the following results.

Theorem A Let $k \geq 2$ be an integer, let $f(z)$ be a meromorphic function of infinite order $\rho(f)$ in \mathbb{C} , and let $a(z) = P(z)e^z \neq 0$, where P is a polynomial. Suppose that

- 1) all zeros of f have multiplicity at least $k+1$, except possibly finitely many, and
- 2) all poles of f are multiple, except possibly finitely many.

Then $f^{(k)}(z) - a(z)$ has infinitely many zeros.

Theorem B Let $k \geq 2$ be an integer, let $f(z)$ be a meromorphic function of finite order $\rho(f)$ in \mathbb{C} , and let $a(z) = P(z)e^z \neq 0$, where P is a polynomial. Suppose that

- 1) all zeros of f have multiplicity at least $k+1$, except possibly finitely many, and

- 2)
$$\lim_{r \rightarrow \infty} \left(\frac{T(r, a)}{T(r, f)} + \frac{T(r, f)}{T(r, a)} \right) = \infty.$$

Then $f^{(k)}(z) - a(z)$ has infinitely many zeros.

In the present paper, we prove the following result, which is a significant improvement of Theorem 1.

Theorem 1 *Let $k \geq 1$ be an integer, let f be a meromorphic function of order $\rho(f) > 2$ in \mathbb{C} , and let $a(z) = P(z)e^z \neq 0$, where P is a polynomial. Suppose that all zeros of f have multiplicity at least $k + 1$, except possibly finitely many. Then $f^{(k)}(z) - a(z)$ has infinitely many zeros.*

Theorem 1 and Theorem 2 taken together imply the following result.

Theorem 2 *Let $k \geq 2$ be an integer, let f be a meromorphic function in \mathbb{C} , and let $a(z) = P(z)e^z \neq 0$, where P is a polynomial. Suppose that*

- 1) all zeros of f have multiplicity at least $k + 1$, except possibly finitely many, and
- 2) $T(r, a) = o(T(r, f))$ as $r \rightarrow \infty$.

Then $f^{(k)}(z) - a(z)$ has infinitely many zeros.

2. Notation and Some Lemmas

We use the following notation. Let \mathbb{C} be complex plane and D be a domain in \mathbb{C} . For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z \mid |z - z_0| < r\}$ and $\Delta = \Delta(0, 1)$. We write $f_n \Rightarrow f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric.

Let f be a meromorphic function in \mathbb{C} . Set

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \text{ and } S(D, f) = \frac{1}{\pi} \iint_D [f^\#(z)]^2 dx dy. \tag{1.1}$$

The Ahlfors-Shimizu characteristic is defined by

$$T_0(r, f) = \int_0^r \frac{S(t, f)}{t} dt.$$

Remark *Let $T(r, f)$ denote the usual Nevanlinna characteristic function. Since $T(r, f) - T_0(r, f)$ is bounded as a function of r , we can replace $T_0(r, f)$ with $T(r, f)$ in the paper.*

The order $\rho(f)$ of the meromorphic function f is defined as

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \text{ or } \rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T_0(r, f)}{\log r}.$$

Lemma 1 [3] *Let $\{h_n\}$ a sequence of holomorphic functions in D such that $h_n \Rightarrow h = H'$ locally uniformly in D , where H is univalent in D . Let $\{f_n\}$ be a sequence of functions meromorphic in D such that for each n ,*

- 1) all zeros of f_n have multiplicity at least $k + 1$; and
- 2) $f_n^{(k)}(z) \neq h_n(z)$.

Then $\{f_n\}$ is quasinormal of order 1 in D . If, moreover, no subsequence of $\{f_n\}$ is normal at $z_0 \in D$, then

$$f_n(z) \xrightarrow{x} f^*(z) = \int_{z_0}^z \int_{z_0}^{\zeta_1} \cdots \int_{z_0}^{\zeta_{k-1}} h(\zeta_k) d\zeta_k d\zeta_{k-1} \cdots d\zeta_1$$

locally uniformly in $D \setminus \{z_0\}$ and there exists $\delta > 0$ such that $S(\Delta(z_0, \delta), f_n) \leq k + 1$ for all n .

Remark *Since Lemma 1 is not stated explicitly in [3], let us indicate how it follows from the results of that paper. The proof that $\{f_n\}$ is quasinormal of order 1 is essentially identical to that of Theorem 1 of [3]. That proof also shows that condition (b) of Lemma 7 in [3] holds for $a_1 = z_0$. It then follows from Lemma 7 that $f_n(z) \Rightarrow f^*(z)$ locally uniformly on $D \setminus \{z_0\}$. The bound on $S(\Delta(z_0, \delta), f_n)$ follows from Lemma 9 of [3]. See also [4, Remark on page 484].*

Lemma 2 [5, Lemma 2] *Let \mathcal{F} be a family of functions meromorphic in D , all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if \mathcal{F} is not normal at z_0 , there exist, for each $0 \leq \alpha \leq k$,*

- 1) points $z_n, z_n \rightarrow z_0$;
- 2) functions $f_n \in \mathcal{F}$; and

3) positive numbers $\rho_n \rightarrow 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \xrightarrow{\chi} g(\zeta)$ in \mathbb{C} , where g is a nonconstant meromorphic function in \mathbb{C} , all of whose zeros have multiplicity at least k , such that $g^\#(\zeta) \leq g^\#(0) = kA + 1$.

Lemma 3 *Let $f(z)$ be a meromorphic function of order $\rho(f) > 2$ in \mathbb{C} , then there exist $a_n \rightarrow \infty$ and $\delta_n \rightarrow 0$ such that*

$$f^\#(a_n) \rightarrow \infty \text{ and } S(\Delta(a_n, \delta_n), f) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Proof *We claim that there exist $t_n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ such that*

$$S(\Delta(t_n, \varepsilon_n), f) = \frac{1}{\pi} \iint_{|z-t_n| < \varepsilon_n} [f^\#(z)]^2 dx dy \rightarrow \infty. \tag{1.2}$$

Otherwise there would exist $\varepsilon > 0$ and $M > 0$ such that

$$S(\Delta(z_0, \varepsilon), f) < M$$

for all $z_0 \in \mathbb{C}$. From this follows

$$S(r, f) = \frac{1}{\pi} \iint_{|z| < r} [f^\#(z)]^2 dx dy = O(r^2),$$

and hence

$$T_0(r, f) = \int_0^r \frac{S(t)}{t} dt = O(r^2).$$

Now we have $\rho(f) = \lim_{r \rightarrow \infty} \frac{-\log T_0(r, f)}{\log r} \leq 2$ which contradicts the hypothesis that $\rho(f) > 2$.

Observing that $S(\Delta(t_n, \varepsilon_n), f) = \frac{1}{\pi} \iint_{|z-t_n| < \varepsilon_n} [f^\#(z)]^2 dx dy \rightarrow \infty$, hence there exists a sequence $\{a_n\}$ such that $|a_n - t_n| \rightarrow 0$ and $f^\#(a_n) \rightarrow \infty$ as $n \rightarrow \infty$. Let $\delta_n = \varepsilon_n + |a_n - t_n|$. Obviously, $\delta_n \rightarrow 0$ and $\Delta(t_n, \varepsilon_n) \subset \Delta(a_n, \delta_n)$, and hence $S(\Delta(a_n, \delta_n), f) \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 4 *Let $k \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$. Let f be a transcendental meromorphic function, all of whose zeros have multiplicity at least $k + 1$. Set $g(z) = \frac{f(z)}{z^d e^z}$. Suppose that $\rho(f) > 2$. Then there exists a sequence $a_n \rightarrow \infty$ and $\delta_n \rightarrow 0$ such that*

$$\frac{f(a_n)}{a_n^d e^{a_n}} \rightarrow 0, \frac{f^{(k)}(a_n)}{a_n^d e^{a_n}} \rightarrow \infty \text{ and } S(\Delta(a_n, \delta_n), g) \rightarrow \infty$$

as $n \rightarrow \infty$.

Proof *Since $\rho(f) > 2$ and $\rho(z^d e^z) = 1$, we have $\rho(g) > 2$. By Lemma 3, there exist $b_n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ such that*

$$g^\#(b_n) \rightarrow \infty \text{ and } S(\Delta(b_n, \varepsilon_n), g) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Set $g_n(z) = g(z + b_n)$. Clearly, $g_n^\#(0) = g^\#(b_n) \rightarrow \infty$. Thus $\{g_n\}$ is not normal at 0. Obviously, all zeros of g have multiplicity at least $k + 1$ in $\mathbb{C} \setminus \{0\}$, and hence all zeros of $g_n(z)$ have multiplicity at least $k + 1$ in Δ for sufficiently large n . Using Lemma 2 for $\alpha = k - (1/2)$, there exist points $z_n \rightarrow 0$, and positive numbers $\rho_n \rightarrow 0$ and a subsequence of $\{g_n\}$ (that we continue to call $\{g_n\}$) such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n^{k-(1/2)}} \xrightarrow{\chi} G(\zeta)$$

in \mathbb{C} , where G is a nonconstant meromorphic function in \mathbb{C} , all of whose zeros have multiplicity at least $k + 1$.

We claim that $G^{(k)}(\zeta) \neq c_0$, where c_0 is a constant. Otherwise, $G(\zeta) = c_0 \zeta^k + c_1 \zeta^{k-1} + \dots + c_k$, where c_1, c_2, \dots , and c_k are constants. Then, either G is a constant function, or all zeros of G have multiplicity

at most k . A contradiction.

Let ζ_0 be not a zero or pole of $G^{(k)}(\zeta)$, and let $a_n = b_n + z_n + \rho_n \zeta_0$. Now we have

$$g^{(i)}(a_n) = g_n^{(i)}(z_n + \rho_n \zeta_0) = \rho_n^{k-i-(1/2)} G_n^{(i)}(\zeta_0),$$

where $i = 0, 1, \dots, k$. Since $\rho_n \rightarrow 0$ and ζ_0 is not a zero or pole of $G^{(k)}(\zeta)$, we have $a_n \rightarrow \infty$, $g^{(i)}(a_n) \rightarrow 0$ and $g^{(k)}(a_n) \rightarrow \infty$ as $n \rightarrow \infty$, where $i = 0, 1, \dots, k-1$.

Set $T_0 = 1$ and $T_i = d \cdot (d-1) \cdots (d+1-i)$, where $i = 1, 2, 3, \dots, k$. Clearly,

$$\begin{aligned} (z^d e^z)^{(n)} &= \sum_{i=0}^{i=n} C_n^i (z^d)^{(i)} e^z = \sum_{i=0}^{i=n} \frac{C_n^i T_i z^d e^z}{z^i} \\ &= z^d e^z \sum_{i=0}^{i=n} \frac{C_n^i T_i}{z^i} = z^d e^z R_n(z), \end{aligned}$$

where $R_n(z) = \sum_{i=0}^{i=n} \frac{C_n^i T_i}{z^i}$ satisfying $R_n(z) \rightarrow 1$ as $z \rightarrow \infty$.

Now, we have $\frac{f(a_n)}{a_n^d e^{a_n}} = g(a_n) \rightarrow 0$ and

$$\begin{aligned} \frac{f^{(k)}(a_n)}{a_n^d e^{a_n}} &= \frac{(z^d e^z g(z))^{(k)}}{a_n^d e^{a_n}} \Big|_{z=a_n} = \frac{\sum_{i=0}^{i=k} C_n^i (z^d e^z)^{(i)} g^{(k-i)}(z)}{a_n^d e^{a_n}} \Big|_{z=a_n} \\ &= \frac{\sum_{i=0}^{i=k} C_n^i z^d e^z R_i(z) g^{(k-i)}(z)}{a_n^d e^{a_n}} \Big|_{z=a_n} = \frac{z^d e^z \sum_{i=0}^{i=k} C_n^i R_i(z) g^{(k-i)}(z)}{a_n^d e^{a_n}} \Big|_{z=a_n} \\ &= \sum_{i=0}^{i=k} C_n^i R_i(a_n) g^{(k-i)}(a_n) \rightarrow \infty. \end{aligned}$$

Set $\delta_n = \varepsilon_n + |a_n - b_n| = \varepsilon_n + |z_n + \rho_n \zeta_0|$. Obviously, $\delta_n \rightarrow 0$ and $\Delta(b_n, \varepsilon_n) \subset \Delta(a_n, \delta_n)$, and hence $S(\Delta(a_n, \delta_n), g) \rightarrow \infty$ as $n \rightarrow \infty$.

3. Proof of Theorem

Proof We assume that $f^{(k)}(z) - a(z)$ has at most finitely many zeros and derive a contradiction. Let $R(z) \sim cz^d$ as $z \rightarrow \infty$, where $c \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{Z}$.

Set $g(z) = \frac{f(z)}{z^d e^z}$. By Lemma 4, there exists a sequence $a_n \rightarrow \infty$ and $\delta_n \rightarrow 0$ such that

$$S(\Delta(a_n, \delta_n), g) \rightarrow \infty \text{ as } n \rightarrow \infty \tag{1.3}$$

and

$$\frac{f(a_n)}{a_n^d e^{a_n}} \rightarrow 0 \text{ and } \frac{f^{(k)}(a_n)}{a_n^d e^{a_n}} \rightarrow \infty \text{ as } n \rightarrow \infty. \tag{1.4}$$

Set $f_n(z) = \frac{f(z + a_n)}{a_n^d e^{a_n}}$. By (1.4),

$$f_n(0) = \frac{f(a_n)}{a_n^d e^{a_n}} \rightarrow 0 \text{ and } f_n^{(k)}(0) = \frac{f^{(k)}(a_n)}{a_n^d e^{a_n}} \rightarrow \infty \text{ as } n \rightarrow \infty. \tag{1.5}$$

Hence, no subsequence of $\{f_n\}$ is normal at $z = 0$.

Since $f^{(k)}(z) - a(z)$ has at most finitely many zeros, we have for sufficiently large n ,

$$f_n^{(k)}(z) \neq \frac{P(z+a_n)e^{z+a_n}}{a_n^d e^{a_n}} \quad \text{for all } z \in \Delta.$$

Observing that

$$\frac{P(z+a_n)e^{z+a_n}}{a_n^d e^{a_n}} \Rightarrow ce^z \quad \text{as } n \rightarrow \infty$$

in Δ . It follows from Lemma 1 (applied to $h_n(z) = \frac{P(z+a_n)e^{z+a_n}}{a_n^d e^{a_n}}$ in Δ), and there exists $\delta^* \in (0,1)$ such that for all n

$$S(\Delta(0, \delta^*), f_n) \leq k+1. \tag{1.6}$$

Set $g_n(z) = g(z+a_n) = f_n(z) \left(1 + \frac{z}{a_n}\right)^{-d} e^{-z}$. Then

$$g_n^\#(z) = \frac{\left| \left(1 + \frac{z}{a_n}\right)^d e^z f_n'(z) - \left(1 + \frac{z}{a_n}\right)^d e^z \left(1 + \frac{d}{a_n+z}\right) f_n(z) \right|}{\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2 + |f_n(z)|^2}$$

and hence

$$[g_n^\#(z)]^2 \leq \frac{2 \left| \left(1 + \frac{z}{a_n}\right)^d e^z f_n'(z) \right|^2}{\left(\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2 + |f_n(z)|^2 \right)^2} + \frac{2 \left| \left(1 + \frac{z}{a_n}\right)^d e^z \left(1 + \frac{d}{a_n+z}\right) f_n(z) \right|^2}{\left(\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2 + |f_n(z)|^2 \right)^2}. \tag{1.7}$$

Using the simple inequality

$$\frac{C}{C^2+x^2} \leq 2 \max(C, 1/C) \frac{1}{1+x^2}$$

for $C > 0$, we have

$$\frac{2 \left| \left(1 + \frac{z}{a_n}\right)^d e^z f_n'(z) \right|^2}{\left(\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2 + |f_n(z)|^2 \right)^2} \leq 2 \max \left(\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2, \frac{1}{\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2} \right) [f_n^\#(z)]^2. \tag{1.8}$$

The second term on the right of (1.7) is

$$\frac{1}{2} \left| 1 + \frac{d}{a_n+z} \right|^2 \frac{\left| \left(1 + \frac{z}{a_n}\right)^d e^z f_n(z) \right|^2}{\left(\left| \left(1 + \frac{z}{a_n}\right)^d e^z \right|^2 + |f_n(z)|^2 \right)^2} \leq \frac{1}{2} \left| 1 + \frac{d}{a_n+z} \right|^2. \tag{1.9}$$

Putting (1.7), (1.8), and (1.9) together, we have for $|z| < \delta^*$ and sufficiently large n ,

$$\left[g_n^\#(z) \right]^2 \leq 2 \cdot (2e)^2 \cdot \left[f_n^\#(z) \right]^2 + \frac{1}{2} \times 2^2. \quad (1.10)$$

It follows from (1.1), (1.6), and (1.10),

$$S\left(\Delta(0, \delta^*), g_n\right) \leq 2 \cdot (2e)^2 (k+1) + \frac{1}{2} \times 2^2 = M_2.$$

Thus,

$$S\left(\Delta(a_n, \delta^*), g\right) = S\left(\Delta(0, \delta^*), g_n\right) \leq M_2$$

which contradicts (1.3).

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