

Nil 3-Armendariz Rings

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ABSTRACT

We introduce nil 3-Armendariz rings, which are generalization of 3-Armendariz rings and nil Armendariz rings and investigate their properties. We show that a ring R is nil 3-Armendariz ring if and only if for any $n \in \mathbb{N}$, $T_n(R)$ is nil 3-Armendariz ring. Also we prove that a right Ore ring R is nil 3-Armendariz if and only if so is Q , where Q is the classical right quotient ring of R . With the help of this result, we can show that a commutative ring R is nil 3-Armendariz if and only if the total quotient ring of R is nil 3-Armendariz.

Keywords: Armendariz Ring; 3-Armendariz Ring; Nil Armendariz Ring; Nil 3-Armendariz Ring

1. Introduction

Throughout this article, R denotes an associative ring, not necessary with identity. Given a ring R , the polynomial ring over R is denoted by $R[x]$. The study of Armendariz ring was initiated by Armendariz [1] and Rege and Chhawchharia [2]. A ring R is called Armendariz if whenever polynomials

$$f(x) = a_0 + a_1x + \cdots + a_nx^n,$$

$$g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \text{ satisfy}$$

$f(x)g(x) = 0$, then $a_ib_j = 0$, for all i, j . (The converse is always true.) Some properties of Armendariz rings have been studied in Rege and Chhawchharia [2], Anderson and Camillo [3], Kim and Lee [4], Huh *et al.* [5], and Lee and Wong [6]. Suiyi [7] introduced the notion of 3-Armendariz ring. A ring R is called a 3-Armendariz if whenever polynomials

$$f(x) = a_0 + a_1x + \cdots + a_nx^n,$$

$$g(x) = b_0 + b_1x + \cdots + b_mx^m,$$

$$h(x) = c_0 + c_1x + \cdots + c_rx^r \in R[x] \text{ satisfy}$$

$f(x)g(x)h(x) = 0$, then $a_ib_jc_k = 0$, for all i, j, k . Due to Ramon Antoine [8], a ring R is said to be nil Armendariz if whenever two polynomials

$$f(x), g(x) \in R[x], \text{ satisfy } f(x)g(x) \in \text{nil}(R)[x]$$

then $ab \in \text{nil}(R)$, for all $a \in \text{coef}(f(x))$ and $b \in \text{coef}(g(x))$. There is a nil Armendariz ring but not Armendariz by [8, Example 4.11]. A ring R is called

reduced if it has no nonzero nilpotent elements. Armendariz rings are thus a generalization of reduced rings, and therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are Armendariz. In fact, in [3], Anderson and Camillo prove that if $n \geq 2$, then $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced. In [9], Liu and Zhao introduced weak Armendariz rings as a generalization of Armendariz rings. A ring is weak Armendariz if whenever the product of two polynomials is zero then the product of their coefficients is nilpotent. In [10], Wu Hui-feng introduced the concept of weak 3-Armendariz ring as a generalization of 3-Armendariz rings and weak Armendariz ring and investigated their properties. A ring is weak 3-Armendariz if whenever the product of three polynomials is zero then the product of their coefficients is nilpotent. Motivated by results in Suiyi [7], Liu and Zhao [9], Antoine [8], Kim and Lee [4], Rege and Chhawchharia [2], and Wu Hui-feng [10,11], we investigate a generalization of nil Armendariz rings and 3-Armendariz rings which we call nil 3-Armendariz rings.

2. Nil 3-Armendariz Rings

If R is a ring, $\text{nil}(R)$ denotes the set of all nilpotent elements in R , and if $f(x) \in R[x]$, $\text{coef}(f(x))$

denotes the subset of R of the coefficients of $f(x)$.

Condition (P): For all $a, b, c \in R$, if $(abc)^2 = 0$, then $abc = 0$. (See [7])

Lemma 2.1. [11, Proposition 1]. If R is a reduced ring, then R satisfies the condition (P), but the converse is not true.

Lemma 2.2. [7, Theorem 1]. If a ring R satisfies condition (P), then R is a 3-Armendariz ring.

Proposition 2.3. Let R be a ring such that $nil(R) \triangleleft R$. If $f(x)g(x)h(x) \in nil(R)[x]$, then $abc \in nil(R)$ for all $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$.

Proof. Observe that $R/nil(R)$ is reduced. By Lemma 2.1, $R/nil(R)$ satisfies condition (P) and by Lemma 2.2, $R/nil(R)$ is 3-Armendariz. Suppose $f(x)g(x)h(x) \in nil(R)[x]$. Then, if we denote by $\bar{f}(x), \bar{g}(x), \bar{h}(x)$ the corresponding polynomials in $R/nil(R)[x]$, $\bar{f}(x)\bar{g}(x)\bar{h}(x) = \bar{0}$. Since $R/nil(R)$ is 3-Armendariz, $\bar{a}\bar{b}\bar{c} = \bar{0}$ for all $\bar{a} \in coef(\bar{f}(x))$, $\bar{b} \in coef(\bar{g}(x))$, and $\bar{c} \in coef(\bar{h}(x))$. Hence abc is nil for all $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$.

Wu Hui-feng gives the following generalization of 3-Armendariz rings.

Definition 2.4. [10, Definition 1]. A ring R is said to be a weak 3-Armendariz ring if whenever polynomials $f(x), g(x), h(x) \in R[x]$, satisfy $f(x)g(x)h(x) = 0$, then $abc \in nil(R)$ for all $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$.

Clearly, 3-Armendariz rings are weak 3-Armendariz. We now present here a stronger condition, given by the property obtained in Proposition 2.3.

Definition 2.5. A ring R is said to be nil 3-Armendariz if whenever polynomials $f(x), g(x), h(x) \in R[x]$, satisfy $f(x)g(x)h(x) \in nil(R)[x]$ then $abc \in nil(R)$ for all $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$.

Observe that if $nil(R) \triangleleft R$ then by Proposition 2.3, R is nil 3-Armendariz. More generally we obtain the following.

Proposition 2.6. Let R be a ring that satisfies the condition (P), and $I \triangleleft R$ a nil ideal. Then R is nil 3-Armendariz if and only if R/I is nil 3-Armendariz.

Proof. We denote $\bar{R} = R/I$. Since I is nil, then $nil(\bar{R}) = nil(R)$. Hence $f(x)g(x)h(x) \in nil(R)[x]$ if and only if $\bar{f}(x)\bar{g}(x)\bar{h}(x) \in nil(\bar{R})[x]$. And, if $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$ then $abc \in nil(R)$ if and only if $\bar{a}\bar{b}\bar{c} \in nil(\bar{R})$. Therefore R is nil 3-Armendariz if and only if \bar{R} is nil 3-Armendariz.

The next results can be proved by using the technique used in the proof of [8, Lemma 2.5, Lemma 2.6].

Lemma 2.7. Let R be a nil 3-Armendariz ring and

$n \geq 3$. If $f_1(x), f_2(x), \dots, f_n(x) \in R[x]$ such that $f_1(x)f_2(x)\dots f_n(x) \in nil(R)[x]$, then if $a_k \in coef(f_k(x))$ for $k = 1, \dots, n$, we have $a_1a_2\dots a_n \in nil(R)$.

Lemma 2.8. If R is a 3-Armendariz ring then $nil(R)[x] \subseteq nil(R[x])$.

Proposition 2.9. If R is a 3-Armendariz ring then R is nil 3-Armendariz.

Proof. Suppose $f(x), g(x), h(x) \in R[x]$ be such that $f(x)g(x)h(x) \in nil(R)[x]$. Since R is 3-Armendariz, by Lemma 2.8, $f(x)g(x)h(x)$ is nilpotent and there exists $n \geq 1$ such that

$(f(x)g(x)h(x))^n = 0$. Hence, since R is 3-Armendariz, for all $a \in coef(f(x))$, $b \in coef(g(x))$ and $c \in coef(h(x))$ by choosing the corresponding coefficient in each polynomial, we have $abcabc\dots abc = 0$ and thus, $abc \in nil(R)$. Therefore R is nil 3-Armendariz.

Proposition 2.10. The class of nil 3-Armendariz rings is closed under finite direct products.

Proof. Let $R = \prod_{s \in \beta} R_s$ be the finite direct product of R_s where $\beta = \{1, 2, \dots, p\}$, R_s is nil 3-Armendariz. Suppose $f(x)g(x)h(x) = 0$ for some polynomials $f(x) = a_0 + a_1x + \dots + a_nx^n$,

$g(x) = b_0 + b_1x + \dots + b_mx^m$,
 $h(x) = c_0 + c_1x + \dots + c_r x^r \in R[x]$, where
 $a_i = (a_{i1}, a_{i2}, \dots, a_{ip})$, $b_j = (b_{j1}, b_{j2}, \dots, b_{jp})$,
 $c_k = (c_{k1}, c_{k2}, \dots, c_{kp})$, are elements of the product ring R . Set $f_s(x) = \sum_{i=0}^n a_{is}x^i$, $g_s(x) = \sum_{j=0}^m b_{js}x^j$ and

$h_s(x) = \sum_{k=0}^r c_{ks}x^k \in R[x]$. Since $f(x)g(x)h(x) = 0$ then $\sum_{i+j+k=l} a_i b_j c_k = 0$, $0 \leq l \leq n+m+r$. So

$\sum_{i+j+k=l} (a_{i1} b_{j1} c_{k1}, \dots, a_{ip} b_{jp} c_{kp}) = 0$, and so

$\sum_{i+j+k=l} (a_{is} b_{js} c_{ks}) = 0$, $1 \leq s \leq p$. Thus

$f_s(x)g_s(x)h_s(x) = 0$ in $R_s[x]$, $1 \leq s \leq p$. Since R_s is nil 3-Armendariz, then we have $a_{is} b_{js} c_{ks} \in nil(R_s)$. Now, for each i, j, k , there exist positive integers m_{ijks} such that $(a_{is} b_{js} c_{ks})^{m_{ijks}} = 0$, in the ring R_s , $1 \leq s \leq p$.

If we take $m_{ijk} = \max\{m_{ijks} : 1 \leq s \leq p\}$, then it is clear that $(a_{is} b_{js} c_{ks})^{m_{ijk}} = 0$. Therefore $a_i b_j c_k \in nil(R)$. This means that R is nil 3-Armendariz.

Lemma 2.11. Let S be a subring of R . If R is nil 3-Armendariz. Then so is S .

Proof. Let $f(x), g(x), h(x) \in S[x]$, be such that $f(x)g(x)h(x) \in nil(S[x])$. Then $f(x)g(x)h(x) \in nil(R[x])$. Since R is nil 3-Armendariz, then $a_i b_j c_k \in nil(R)$, i.e., $(a_i b_j c_k)^{n_{ijk}} = 0 \Rightarrow a_i b_j c_k \in nil(S)$. This means that S is nil 3-Armendariz.

We denote by $T_n(R)$ the ring consisting of all n -by- n upper triangular matrices over R . In [10, Theorem 1], showed that R is a weak 3-Armendariz if and only if $T_n(R)$ is a weak 3-Armendariz ring for all $n \in \mathbb{N}$. Here we have a similar results for nil 3-Armendariz rings.

Proposition 2.12. Let R be a ring. The following conditions are equivalent:

- 1) R is nil 3-Armendariz;
- 2) for any n , $T_n(R)$ is nil 3-Armendariz.

Proof. (2) \Rightarrow (1) We note that any subring of nil 3-Armendariz rings is nil 3-Armendariz by Lemma 2.11. Thus if $T_n(R)$ is nil 3-Armendariz ring, then so is R .

(1) \Rightarrow (2) Let $f(x) = A_0 + A_1x + \dots + A_px^p$, $g(x) = B_0 + B_1x + \dots + B_qx^q$, and $h(x) = C_0 + C_1x + \dots + C_dx^d$ be elements of $T_n(R)[x]$. It is easy to see that there exists an isomorphism of rings $T_n(R)[x] \rightarrow T_n(R[x])$ define by:

$$\sum_{i=0}^p \begin{pmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \dots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \dots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \dots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^i \end{pmatrix} x^i \mapsto \begin{pmatrix} \sum_{i=0}^p a_{11}^i x^i & \sum_{i=0}^p a_{12}^i x^i & \sum_{i=0}^p a_{13}^i x^i & \dots & \sum_{i=0}^p a_{1n}^i x^i \\ 0 & \sum_{i=0}^p a_{22}^i x^i & \sum_{i=0}^p a_{23}^i x^i & \dots & \sum_{i=0}^p a_{2n}^i x^i \\ 0 & 0 & \sum_{i=0}^p a_{33}^i x^i & \dots & \sum_{i=0}^p a_{3n}^i x^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{i=0}^p a_{nn}^i x^i \end{pmatrix}.$$

Assume that $f(x)g(x)h(x) \in \text{nil}(T_n(R)[x])$. Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \dots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \dots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \dots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^i \end{pmatrix}, B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & b_{13}^j & \dots & b_{1n}^j \\ 0 & b_{22}^j & b_{23}^j & \dots & b_{2n}^j \\ 0 & 0 & a_{33}^j & \dots & b_{3n}^j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{nn}^j \end{pmatrix} \text{ and } C_k = \begin{pmatrix} c_{11}^k & c_{12}^k & c_{13}^k & \dots & c_{1n}^k \\ 0 & c_{22}^k & c_{23}^k & \dots & c_{2n}^k \\ 0 & 0 & c_{33}^k & \dots & c_{3n}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn}^k \end{pmatrix}.$$

Then

$$\begin{pmatrix} \sum_{i=0}^p a_{11}^i x^i & \sum_{i=0}^p a_{12}^i x^i & \sum_{i=0}^p a_{13}^i x^i & \dots & \sum_{i=0}^p a_{1n}^i x^i \\ 0 & \sum_{i=0}^p a_{22}^i x^i & \sum_{i=0}^p a_{23}^i x^i & \dots & \sum_{i=0}^p a_{2n}^i x^i \\ 0 & 0 & \sum_{i=0}^p a_{33}^i x^i & \dots & \sum_{i=0}^p a_{3n}^i x^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{i=0}^p a_{nn}^i x^i \end{pmatrix} \times \begin{pmatrix} \sum_{j=0}^q b_{11}^j x^j & \sum_{j=0}^q b_{12}^j x^j & \sum_{j=0}^q b_{13}^j x^j & \dots & \sum_{j=0}^q b_{1n}^j x^j \\ 0 & \sum_{j=0}^q b_{22}^j x^j & \sum_{j=0}^q b_{23}^j x^j & \dots & \sum_{j=0}^q b_{2n}^j x^j \\ 0 & 0 & \sum_{j=0}^q b_{33}^j x^j & \dots & \sum_{j=0}^q b_{3n}^j x^j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{j=0}^q b_{nn}^j x^j \end{pmatrix} \times \begin{pmatrix} \sum_{k=0}^d c_{11}^k x^k & \sum_{k=0}^d c_{12}^k x^k & \sum_{k=0}^d c_{13}^k x^k & \dots & \sum_{k=0}^d c_{1n}^k x^k \\ 0 & \sum_{k=0}^d c_{22}^k x^k & \sum_{k=0}^d c_{23}^k x^k & \dots & \sum_{k=0}^d c_{2n}^k x^k \\ 0 & 0 & \sum_{k=0}^d c_{33}^k x^k & \dots & \sum_{k=0}^d c_{3n}^k x^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{k=0}^d c_{nn}^k x^k \end{pmatrix}$$

corresponds a polynomial with coefficients in $\text{nil}(T_n(R))$ under the isomorphism

$T_n(R)[x] \rightarrow T_n(R[x])$. Because $T_n(R)[x] \cong T_n(R[x])$ and

$$\text{nil}(T_n(R)) = \begin{pmatrix} n \text{ l}(R) & R & R & R \\ 0 & \text{nil}(R) & R & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{nil}(R) \end{pmatrix},$$

$$(A_i B_j C_k)^{m_{ijk}} = \begin{pmatrix} (a_{11}^i b_{11}^j c_{11}^k)^{m_{ijk}} & * & \cdots & * \\ 0 & (a_{22}^i b_{22}^j c_{22}^k)^{m_{ijk}} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & (a_{nn}^i b_{nn}^j c_{nn}^k)^{m_{ijk}} \end{pmatrix} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, $((A_i B_j C_k)^{m_{ijk}})^n = 0$ and so

$A_i B_j C_k \in \text{nil}(T_n(R))$. This shows that $T_n(R)$ is nil 3-Armendariz.

Corollary 2.13. If R is a 3-Armendariz ring, then, for any n , $T_n(R)$ is nil 3-Armendariz ring.

In [10, Corollary 1], it is shown that a ring is a weak 3-Armendariz ring if and only if $R[x]/(x^n)$ is a weak 3-Armendariz ring, where (x^n) is the ideal of $R[x]$ generated by x^n and n is a positive integer. For nil 3-Armendariz rings, we have the following result.

Proposition 2.14. Let R be a ring and n any positive integer. Then R is nil 3-Armendariz if and only if $R[x]/(x^n)$ is nil 3-Armendariz, where (x^n) is the ideal of $R[x]$ generated by x^n .

Proof. As $R[x]/(x^n) \cong S$ where

$$S = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R, i = 0, 1, \dots, n-1 \right\}$$

is a subring of $T_n(R)$. If R is nil 3-Armendariz, then, by Proposition 2.12, we have that $T_n(R)$ is nil 3-Armendariz, and so is S . Thus, $R[x]/(x^n)$ is nil 3-Armendariz. Conversely, if $R[x]/(x^n)$ is nil 3-Armendariz, then R as a subring of $R[x]/(x^n)$, is nil 3-Armendariz too.

Corollary 2.15. A ring R is nil 3-Armendariz if and only if the trivial extension $T(R, R)$ is nil 3-Armendariz.

Proof. It follows from Proposition 2.12.

From Proposition 2.12, one may suspect that if R is nil 3-Armendariz then every n -by- n full matrix ring $M_n(R)$ over R is nil 3-Armendariz, where $n \geq 2$. But the following example erases the possibility.

Example 2.16. Let R be a ring and let

we have

$$\left(\sum_{i=0}^p a_{ss}^i x^i \right) \left(\sum_{j=0}^q b_{ss}^j x^j \right) \left(\sum_{k=0}^d c_{ss}^k x^k \right) \in n \text{ l}(R)[x],$$

for $s = 1, 2, \dots, n$.

Since R is nil 3-Armendariz, there exists $m_{ijks} \in \mathbb{N}$ such that $(a_{ss}^i b_{ss}^j c_{ss}^k)^{m_{ijks}} = 0$ for any s and any i, j, k . Let $m_{ijk} = \max\{m_{ijk1}, m_{ijk2}, \dots, m_{ijkn}\}$. Then

$S = M_2(F)$. Let

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x,$$

$$g(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x, h(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

be polynomials in $S[x]$. Then $f(x)g(x)h(x) = 0$. But

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is not nilpotent. Thus S is not nil 3-Armendariz. Now we can give the example of nil 3-Armendariz rings which are not 3-Armendariz.

Example 2.17. Let R be a nil 3-Armendariz ring. Then the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

is not 3-Armendariz by [7, Example 4], for $n \geq 4$, but R_n is a nil 3-Armendariz ring by Proposition 2.12, because R_n is a subring of $T_n(R)$.

Proposition 2.18. Let R be a ring and e an idempotent of R . If e is central in R , then the following statements are equivalent:

- 1) R is nil 3-Armendariz;
- 2) eR and $(1-e)R$ are nil 3-Armendariz.

Proof. (2) \Rightarrow (1). Is obvious since eR and $(1-e)R$ are subrings of R .

(1) \Rightarrow (2). Note that $R = eR \oplus (1-e)R$ as rings. Thus the result follows from Proposition 2.10.

In [5, Theorem 11], it was shown that if I is a re-

duced ideal of R such that R/I is Armendariz, then R is Armendariz. In [10, Proposition 4], it is shown that if R/I is a weak 3-Armendariz ring, then so is R , where I is a nilpotent ideal of R . We show that this result also holds for nil 3-Armendariz rings in the following.

Proposition 2.19. Let R be a ring such that R/I is a nil 3-Armendariz ring for some proper ideal I of R . If $I \subseteq \text{nil}(R)$, then R is nil 3-Armendariz.

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j$,

$h(x) = \sum_{k=0}^r c_k x^k \in R[x]$ such that

$f(x)g(x)h(x) \in \text{nil}(R)[x]$. Then

$$\left(\sum_{i=0}^n \bar{a}_i x^i\right) \left(\sum_{j=0}^m \bar{b}_j x^j\right) \left(\sum_{k=0}^r \bar{c}_k x^k\right) \in \text{nil}(R/I)[x].$$

Since R/I is nil 3-Armendariz, we have that $\bar{a}_i \bar{b}_j \bar{c}_k \in \text{nil}(R/I)$. Hence $a_i b_j c_k \in I$. Since $I \subseteq \text{nil}(R)$, then $a_i b_j c_k \in \text{nil}(R)$. This means that R is a nil 3-Armendariz ring.

Anderson and Camillo in [3, Theorem 2], prove that a ring R is Armendariz if and only if the polynomial ring $R[x]$ is Armendariz. Yang Suiyi [7], prove that a ring R is 3-Armendariz if and only if the polynomial ring $R[x]$ is 3-Armendariz. In [11], it is shown that if R is reduced ring, then $R[x]$ and $R[[x]]$ is 3-Armendariz ring. For nil 3-Armendariz rings we will give the following results.

Proposition 2.20. If R is nil 3-Armendariz, then $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$.

Proof. Suppose $f(x) \in \text{nil}(R[x])$ and $f(x)^k = 0$. By Lemma 2.7, we have that $a_0 \cdots a_k \in \text{nil}(R)$ where $a_i \in \text{coef}(f(x))$ for $i = 1, \dots, k$. In particular, for every $a \in \text{coef}(f)$, a^k is nilpotent. Therefore $a \in \text{nil}(R)$ for all $a \in \text{coef}(f(x))$ and hence $f(x) \in \text{nil}(R)[x]$.

Theorem 2.21. If R is a 3-Armendariz ring, then $R[x]$ is a nil 3-Armendariz ring.

Proof. Let R be 3-Armendariz ring. Then by [7, Theorem 3], $R[x]$ is 3-Armendariz. Thus by Proposition 2.9, $R[x]$ is nil 3-Armendariz.

Proposition 2.22. Let R be a reduced ring. Then $R[[x]]$ is a nil 3-Armendariz ring.

Proof. It follows from the method in the proof of [11, Theorem 1].

Corollary 2.23. If R is a reduced ring, then $R[x]$ is a nil 3-Armendariz ring.

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

A ring R is called right (resp., left) Ore if given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with

b_1 regular such that $ab_1 = ba_1$ (resp., $b_1a = a_1b$). It is a well-known fact that R is a right (resp., left) Ore ring if and only if the classical right (resp., left) quotient ring of R exists.

Lemma 2.24. If $f(x) \in \text{nil}(R)[x]$, then for any central element $\lambda \in R$, $\lambda f(x) \in \text{nil}(R)[x]$.

Proof. Set $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $a_i \in \text{nil}(R)$. Thus $\lambda a_i \in \text{nil}(R)$. This means that $\lambda f(x) \in \text{nil}(R)[x]$.

Theorem 2.25. Let R be a right Ore ring with the classical right quotient ring Q . If all right regular elements are central, then R is nil 3-Armendariz if and only if so is Q .

Proof. It suffices to show by Lemma 2.11, that if R is nil 3-Armendariz rings so is Q . We apply the proof of [5, Theorem 12]. Consider $f(x) = \sum_{i=0}^n \alpha_i x^i$,

$g(x) = \sum_{j=0}^m \beta_j x^j$, $h(x) = \sum_{k=0}^r \gamma_k x^k \in Q[x]$, such that

$f(x)g(x)h(x) \in \text{nil}(Q)[x]$. By [12, Proposition 2.1.16], we can assume that $\alpha_i = a_i u^{-1}$,

$\beta_j = b_j v^{-1}$, $\gamma_k = c_k w^{-1}$ with $a_i, b_j, c_k \in R$ for all i, j, k and a right regular elements $u, v, w \in R$. Put

$$\phi(x) = \sum_{i=0}^n a_i x^i, \quad \varphi(x) = \sum_{j=0}^m b_j x^j,$$

$\psi(x) = \sum_{k=0}^r c_k x^k \in R[x]$. Then we have

$$\begin{aligned} f(x)g(x)h(x) &\in \text{nil}(Q)[x] \\ &= \sum_{p=0}^{n+m+r} \left(\sum_{i+j+k=p} \alpha_i \beta_j \gamma_k\right) x^p \\ &= \sum_{p=0}^{n+m+r} \left(\sum_{i+j+k=p} a_i u^{-1} b_j v^{-1} c_k w^{-1}\right) x^p \\ &= \sum_{p=0}^{n+m+r} \left(\sum_{i+j+k=p} a_i b_j c_k (uvw)^{-1}\right) x^p \\ &= \phi(x)\varphi(x)\psi(x)(uvw)^{-1}. \end{aligned}$$

Since $\phi(x)\varphi(x)\psi(x)(uvw)^{-1} \in \text{nil}(R)[x]$, by Lemma 2.24, $\phi(x)\varphi(x)\psi(x) \in \text{nil}(R)[x]$. Since R is nil 3-Armendariz, $a_i b_j c_k \in \text{nil}(R)$ for each i, j, k and so

$\alpha_i \beta_j \gamma_k = a_i u^{-1} b_j v^{-1} c_k w^{-1} = a_i b_j c_k (uvw)^{-1} \in \text{nil}(Q)$, for all i, j, k . Therefore Q is nil 3-Armendariz ring.

Corollary 2.26. Let R be a ring and Δ be a multiplicative closed subset in R consisting of central regular elements. Then R is nil 3-Armendariz rings if and only if $\Delta^{-1}R$ is nil 3-Armendariz rings.

Corollary 2.27. A commutative ring R is nil 3-Armendariz if and only if so is the total quotient ring of R .

Proof. It suffices to show the necessity by Lemma 2.11. Let Δ be the multiplicative closed subset of all regular elements in R . Then $\Delta^{-1}R$ is the total quotient ring of R and hence the result holds by Corollary 2.26.

The ring of Laurent polynomials in x , with coefficients in a ring R , consists of all formal sum

$\sum_{i=k}^n m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers and denote it by $R[x; x^{-1}]$.

Corollary 2.28. Let R be a ring. $R[x]$ is nil 3-Armendariz if and only if $R[x; x^{-1}]$ is nil 3-Armendariz.

Proof. It suffices to establish necessity since $R[x]$ is a subring of $R[x; x^{-1}]$. Let $\Delta = \{1, x, x^2, \dots\}$, then clearly Δ is a multiplicatively closed subset in $R[x]$ consisting of central regular elements. Note that $R[x; x^{-1}] = \Delta^{-1}R[x]$. If $R[x]$ is nil 3-Armendariz, so is $\Delta^{-1}R[x]$ by Corollary 2.26.

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