

# Time-Optimal Control Problem for $n \times n$ Co-Operative Parabolic Systems with Control in Initial Conditions

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## ABSTRACT

In this paper, time-optimal control problem for a linear  $n \times n$  co-operative parabolic system involving Laplace operator is considered. This problem is, steering an initial state  $y(0) = u$ , with control  $u$ , so that an observation  $y(t)$  hits a given target set in minimum time. First, the existence and uniqueness of solutions of such system under conditions on the coefficients are proved. Afterwards necessary and sufficient conditions of optimality are obtained. Finally a scalar case is given.

**Keywords:** Time-Optimal Control Problems; Bang-Bang Controls; Parabolic System; Co-Operative Systems

## 1. Introduction

The “time optimal” control problem is one of the most important problems in the field of control theory. The simple version is that steering the initial state  $y_0$  in a Hilbert space  $H$  to hit a target set  $K \subset H$  in minimum time, with control subject to constraints ( $u \in U \subset H$ ).

In this paper, we will focus our attention on some special aspects of minimum time problems for co-operative parabolic system involving Laplace operator with control acts in the initial conditions. In order to explain the results we have in mind, it is convenient to consider the abstract form.

Let  $V$  and  $H$  be two real Hilbert spaces such that  $V$  is a dense subspace of  $H$ . Identifying the dual of  $H$  with  $H$ , we may consider  $V \subset H \subset V'$ , where the embedding is dense in the following space. Let  $A(t)$  ( $t \in ]0, T[$ ) be a family of continuous operators associated with a bilinear forms  $\pi(t, \cdot, \cdot)$  defined on  $V \times V$  which are satisfied with Gårding’s inequality

$$\pi(t, y, y) + c_0 \|y\|_H^2 \geq c_1 \|y\|_V^2 \quad (1)$$

for  $y \in V, t \in [0, T]$  and  $c_0 \geq 0, c_1 > 0$ .

Then, from [1] and [2], for  $t \in ]0, T[$  and  $B$  being a bounded linear operator on  $H$ , the following abstract

systems:

$$\left. \begin{aligned} \frac{d}{dt} y(t) + A(t) y(t) &= f, f \in L^2(0, T; V'), \\ y(0) &= Bu \end{aligned} \right\} \quad (2)$$

have a unique weak solution  $y$  such that  $y \in C([0, T]; H)$ . We shall denote by  $y(t; u)$  the unique solution of the Equation (2) corresponding to the control  $u$ . The time optimal control problem that we shall concern reads:

$$\min \{ \tau : y(\tau; u) \in K, u \in U \} \quad (3)$$

where  $K$  is a given subset of  $H$ , which is called the target set of the Problem (3). A control  $u^0$  is called a time optimal control if  $u^0 \in U$  and if there is a number  $\tau^0 > 0$  such that  $y(\tau^0; u^0) \in K$  and

$$\tau^0 = \min \{ \tau : y(\tau; u) \in K, u \in U \}. \quad (4)$$

We call the number  $\tau^0$  as the optimal time for the time optimal control Problem (3).

Three questions (problems) arise naturally in connection with this problem:

1) Is there a control  $u$ , and  $\tau > 0$  such that  $y(\tau; u) \in K$  ? (this is an approximate controllability problem).

2) Assume that the answer to 1) is in the affirmative and

$$\tau^0 = \min \{ \tau : y(\tau; u) \in K, u \in U \}.$$

Is there a control  $u^0$  which steers  $y(\tau^0)$  to hit a target set  $K$  in minimum time?

3) If  $u^0$  exists, is it unique? What additional properties does it have?

Let  $\Omega \subset R^N$  be a bounded open domain with smooth boundary  $\Gamma$ . and set  $Q = \Omega \times ]0, T[$   $\Sigma = \Gamma \times ]0, T[$ . In the works [1] and [3], the existence of time optimal controls of the following controlled linear parabolic equations with distributed control  $u$  was obtained:

$$\left. \begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + u && \text{in } Q, \\ y(x, 0) &= y_0(x) && \text{in } \Omega, \\ y(x, t) &= 0 && \text{on } \Sigma, \end{aligned} \right\} \quad (5)$$

where  $y_0(x)$  is a given function in  $L^2(\Omega)$ ,  $u \in U$  and  $U$  is a closed bounded set in  $L^2(\Omega)$ . The results in [3] partly overlap with results in [1] and they were shown that if the system (5) is controllable and if  $K = \{0\}$  then the corresponding time optimal control problem has at least one solution and it is bangbang.

In the work [4], the authors gave a sufficient and necessary condition for the existence of time optimal control for the problem with the target set  $K = \{0\}$  and certain controlled systems. These results will be stated as follows. Consider the following controlled system

$$\left. \begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + ay + u && \text{in } Q, \\ y(x, 0) &= y_0(x) && \text{in } \Omega, \\ y(x, t) &= 0 && \text{on } \Sigma, \end{aligned} \right\} \quad (6)$$

where  $a$  is a real number. Let  $\{\lambda_i\}_{i \geq 1}$ ,  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ , be the eigenvalues of  $-\Delta$  with the Dirichlet boundary condition and  $\{e_i\}_{i \geq 1}$  be the corresponding eigenfunctions, which forms an orthogonal basis of  $L^2(\Omega)$ . We take the target set  $K$  to be the origin  $\{0\}$  in  $L^2(\Omega)$  and the control set  $U$  to be the set

$$U_\varepsilon = \left\{ u(., t) \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \leq \varepsilon \right\}$$

where  $\varepsilon$  is a positive number, namely,  $U_\varepsilon = B(0, \varepsilon)$ , the closed ball in  $L^2(\Omega)$  centered at 0 and of radius  $\varepsilon$ . It was proved that if  $K = \{0\}$  and  $U = U_\varepsilon$ , then the corresponding time optimal control problem has at least one solution if and only if  $a \leq \lambda_1$ .

More early, in the works [5-7], the time optimal controls problem for globally controlled linear and semi-linear parabolic equations was considered.

In our papers [8,9], the time optimal control problem of  $n \times n$  co-operative hyperbolic systems with different cases of the observation and distributed or boundary controls constraints was considered.

In [10], optimal control of infinite order hyperbolic equation with control via initial conditions was considered.

In the present paper, the above results for the time optimal control of systems governed by parabolic equations are extended to the case of  $n \times n$  co-operative parabolic systems as well as control via initial conditions. First, the existence and uniqueness of solutions for  $n \times n$  co-operative parabolic system are proved under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem, then the time optimal control problem is formulated and the existence of a time optimal control is proved. Then the necessary and sufficient conditions which the optimal controls must satisfy are derived in terms of the adjoint. Finally, the scalar case is given.

## 2. $n \times n$ Co-Operative Parabolic Systems

Let  $H_0^1(\Omega)$ , be the usual Sobolev space of order one which consists of all  $\phi \in L^2(\Omega)$  whose distributional derivatives  $\frac{\partial \phi}{\partial x_i} \in L^2(\Omega)$  and  $\phi|_\Gamma = 0$  with the scalar product norm

$$\begin{aligned} \langle y, \phi \rangle_{H_0^1(\Omega)} &= \langle y, \phi \rangle_{L^2(\Omega)} + \langle \nabla y, \nabla \phi \rangle_{L^2(\Omega)}, \\ \text{where } \nabla &= \sum_{k=1}^N \frac{\partial}{\partial x_k}. \end{aligned}$$

We have the following dense embedding chain [11]

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H_0^{-1}(\Omega))^n.$$

where  $H_0^{-1}(\Omega)$  is the dual of  $H_0^1(\Omega)$ .

Here and everywhere below the vectors are denoted by bold letters. For  $\mathbf{y} = (y_i)_{i=1}^n$ ,  $\boldsymbol{\phi} = (\phi_i)_{i=1}^n \in (H_0^1(\Omega))^n$  and  $t \in ]0, T[$ , let us define a family of continues bilinear forms

$$\pi(t, \cdot, \cdot) : (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \rightarrow \mathfrak{R} \text{ by}$$

$$\begin{aligned} \pi(t, \mathbf{y}, \boldsymbol{\phi}) &= \sum_{i=1}^n \int_{\Omega} [(\nabla y_i)(\nabla \phi_i) - a_i(x, t) y_i \phi_i] dx \\ &\quad - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t) y_j \phi_i dx \end{aligned} \quad (7)$$

where

$$\left. \begin{aligned} a_i(x, t) \text{ and } a_{ij}(x, t) &\text{ are positive functions in } L^\infty(Q), \\ a_{ij} &= 0 \text{ when } i = j \text{ and } a_{ij} \leq \sqrt{a_i a_j} \text{ when } i \neq j \end{aligned} \right\} \quad (8)$$

The bilinear form (7) can be but in the operator form:

$$\begin{aligned} \pi(t; \mathbf{y}, \boldsymbol{\phi}) &= \sum_{i=1}^n \int_{\Omega} [(-\Delta y_i) - a_i(x, t) y_i] \phi_i dx \\ &\quad - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t) y_j \phi_i dx \\ &= \sum_{i=1}^n \langle -(A(t) \mathbf{y})_i, \phi_i \rangle_{L^2(\Omega)} \end{aligned}$$

where  $A(t)$  is  $n \times n$  matrix operator which maps  $(H_0^1(\Omega))^n$  onto  $(H_0^{-1}(\Omega))^n$  and takes the form

$$A(t) \mathbf{y} = \begin{pmatrix} \Delta + a_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & \Delta + a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

**Lemma 2.1.** If  $\Omega$  is a regular bounded domain in  $R^N$ , with boundary  $\Gamma$ , and if  $m$  is positive on  $\Omega$  and smooth enough ( in particular  $m \in L^\infty(\Omega)$ , ) then the eigenvalue problem:

$$\begin{cases} -\Delta y = \lambda m(x) y & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \end{cases}$$

possesses an infinite sequence of positive eigenvalues:

$$\begin{aligned} 0 < \lambda_1(m) < \lambda_2(m) \leq \cdots \leq \lambda_k(m) \leq \cdots; \\ \lambda_k(m) &\rightarrow \infty, \text{ as } k \rightarrow \infty. \end{aligned}$$

Moreover  $\lambda_1(m)$  is simple, its associate eigenfunction  $e_m$  is positive, and  $\lambda_1(m)$  is characterized by:

$$\lambda_1(m) \int_{\Omega} m y^2 dx \leq \int_{\Omega} |\nabla y|^2 dx \tag{9}$$

**Proof.** See [12].  $\square$ .

Now, let

$$\lambda_1(a_i) \geq n, \quad i = 1, 2, \dots, n \tag{10}$$

**Lemma 2.2.** If (8) and (10) hold then, the bilinear form (7) satisfy the Gårding inequality

$$\begin{aligned} \pi(t; \mathbf{y}, \mathbf{y}) + c_0 \|y\|_{(L^2(\Omega))^n}^2 &\geq c_1 \|y\|_{(H_0^1(\Omega))^n}^2, \\ c_0, c_1 &> 0 \end{aligned}$$

**Proof.** In fact

$$\begin{aligned} \pi(t; \mathbf{y}, \mathbf{y}) &= \sum_{i=1}^n \int_{\Omega} [|\nabla y_i|^2 - a_i(x, t) y_i^2] dx \\ &\quad - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t) y_i y_j dx \\ &\geq \sum_{i=1}^n \int_{\Omega} [|\nabla y_i|^2 - a_i(x, t) y_i^2] dx \\ &\quad - 2 \sum_{i>j}^n \int_{\Omega} \sqrt{a_i(x, t) a_j(x, t)} y_i y_j dx \end{aligned}$$

By Cauchy Schwarz inequality and (9), we obtain

$$\begin{aligned} \pi(t; \mathbf{y}, \mathbf{y}) &\geq \sum_{i=1}^n \left(1 - \frac{1}{\lambda_1(a_i)}\right) \int_{\Omega} |\nabla y_i|^2 dx \\ &\quad - 2 \sum_{i>j}^n \int_{\Omega} \frac{1}{\sqrt{\lambda_1(a_i) \lambda_1(a_j)}} \left(\int_{\Omega} |\nabla y_i|^2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla y_j|^2 dx\right)^{\frac{1}{2}} \\ &\geq \sum_{i=1}^n \left(\frac{\lambda_1(a_i) - n}{\lambda_1(a_i)}\right) \int_{\Omega} |\nabla y_i|^2 dx \end{aligned}$$

From (10) we have

$$\pi(t; \mathbf{y}, \mathbf{y}) \geq \alpha \left[ \sum_{i=1}^n \int_{\Omega} |\nabla y_i|^2 dx \right] \quad \alpha > 0$$

Add  $\|y\|_{(L^2(\Omega))^n}$  to two sides, then we have the result.  $\square$ .

We can now apply Theorem 1.1 and Theorem 1.2 Chapter 3 in [1] (with  $V = (H_0^1(\Omega))^n$  and  $H = (L^2(\Omega))^n$ ) to obtain the following theorem:

**Theorem 2.3.** If (8) and (10) hold, then there exist a unique solution

$$y \in W(0, T)$$

$$= \left\{ \mathbf{y} : \mathbf{y} \in L^2(0, T; (H_0^1(\Omega))^n), \frac{\partial \mathbf{y}}{\partial t} \in (H_0^1(\Omega))^n \right\}$$

satisfying the following  $n \times n$  system:  $i = 1, 2, \dots, n$

$$\left. \begin{aligned} \frac{\partial y_i}{\partial t} &= (A(t) \mathbf{y})_i + f_i, \quad f_i \in L^2(0, T; H_0^{-1}(\Omega)) \text{ in } \mathcal{Q}, \\ y_i(x, 0) &= u_i(x), \quad u_i(x) \in L^2(\Omega) \quad \text{in } \Omega, \\ y_i(x, t) &= 0 \quad \text{on } \Sigma. \end{aligned} \right\} \tag{11}$$

Moreover  $\mathbf{y}$  is continuous from  $[0, T] \rightarrow (L^2(\Omega))^n$ .

### 3. Minimum Time and Controllability

We denote the unique solution of (11), at time  $t$  for each control  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  by  $\mathbf{y}(t; \mathbf{u})$ . Occasionally, we write  $\mathbf{y}(x, t; \mathbf{u})$  when the explicit dependence on  $x$  is required. We can now formulate the time optimal control problem corresponding to the  $n \times n$  cooperative parabolic system (11):

$$\min \{t : \mathbf{y}(t; \mathbf{u}) \in K_\varepsilon^n, \mathbf{u} \in U_\varepsilon^n\}, \tag{12}$$

with constraints

$$\left. \begin{aligned} &\mathbf{y}(t; \mathbf{u}) \text{ is the solution of (11),} \\ U_\varepsilon^n &= \left\{ \mathbf{u} = (u_1, \dots, u_n) \in (L^2(\Omega))^n : \|u_i\|_{L^2(\Omega)} \leq \varepsilon \right\}, \\ K_\varepsilon^n &= \left\{ \mathbf{z} = (z_1, \dots, z_n) \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} \leq \varepsilon \right\}, \end{aligned} \right\} \tag{13}$$

and  $\varepsilon, \mathcal{E} > 0$  and  $z_{id} \in L^2(\Omega)$  are given.

**Theorem 3.1.** If (8) and (10) are hold, then the system whose state is given by (11) is controllable, *i.e.*,

there exists a  $\tau \in ]0, T]$  and  $\mathbf{u} \in U_\varepsilon^n$  with  $\mathbf{y}(\tau; \mathbf{u}) \in K_\varepsilon^n$  (14)

**Proof.** Let us first remark that by translation we may always reduce the problem of controllability to the case were the system (11) with  $f_i = 0$ . We can show quit easily that (11) is approximately controllable in  $(L^2(\Omega))^n$  in any finite time  $\tau > 0$ , if and only if,  $\{\mathbf{y}(\tau; \mathbf{u}) : \mathbf{u} \in (L^2(\Omega))^n\}$  is dense in  $(L^2(\Omega))^n$ . By the Hahn-Banach theorem, this will be the case if

$$\int_{\Omega} \bar{z}_i(x) y_i(x, \tau; \mathbf{u}) dx = 0, \quad \bar{z}_i \in L^2(\Omega), \quad (15)$$

for all  $\mathbf{u} \in (L^2(\Omega))^n$ , implies that  $\bar{z}_i(x) = 0, \quad i = 1, 2, \dots, n$ .

Let us introduce the adjoint state  $p(t; \mathbf{u})$  by the solution of the following system

$$\left. \begin{aligned} -\frac{\partial p_i}{\partial t}(t; \mathbf{u}) - (A^*(t) \mathbf{p}(t; \mathbf{u}))_i &= 0 && \text{in } \Omega \times ]0, \tau[, \\ p_i(x, \tau) &= \bar{z}_i(x) && \text{in } \Omega, \\ p_i(x, t) &= 0 && \text{on } \Gamma \times ]0, \tau[, \end{aligned} \right\} (16)$$

where  $A^*(t)$  is the adjoint of  $A(t)$  which is defined by

$$\langle A^*(t) \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, A(t) \boldsymbol{\psi} \rangle, \quad \boldsymbol{\phi}, \boldsymbol{\psi} \in (H_0^1(\Omega))^n.$$

The existence of a unique solution for the Problem (16) can be proved using Theorem 2.3, with an obvious change of variables.

Multiply the first equation in (16) by  $y_i(t; \mathbf{u})$  and integrate by parts from 0 to  $\tau$ , we obtain the following identity:

$$\begin{aligned} 0 &= \int_0^\tau \int_{\Omega} \left[ -\frac{\partial p_i}{\partial t} - (A^*(t) \mathbf{p}(t; \mathbf{u}))_i \right] y_i(t; \mathbf{u}) dx dt \\ &= - \int_{\Omega} p_i(t; \mathbf{u}) y_i(t; \mathbf{u}) \Big|_0^\tau dx \\ &\quad + \int_0^\tau \int_{\Omega} p_i(t; \mathbf{u}) \left[ \frac{\partial}{\partial t} y_i(t; \mathbf{u}) - (A(t) \mathbf{y}(t; \mathbf{u}))_i \right] dx dt \\ &= \int_{\Omega} \bar{z}_i(x) y_i(x, \tau; \mathbf{u}) dx - \int_{\Omega} p_i(0; \mathbf{u}) u_i dx. \end{aligned}$$

and so, if (15) holds, then

$$\int_{\Omega} p_i(x, 0; \mathbf{u}) u_i dx = 0 \quad \forall u_i \in L^2(\Omega)$$

hence  $p_i(x, 0; \mathbf{u}) = 0$ . But from the backward uniqueness property,  $\mathbf{p} = (p_i)_{i=1}^n \equiv 0$  and hence  $\bar{z}_i(x) = 0$ .  $\square$ .

Now set

$$\tau^0 = \inf \{ \tau : \mathbf{y}(\tau; \mathbf{u}) \in K_\varepsilon^n \text{ fore some } \mathbf{u} \in U_\varepsilon^n \}. \quad (17)$$

Then, the following result holds.

**Theorem 3.2.** If (8) and (10) are hold, then there exist an admissible control  $\mathbf{u}^0$  to the problem (12)-(17), which steering  $\mathbf{y}(t; \mathbf{u}^0)$  to hitting a target set  $K_\varepsilon^n$  in

minimum time  $\tau^0$  (defined by (17)). Moreover

$$\sum_{i=1}^n \int_{\Omega} (y_i(\tau^0; \mathbf{u}^0) - z_{id})(y_i(\tau^0; \mathbf{u}) - y_i(\tau^0; \mathbf{u}^0)) dx \geq 0 \quad (18)$$

$$\forall \mathbf{u} \in U_\varepsilon^n$$

**Proof.** Fixe  $x$ , we can choose  $\tau^m \rightarrow \tau^0$  and admissible controls  $\{\mathbf{u}^m\}$  such that

$$\mathbf{y}(\tau^m; \mathbf{u}^m) \in K_\varepsilon^n, \quad m = 1, 2, \dots.$$

Set  $\mathbf{y}^m = \mathbf{y}(\mathbf{u}^m)$ . Since  $U_\varepsilon^n$  is bounded, we may verify that  $\mathbf{y}^m$  ranges in a bounded set in

$$\left( L^2\left(0, T; (L^2(\Omega))^n\right) = (L^2(Q))^n \right).$$

We may then extract a subsequence, again denoted by  $\{\mathbf{u}^m, \mathbf{y}^m\}$  such that

$$\left. \begin{aligned} \mathbf{u}^m &\rightarrow \mathbf{u}^0 \text{ weakly in } (L^2(\Omega))^n, \quad \mathbf{u}^0 \in U_\varepsilon^n, \\ \mathbf{y}^m &\rightarrow \mathbf{y} \text{ weakly in } L^2\left(0, T; (H_0^1(\Omega))^n\right). \end{aligned} \right\} (19)$$

We deduce from the equality

$$\frac{d\mathbf{y}^m}{dt} = \mathbf{f} - A(t) \mathbf{y}^m$$

that

$$\frac{d\mathbf{y}^m}{dt} \rightarrow \frac{d\mathbf{y}}{dt} = \mathbf{f} - A(t) \mathbf{y} \quad \text{in } L^2\left(0, T; (H^{-1}(\Omega))^n\right),$$

and

$$\mathbf{y}^m(0) \rightarrow \mathbf{y}(0) = \mathbf{u}^0 \quad \text{in } U_\varepsilon^n.$$

But

$$\begin{aligned} &\mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau^0; \mathbf{u}^0) \\ &= \mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau^0; \mathbf{u}^m) + \mathbf{y}(\tau^0; \mathbf{u}^m) - \mathbf{y}(\tau^0; \mathbf{u}^0) \end{aligned}$$

Now from (19)

$$\mathbf{y}(\tau^0; \mathbf{u}^m) \rightarrow \mathbf{y}(\tau^0; \mathbf{u}^0) \text{ weakly in } (L^2(\Omega))^n \quad (20)$$

and

$$\begin{aligned} &\left\| \mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau^0; \mathbf{u}^m) \right\|_{(H^{-1}(\Omega))^n} \\ &= \left\| \int_{\tau^0}^{\tau^m} \frac{d}{dt} \mathbf{y}(t; \mathbf{u}^m) dt \right\|_{(H^{-1}(\Omega))^n} \\ &\leq \sqrt{\tau^m - \tau^0} \left( \int_{\tau^0}^{\tau^m} \left\| \frac{d}{dt} \mathbf{y}(t; \mathbf{u}^m) \right\|_{(H^{-1}(\Omega))^n}^2 dt \right)^{\frac{1}{2}} \\ &\leq c \sqrt{\tau_n - \tau^0} \end{aligned} \quad (21)$$

Combine (20) and (21) show that

$$\mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau^0; \mathbf{u}^0) \rightarrow 0 \text{ weakly in } (H_0^{-1}(\Omega))^n. \quad (22)$$

and so,  $y(\tau^0; \mathbf{u}^0) \in K_\varepsilon^n$  as  $K_\varepsilon^n$  is closed and convex, hence weakly closed. This shows that  $K_\varepsilon^n$  is reached in time  $\tau^0$  by admissible control  $\mathbf{u}^0$ .

For the second part of the theorem, really, from Theorem 2.3, the mapping  $t \rightarrow y(t; \mathbf{u})$  from  $[0, T] \rightarrow (L^2(\Omega))^n$  is continuous for each fixed  $\mathbf{u}$  and so  $y(\tau^0; \mathbf{u}) \notin \text{int } K_\varepsilon^n$ , for any  $\mathbf{u} \in U_\varepsilon^n$ , by minimality of  $\tau^0$ .

Using Theorem 2.3, it is easy to verify that the mapping  $\mathbf{u} \rightarrow y(\tau^0; \mathbf{u})$ , from  $(L^2(\Omega))^n \rightarrow (L^2(\Omega))^n$ , is continuous and linear. then, the set

$$\mathcal{A}(\tau^0) = \{y(\tau^0; \mathbf{u}) : \mathbf{u} \in U_\varepsilon^n\}$$

is the image under a linear mapping of a convex set hence  $\mathcal{A}(\tau^0)$  is convex. Thus we have

$\mathcal{A}(\tau^0) \cap \text{int } K_\varepsilon^n = \emptyset$  and  $y(\tau^0; \mathbf{u}^0) \in \partial K_\varepsilon^n$  (boundary of  $K_\varepsilon^n$ ). Since  $\text{int } K_\varepsilon^n \neq \emptyset$  (from (14)) so there exists a closed hyperplane separating  $\mathcal{A}(\tau^0)$  and  $K_\varepsilon^n$  containing  $y(\tau^0; \mathbf{u}^0)$ , i.e. there is a nonzero  $\mathbf{g} \in (L^2(\Omega))^n$  such as

$$\int_\Omega \langle \mathbf{g}, y(\tau^0; \mathbf{u}) \rangle_{(L^2(\Omega))^n} \leq \langle \mathbf{g}, y(\tau^0; \mathbf{u}^0) \rangle_{(L^2(\Omega))^n} \tag{23}$$

$$\leq \inf_{y \in K_\varepsilon^n} \langle \mathbf{g}, y(\tau^0; \mathbf{u}) \rangle_{(L^2(\Omega))^n}$$

From the second inequality in (23),  $\mathbf{g}$  must support the set  $K_\varepsilon^n$  at  $y(\tau^0; \mathbf{u}^0)$  i.e.

$$\langle \mathbf{g}, (y(\tau^0; \mathbf{u}) - y(\tau^0; \mathbf{y}^0)) \rangle_{(L^2(\Omega))^n} \geq 0$$

$$\forall \mathbf{u} \in U_\varepsilon^n$$

and since  $(L^2(\Omega))^n$  is a Hilbert space,  $\mathbf{g}$  must be of the form

$$\mathbf{g} = \lambda (y(\tau^0; \mathbf{u}^0) - z_{id})$$

for some  $\lambda > 0$ .

Dividing the inequality (23) by  $\lambda$  gives the desired result.  $\square$ .

Now Inequality (18) can be interpreted as follows: let us introduce the adjoint state  $p(t; \mathbf{u}^0)$  by the solution of the following system

$$\left. \begin{aligned} -\frac{\partial p_i}{\partial t}(t; \mathbf{u}^0) + (A^*(t)p(t; \mathbf{u}^0))_i &= 0 && \text{in } \Omega \times ]0, \tau^0[ , \\ p_i(x, \tau^0) &= (y_i(x, \tau^0) - z_{id}) && \text{in } \Omega, \\ p_i(x, t) &= 0 && \text{on } \Gamma \times ]0, \tau^0[ . \end{aligned} \right\} \tag{24}$$

As the proof of Theorem 3.1, we multiply the first equation int (24) by  $y_i(t; \mathbf{u}) - y_i(t; \mathbf{u}^0)$  and integrate by parts from 0 to  $\tau^0$ , we obtain the following identity:

$$\int_\Omega (y_i(\tau^0; \mathbf{u}^0) - z_{id})(y_i(x, \tau^0; \mathbf{u}) - y_i(x, \tau^0; \mathbf{u}^0)) dx = \int_\Omega p_i(0; \mathbf{u}^0)(\mathbf{u} - \mathbf{u}^0) dx.$$

hence condition (18) becomes

$$\sum_{i=1}^n \int_\Omega p_i(x, 0; \mathbf{u}^0)(\mathbf{u} - \mathbf{u}^0) dx \geq 0 \tag{25}$$

$$\forall \mathbf{u} \in U_\varepsilon^n.$$

Using controllability condition (14), the backward uniqueness property implies  $p_i(x, 0; \mathbf{u}^0) = 0$ . then the optimal control is bang-bang, i.e.,  $\|u_i^0\|_{L^2(\Omega)} = \varepsilon$  and since  $U_\varepsilon^n$  is strictly convex, then the optimal control is unique. We have thus proved:

**Theorem 3.3.** If (8) and (10) are hold, then there exist the adjoint state

$$p \in W(0, \tau^0) = \left\{ p : p \in L^2(0, \tau^0; (H_0^1(\Omega))^n), \frac{\partial p}{\partial t} \in (H_0^{-1}(\Omega))^n \right\}$$

such that the optimal control  $\mathbf{u}^0$  of problem (12)-(17) is bang-bang unique and it is determined by (24), (25) together with (11) (with  $u_i = u_i^0, i = 1, 2, \dots, n$ ).

### 4. Scaler Case

Here, we take the case where  $n = 2$ , in this case, the time optimal problem therefore is

$$\min \{t : y(x, t; \mathbf{u}) \in K_\varepsilon^2, \mathbf{u} = (u_1, u_2) \in U_\varepsilon^2\}$$

The state  $y = (y_1, y_2)$  is solution of the following equations

$$\left. \begin{aligned} \frac{\partial y_1}{\partial t} - \Delta y_1 &= a_{11}(x, t)y_1 + a_{12}(x, t)y_2 \\ &+ f_1, \quad x \in \Omega, t \in ]0, \tau^0[ , \\ \frac{\partial y_2}{\partial t} - \Delta y_2 &= a_{21}(x, t)y_1 + a_{22}(x, t)y_2 \\ &+ f_2, \quad x \in \Omega, t \in ]0, \tau^0[ , \\ y_1(x, 0) &= u_1^0(x), \quad y_2(x, 0) = u_2^0(x), \\ x \in \Omega, \quad y_1(x, t) &= y_2(x, t) = 0, \\ x \in \Gamma, t \in ]0, \tau^0[ , \end{aligned} \right\}$$

with

$$\left. \begin{aligned} a_{ij}(x, t), i, j = 1, 2 &\text{ are positive functions in } L^\infty(Q), \\ \lambda_1(a_{11}) \geq 2, \quad \lambda_1(a_{22}) &\geq 2. \end{aligned} \right\}$$

The adjoint is solution of the following equations

$$\left. \begin{aligned} -\frac{\partial p_1}{\partial t} - \Delta p_1 &= a_{22}(x,t)p_1 - a_{12}(x,t)p_2 \\ &\quad + f_1, \quad x \in \Omega, t \in ]0, \tau^0[, \\ -\frac{\partial p_2}{\partial t} - \Delta p_2 &= -a_{21}(x,t)p_1 + a_{11}(x,t)p_2 \\ &\quad + f_2, \quad x \in \Omega, t \in ]0, \tau^0[, \\ p_1(x, \tau^0) &= (y_1(x, \tau^0) - z_{1d}), \quad x \in \Omega, \\ p_2(x, \tau^0) &= (y_2(x, \tau^0) - z_{2d}), \quad x \in \Omega, \\ p_1(x, t) = p_2(x, t) &= 0, \quad x \in \Gamma, t \in ]0, \tau^0[. \end{aligned} \right\}$$

The maximum condition is

$$\int_0^{\tau^0} \int_{\Omega} [p_1(x, 0)(u_1 - u_1^0) + p_2(x, 0)(u_2 - u_2^0)] dx dt \geq 0$$

$$\forall \mathbf{u} = (u_1, u_2) \in U_{\varepsilon}^2.$$

## 5. Comments

We note that, in this paper, we have chosen to treat a special systems involving Laplace operator just for simplicity. Most of the results we described in this paper apply without any change on the results to more general parabolic systems involving the following second order operator:

$$L(x, \cdot) = \sum_{i,j=1}^n b_{ij}(x, \cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x, \cdot) \frac{\partial}{\partial x_j} + b_0(x, \cdot)$$

with sufficiently smooth coefficients (in particular,  $b_{ij}, b_j, b_0 \in L^{\infty}(Q), b_j, b_0 > 0$ ) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^n \eta_i \eta_j \geq \sigma \sum_{i=1}^n \eta_i^2 \quad \forall (x, t) \in Q,$$

for all  $\eta_i \in \mathfrak{R}$  and some constants  $\sigma > 0$ .

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator  $L$  (see [12]).

In this paper, we have chosen to treat a co-operative parabolic system with Dirichlet boundary conditions. The results can be extended to the case of  $n \times n$  co-operative parabolic system with Neumann boundary conditions: if we take  $H^1(\Omega)$  instead of  $H_0^1(\Omega)$ , we have to replace the Dirichlet boundary conditions  $y_i = 0, p_i = 0$  on the boundary by Neumann boundary conditions  $\frac{\partial y_i}{\partial \nu} = 0, \frac{\partial p_i}{\partial \nu} = 0$ , where  $\nu$  is the outward normal.

The results in this paper carry over to the fixed-time problem ([1] chapter 3).

$$\text{minimize } \sum_{i=1}^n \int_{\Omega} |y_i(x, T; \mathbf{u}) - z_{id}(x)|^2 dx, \quad T \text{ fixed,}$$

subject to (11) [except in the trivial case where  $z_{id}(x) = y_i(x, T; \mathbf{u}) \forall i = 1, 2, \dots, n$  for some admissible control  $\mathbf{u}$ ]. This can be proven in an analogous manner, as the necessary and sufficient conditions for optimality for this problem coincide with (11), (16) and (25) (with  $u_i = u_i^0, i = 1, 2, \dots, n$ ).

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