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Set-Valued Stochastic Integrals with Respect to Finite Variation Processes*

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ABSTRACT

In a Euclidean space R^d, the Lebesgue-Stieltjes integral of set-valued stochastic processes $F=\{F_t(\omega),t\in[0,T]\}$ with respect to real valued finite variation process $\{A_t(\omega),t\in[0,T]\}$ is defined directly by employing all integrably bounded selections instead of taking the decomposable closure appearing in some existed references. We shall show that this kind of integral is measurable, continuous in t under the Hausdorff metric and L^2-bounded.

Keywords: Set-Valued Stochastic Process; Finite Variation Process; Measurability

1. Introduction

Recently, integrals for set-valued stochastic processes with respect to Brownian motion, martingales and the Lebesgue measure have received much attention.

In 1997, Kisielewicz ([1]) defined the integral of set-valued process as a subset of L^2 space, but he didn't consider the measurability of the integral. In 1999, Kim and Kim [2] used the definition of stochastic integrals of set-valued stochastic process with respect to the Brownian motion. They called it Aumann ([3]) type It$\hat{o}$ integrals. In [4], Jung and Kim modified the definition by taking the decomposable closure such that the integral is measurable. Li and Ren [5] modified Jung and Kim's definition by considering the predictable set-valued stochastic process as a set-valued random variable in the product space $(\mathbb{R}_+\times\Omega)$, and the measurability and decomposability also were based on product σ-algebra. After that, Zhang *et al.* ([6,7]) studied the set-valued integrals with respect to the martingale and Brownian motion.

Stochastic differential inclusions and set-valued stochastic differential (or integral) equations are employed to model the problems with not only randomness but also impreciseness. Recently, there are some references related to set-valued differential equations such as [8-13] etc.

Concerning to the integral with respect to finite variation processes, Malinowski and Michta [12] give the notion of set-valued integral with respect to single valued finite variation but without considering the measurability. Z.Wang and R.Wang [14] defined the Lebesgue-Stieltjes stochastic integral of single valued stochastic processes with respect to set-valued finite variation processes (refer to [14] for the detail).

In this paper, different from the definition in [14], based on the Definition 3.1 in [12], we will study the Lebesgue-Stieltjes integral of set-valued stochastic processes with respect to single valued finite variation process. We shall prove the measurability of integral, namely, it is a set-valued random, which is similar to the classical stochastic integral.

This paper is organized as follows: in Section 2, we present some notions and facts on set-valued random variables; in Section 3, we shall give the definition of integral of set-valued stochastic processes with respect to finite variation process and then prove the measurability and L^2-boundedness.

2. Preliminaries

We denote $\mathbb{N}$ the set of all natural numbers, R the set

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of all real numbers, R^d the d-dimensional Euclidean space with the usual norm $\|\cdot\|$, R_+ the set of all nonnegative numbers. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $\{\mathcal{F}_t : t\in[0,T]\}$ a σ-field filtration satisfying the usual conditions. Let $\mathcal{B}(E)$ be a Borel field of a topological space E.

Let $\mathcal{K}(R^d)$ (resp. $\mathcal{K}_k(R^d),\mathcal{K}_{kc}(R^d)$) be the family of all nonempty, closed (resp. nonempty compact, nonempty compact convex) subsets of R^d. For any $x\in R^d$ and $A\in\mathcal{K}(R^d)$, define the distance between x and A by $d(x,A)=\inf_{y\in A}\|x-y\|$. The *Hausdorff metric* d_H on $\mathcal{K}(\mathcal{R}^d)$ (see e.g. [15]) is defined by

$$d_H(A,B)=\max\left\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\right\} \quad (1)$$

$A,B\in\mathcal{K}(R^d)$.

Denote $\|A\|_k = d_H(\{0\},A)=\sup_{a\in A}\|a\|$. For $A,B,C,D\in\mathcal{K}(R^d)$, we have

$$H(A+B,C+D)\le H(A,C)+H(B,D).$$

For $A\subset R^d$, $x^*\in R^d$ the support function of A is defined as follows:

$$S(x^*,A)=\sup\left\{\langle x^*,x\rangle : x\in A\right\}.$$

$L^p(\Omega,\mathcal{F},P;R^d)=L^p(\Omega;R^d)\,(p\ge 1)$: the set of all R^d—valued Borel measurable functions $f:\Omega\to R^d$ such that the norm

$$\|f\|_p=\left\{\int_\Omega\|f(\omega)\|^p\,\mathrm{d}P\right\}^{\frac{1}{p}},\ \text{if } 1\le p<\infty,$$

$$\|f\|_\infty=\operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega)\|,\ \text{if } p=\infty,$$

is finite. f is called L^p*-integrable* if $f\in L^p(\Omega;R^d)$.

A set-valued function $F:\Omega\to\mathcal{K}(R^d)$ is said to be measurable if for any open set $O\subset R^d$, the inverse $F^{-1}(O):=\{\omega\in\Omega : F(\omega)\cap O\ne\varnothing\}$ belongs to $\mathcal{F}$. Such a function F is called a set-valued random variable.

Let $\mathcal{M}(\Omega,\mathcal{F},P;\mathcal{K}(R^d))$ (*resp.* $\mathcal{M}(\Omega,\mathcal{F},P;\mathcal{K}_c(R^d))$, $\mathcal{M}(\Omega,\mathcal{F},P;\mathcal{K}_{kc}(R^d))$) be the family of all measurable $\mathcal{K}(R^d)$-valued (resp. $\mathcal{K}_c(R^d),\mathcal{K}_{kc}(R^d)$-valued) functions, briefly by $\mathcal{M}(\Omega;\mathcal{K}(R^d))$ (resp. $\mathcal{M}(\Omega,\mathcal{F},P;\mathcal{K}_c(R^d))$, $\mathcal{M}(\Omega,\mathcal{F},P;\mathcal{K}_{kc}(R^d))$.

For $F\in\mathcal{M}(\Omega,K(R^d))$, the family of all L^p-integrable selections is defined by

$$S_F^p(\mathcal{F}):=\left\{f\in L^p(\Omega,\mathcal{F},P;R^d): f(\omega)\in F(\omega)\ a.s.\right\}, \quad (2)$$

$p\ge 1$

In the following, $S_F^p(\mathcal{F})$ is denoted briefly by S_F^p.

A set-valued random variable F is said to be *integrable* if S_F^1 is nonempty. F is called $L^p(p\ge 1)$*-integrably bounded* if there exits $h\in L^p(\Omega,\mathcal{F},P;R^d)$ s.t. for all $x\in F(\omega)$, $\|x\|\le h(\omega)$ almost surely.

An R^d-valued stochastic process $f=\{f_t : t\ge 0\}$ (or denoted by $f=\{f(t): t\ge 0\}$) is defined as a function $f:R_+\times\Omega\to R^d$ with the $\mathcal{F}$-measurable section f_t, for $t\ge 0$. We say f is measurable if f is $\mathcal{B}_+\otimes\mathcal{F}$-measurable. The process $f=\{f_t : t\ge 0\}$ is called $\mathcal{F}_t$-adapted if f_t is $\mathcal{F}_t$-measurable for every $t\ge 0$. Let $\Sigma:=\bigcap_{t\ge 0}\{Z\in\mathcal{B}(R_+)\otimes\mathcal{F}: Z_t\in\mathcal{F}_t\}$, where $Z_t=\{\omega;(t,\omega)\in Z\}$. We know that Σ is a σ-algebra on $R_+\times\Omega$. A function $f:R_+\times\Omega\to R^d$ is measurable and $\mathcal{F}_t$-adapted if and only if it is Σ-measurable ([8]).

In a fashion similar to the R^d-valued stochastic processes, a set-valued stochastic process $F=\{F_t : t\ge 0\}$ is defined as a set-valued function $F:R_+\times\Omega\to\mathcal{K}(R^d)$ with $\mathcal{F}$-measurable section F_t for $t\ge 0$. It is called measurable if it is $\mathcal{B}_+\otimes\mathcal{F}$-measurable, and $\mathcal{F}_t$-adapted if for any fixed t, $F_t(\cdot)$ is $\mathcal{F}_t$-measurable. $F=\{F_t : t\ge 0\}$ is measurable and $\mathcal{F}_t$-adapted if and only if it is Σ-measurable. $F=\{F_t : t\ge 0\}$ is called L^p*-integrable* if every F_t is L^p-integrable.

3. Set-Valued Stochastic Integral w.r.t Finite Variation Processes

Let $A=\{A_t,t\ge 0\}$ be a real valued $\mathcal{F}_t$-adapted measurable process with finite variation and continuous sample trajectories a.s. from the origin. That is to say, for each compact interval $[s,t]\subset[0,\infty)$ and any partition $\Delta=\{t_1,\cdots,t_n\}$ of $[s,t]$, the total variation

$$V_A([s,t])=\sup_\Delta\Sigma_{2=1}^n\left|A_{t_i}(\omega)-A_{t_{i-1}}(\omega)\right|$$

is finite and $A(0,\cdot)=0$ a.s. Then for any $T>0$, the process $A=\{A_t,t\ge 0\}$ can generate a random measure denoted by μ_A in the space $([0,T],(\mathcal{B}[0,T]))$. For any $(s,t]\subset[0,T]$, let

$$\mu_A((s,t]):=|A(t,\omega)|-|A(s,\omega)|$$

where $A(t,\omega)=A^+(t,\omega)+A^-(t,\omega)$ is the decomposition of A, A^+ and A^- are non-negative and non-decreasing processes, $|A(t,\omega)|=A^+(t,\omega)+A^-(t,\omega)$. In the product space $(\Omega\times[0,T],\Sigma)$, set

$$\nu_A(C):=\int_\Omega\int_{[0,T]}\mathbf{1}_C(t,\omega)\mu_A([0,T])\mu_A(\mathrm{d}t)P(\mathrm{d}\omega), \quad (3)$$

for $C\in\Sigma$, where $\mathbf{1}_C$ is the index function. Then the set function ν is a finite measure in the measurable space $(\Omega\times[0,T],\Sigma)$ if and only if

$\int_\Omega(\mu_A([0,T]))^2P(\mathrm{d}\omega)<\infty$. In the following we always assume $\int_\Omega(\mu_A([0,T]))^2P(\mathrm{d}\omega)<\infty$.

Let $L^2(\Omega\times[0,T],\Sigma,\nu_A;R^d)$ be the family of all Σ-measurable R^d-valued stochastic processes f such that

$$\int_{\Omega\times[0,T]}\|f(\omega,t)\|^2\nu_A(\mathrm{d}t)<\infty.$$

For any $f \in L^2(\Omega\times[0,T],\Sigma,\nu_A;R^d)$ and $[s,t]\subset[0,T]$, the stochastic Lebesgue-Stieltjes integral $\int_{[s,t]} f(\tau)\mathrm{d}A_\tau$ is defined by the Bochner integral $\int_{[s,t]} f(\tau)\mu_A(\mathrm{d}\tau)$ path-by-path. One can show that the integral process $\left\{\int_{[0,t]} f(s)\mathrm{d}A_s, t\in[0,T]\right\}$ is Σ-measurable.

Note: in [12], the integrand is assumed being predictable, in fact the integrand can be relaxed to the Σ-measurable class since the integrator A_t is continuous.

Let $M^2(\Omega\times[0,T],\Sigma,\nu_A;\mathcal{K}(R^d))$ be the family of all Σ-measurable $\mathcal{K}(R^d)$-valued stochastic processes F such that

$$\int_{\Omega\times[0,T]}\|F(\omega,t)\|_k^2\,\nu_A(\mathrm{d}t)<\infty,$$

where $\|F(\omega,t)\|_k=\sup_{x\in F(\omega,t)}\|x\|$. For any $F\in M^2(\Omega\times[0,T],\Sigma,\nu_A;\mathcal{K}(R^d))$, set

$$S^2(F):=\left\{f\in M^2(\Omega\times[0,T],\Sigma,\nu_A;R^d):\ f(\omega,t)\in F(\omega,t),\nu_A-a.e.\right\} \tag{4}$$

Definition 1. (*see* [12]) *For a set-valued stochastic process* $F\in M^2(\Omega\times[0,T];\mathcal{K}_{kc}(R^d))$ *the set-valued stochastic Lebesgue-Stieltjes integral* (*over interval* $[s,t]$) *of* F *with respect to the finite variation continuous process* A *is the set*

$$\int_{[s,t]}F(\tau)\mathrm{d}A_\tau:=\left\{\int_{[s,t]}f(\tau)\mathrm{d}A_\tau : f\in S^2(F)\right\}.$$

In [12], the authors call this kind of integral as trajectory integral since they consider it as a $\mathcal{K}(L^2(\Omega\times[0,T],\Sigma,\nu_A;R^d))$-valued random variable. Here, we shall consider it as a subset of R^d and show the measurability with respect to $\mathcal{F}$, which is very different from the way in [12], also different from other references such as [10,16,17] etc. In fact, for almost every $\omega\in\Omega$, the above integral $\int_{[s,t]}F(\tau)\mathrm{d}A_\tau$ is a subset of R^d. In the following, we shall assume the σ-algebra $\mathcal{F}$ is separable w.r.t P. In addition, $\mathcal{B}([0,T])$ is separable and $\Sigma\subset\mathcal{F}\otimes\mathcal{B}([0,T])$, then one can get $S^2(F)$ is separable. Therefore we can find an $\mathcal{F}$-measurable set Ω_F, such that $P(\Omega_F)=1$ and for every $\omega\in\Omega_F$, the integral $\int_{[0,T]}F(\tau)\mathrm{d}A_\tau$ is defined path-by-path. For $\omega\in\Omega/\Omega_F$, set $\int_{[s,t]}F(\tau)\mathrm{d}A_\tau=\{0\}$, therefore it is well defined for every $\omega\in\Omega$.

$\int_{(s,t]}F(\tau)\mathrm{d}A_\tau=\int_{[s,t]}F(\tau)\mathrm{d}A_\tau$ since the continuity of A_t. In the sequel, we shall denote the integral by $\int_s^t F(\tau)\mathrm{d}A_\tau$ instead of $\int_{[s,t]}F(\tau)\mathrm{d}A_\tau$. For any $t\in[0,T]$, denote $\int_0^t F_s\mathrm{d}A_s$ by $I_t(F)$.

Theorem 1. *For* $F\in M^2(\Omega\times[0,T],\Sigma,\mathcal{K}_{kc}(R^d);\nu_A)$, $[s,t]\subset[0,T]$ *and* $\omega\in\Omega$, *the Lebesgue-Stieltjes integral* $\int_s^t F_\tau(\omega)\mathrm{d}A_\tau(\omega)$ *is a compact and convex subset of* R^d.

Proof 1. *In fact,* $S^2(F)$ *is a bounded and convex subset of* $L^2(\Omega\times[0,T],\Sigma,R^d;\nu_A)$, *since* F *is convex and compact,moreover, it is weakly compact since* $L^2(\Omega\times[0,T],\Sigma,R^d;\nu_A)$ *is reflexive. The convexity of the integral is obvious.*

We shall show the linear operator $\mathcal{T}(F):=\int_s^t F_\tau(\omega)\mathrm{d}A_\tau(\omega):\ S^2(F)\to\mathcal{K}_{kc}(R^d)$ *is bounded.*

For any $f\in S^2(F)$, $[s,t]\subset[0,T]$,

$$\begin{aligned}\left\|\int_s^t f(\tau,\omega)\mathrm{d}A_\tau(\omega)\right\| &\le \int_s^t\|f(\tau,\omega)\|\,|\mathrm{d}A_\tau(\omega)| \\ &\le \int_s^t\|F(\tau,\omega)\|_{\mathcal{K}}\,|\mathrm{d}A_\tau(\omega)|<\infty,\end{aligned} \tag{5}$$

which implies the linear operator $\mathcal{T}$ *is bounded. Therefore the integral* $\int_s^t F_\tau\mathrm{d}A_\tau$ *is weakly compact since the bounded linear operator mapping a weakly compact set to a weakly compact one. In* R^d *space, a weakly compact set is compact.*

Lemma 1. (*see* [16] *Corollary* 2.1.1 (5)) *Assume* $(\Omega,\mathcal{A})$ *is a measurable space,* $\mathfrak{X}$ *is a separable Banach space,* $F:\Omega\to\mathcal{K}(\mathfrak{X})$, *and F is a set-valued random variable, then* $S(x^*,F(\omega))$ $(x^*\in\mathfrak{X}^*)$ *is measurable.*

By using Lemma 1, as a manner similar to Theorem 1 in [17], we have the following result:

Lemma 2. *Assume* A *is the corresponding stochastic process,* $F\in M^2(\Omega\times[0,T],\Sigma,\mathcal{K}_{kc}(R^d);\nu_A)$, *for any* $[s,t]\subset[0,T]$, *we have*

1) $\int_s^t\alpha F_\tau\mathrm{d}A_\tau=\alpha\int_s^t F_\tau\mathrm{d}A_\tau$, $\alpha\in R$;

2) $S\left(x^*,\int_s^t F_\tau\mathrm{d}A_\tau\right)=\int_s^t S(x^*,F_\tau)\mathrm{d}A_\tau$, $x^*\in R^d$.

Lemma 3. (*see* [16] *Theorem* 2.1.16) *Assume* $(\Omega,\mathcal{F})$ *is a measurable space,* $\mathfrak{X}$ *is a separable Banach space,* $F:\Omega\to\mathcal{K}_{kc}(\mathfrak{X})$, *and for any fixed* $x^*\in\mathfrak{X}^*$, $S(x^*,F)$ *is measurable, if one of the following conditions is satisfied*:

1) $\mathfrak{X}^*$ *is separable*;

2) *for any* $\omega\in\Omega, F(\omega)\in\mathcal{K}_{kc}(\mathfrak{X})$.

Then F *is a set-valued random variable*.

From Lemma 1 and Lemma 3, when $\mathfrak{X}=R^d$, for any $x^*\in R^d$, $F(\omega)\in K_{kc}(R^d)$ is $\mathcal{F}$-measurable if and only if $S(x^*,F)(\omega)$ is $\mathcal{F}$-measurable.

Lemma 4. ([16] *Theorem* 1.7.7) *If* $(\Omega,\mathcal{F})$ *is a separable space,* $\mathfrak{X},\mathcal{Y}$ *are separable metric space* $F:\Omega\times\mathfrak{X}\to\mathcal{K}(\mathcal{Y})$ *satisfy*:

(a) *for any* $x\in\mathfrak{X},\omega\to F(\omega,x)$ *is measurable*;

(b) *for any* $\omega\in\Omega,x\to F(\omega,x)$ *is continuous or is continuous with respect to Hausdorff metric,*

Then $(\omega,x)\to F(\omega,x)$ *is jointly measurable.*

Then by Lemma 1 we have the following:

Lemma 5. *Assume* $F \in M^2\left(\left[0,T\right]\times\Omega,\Sigma,\mathcal{K}_{kc}\left(R^d\right);\nu_A\right)$. *Then* $S\left(x^*,F\left(t,\omega\right)\right):\left[0,T\right]\times\Omega\to R$ *is* Σ*-measurable.*

Theorem 2. *Assume* $F \in M^2\left(\left[0,T\right]\times\Omega,\Sigma,\mathcal{K}_{kc}\left(R^d\right);\nu_A\right)$. *Then* $I_t\left(F\right)\in L^2\left(\Omega;R^d\right)$ *for each* $t\in\left[0,T\right]$. *Furthermore, the mapping* $\psi\left(t,\omega\right)=I_t\left(F\right)$ *is* $\left(\mathcal{B}\left[0,t\right]\otimes\mathcal{F}_t\right)$*-measurable.*

Proof 2. *Step* 1. *We will show that* $I_t\left(F\right)$ *is* $\mathcal{F}_t$*-measurable for each* $t\in\left[0,T\right]$, $\psi\left(t,\omega\right)=I_t\left(F\right)$ *is* $\left(\mathcal{B}\left[0,T\right]\times\mathcal{F}_T\right)$*-measurable.*

By Theorem 1, *we have*

$$\begin{aligned}I_T\left(F\right)\left(\omega\right)&=\int_0^T F_t\left(\omega\right)\mathrm{d}A_t\left(\omega\right)\\&=\left\{\int_0^T f_t\left(\omega\right)\mathrm{d}A_t\left(\omega\right);\, f\in S^2\left(F\right)\right\}\in\mathcal{K}_{kc}\left(R^d\right)\end{aligned}\tag{6}$$

for all $\omega\in\Omega$. *Furthermore, we obtain*

$$S\left(x^*,I_t\left(F\right)\left(\omega\right)\right)=\int_0^t S\left(x^*,F\left(s,\omega\right)\right)\mathrm{d}A_s$$

for all $\omega\in R^d$. *Moreover, since* $F\left(t,\omega\right):\left[0,t\right]\times\Omega\to\mathcal{K}_k\left(R^d\right)$ *is* $\left(\mathcal{B}\left[0,t\right]\otimes\mathcal{F}_t\right)$*-measurable, from the Lemma* 5 *we can obtain that the function* $S\left(x^*,F\left(s,\omega\right)\right):\left[0,T\right]\times\Omega\to R$ *is* $\left(\mathcal{B}\left(\left[0,T\right]\right)\otimes\mathcal{F}_T\right)$ *measurable. By Fubini theorem,* $\int_0^t S\left(x^*,F\left(s,\omega\right)\right)\mathrm{d}A_s$ *is* $\mathcal{F}_t$*-measurable, based on Lemma* 3, $I_t\left(F\right)$ *is* $\mathcal{F}_T$*-measurable.*

Finally, in the argument above, the function $\phi\left(t,\omega\right)=\int_0^t S\left(x^*,F\left(s,\omega\right)\right)\mathrm{d}A_s$ *is* $\mathcal{F}_T$*-measurable for each* $t\in\left[0,T\right]$. *Since it is continuous in* $t\in\left[0,T\right]$ *for all* $\omega\in\Omega$, *so it is* $\left(\mathcal{B}\left(\left[0,T\right]\right)\otimes\mathcal{F}_T\right)$*-measurable. From Lemma* 4, *we obtain that* $I_t\left(F\right)\left(\omega\right)$ *is* $\left(\mathcal{B}\left(\left[0,t\right]\right)\otimes\mathcal{F}_t\right)$*-measurable.*

Step 2. *In this step, we will show that* $I_t\left(F\right)\left(\omega\right)\in L^2\left(\Omega;\mathcal{K}\left(R^d\right)\right)$ *for each* $t\in\left[0,T\right]$.

For each $\omega\in\Omega,t\in\left[0,T\right]$ *and* $f\in S^2\left(F\right)\left(\omega\right)$, *we have*

$$\begin{aligned}\left\|\int_0^t f_s\mathrm{d}A_s\right\|^2&\le\mu_A\left(\left[0,T\right]\right)\int_0^t\left\|f_s\right\|^2\left|\mathrm{d}A_s\right|\\&\le\mu_A\left(\left[0,T\right]\right)\int_0^t\left\|F_s\right\|_k^2\left|\mathrm{d}A_s\right|\end{aligned}\tag{7}$$

then

$$\sup_{f\in S^2(F)}\left\|\int_0^t f_s\mathrm{d}A_s\right\|^2\le\mu_A\left(\left[0,T\right]\right)\int_0^t\left\|F_s\right\|_k^2\left|\mathrm{d}A_s\right|$$

Hence,

$$\begin{aligned}&E\left[\left\|I_t\left(F\right)\left(\omega\right)\right\|_k^2\right]\\&\le E\left[\int_0^T\mu_A\left(\left[0,T\right]\right)\left\|F\left(t,\omega\right)\right\|_k^2\left|\mathrm{d}A_s\right|\right]\\&\le\int_{\Omega\times[0,T]}\left\|F_s\right\|_k^2\,\nu_A\left(\mathrm{d}s\right)<\infty,\end{aligned}\tag{8}$$

which implies

$$I_t\left(F\right)\in L^2\left(\Omega;\mathcal{K}\left(R^d\right)\right).$$

As a manner similar to Theorem 3.8. in [8], we have the Castaing representation as following:

Theorem 3. *For a set-valued stochastic process* $F\in M^2\left(\left[0,T\right]\times\Omega;\mathcal{K}_{kc}\left(R^d\right),\nu_A\right)$, *there exists a sequence* $\left\{f^n;n\in\mathbb{N}\right\}\subset S^2\left(F\right)$ *such that*

$$F_t\left(\omega\right)=\mathrm{cl}\left\{f_t^n\left(\omega\right);n\in\mathbb{N}\right\}\text{ for } a.e.\left(t,\omega\right),$$

and, for $0\le s\le t\le T$,

$$I_{s,t}\left(F\right)\left(\omega\right)=\mathrm{cl}\left\{\int_s^t f_u^n\left(\omega\right)\mathrm{d}A_u;n\in\mathbb{N}\right\}\ a.s,$$

where cl *denotes the closure in* R^d.

Theorem 4. *For each* $F\in M^2\left(\left[0,T\right]\times\Omega;\mathcal{K}_{kc}\left(R^d\right),\nu_A\right)$, $I_t\left(F\right)\left(\omega\right)$ *is continuous a.s. with respect to the Hausdorff metric* d_H.

Proof 3. *Let* $0\le r<t\le T$ *and* $\omega\in\Omega$. *We then have*

$$\begin{aligned}I_T\left(\mathbf{1}_{[0,t]}F\right)\left(\omega\right)&=I_T\left(\mathbf{1}_{[0,r]}F+\mathbf{1}_{[r,t]}F\right)\left(\omega\right)\\&=I_T\left(\mathbf{1}_{[0,r]}F\right)\left(\omega\right)+I_T\left(\mathbf{1}_{[r,t]}F\right)\left(\omega\right)\end{aligned}\tag{9}$$

Hence,

$$\begin{aligned}&d_H\left(I_t\left(F\right)\left(\omega\right),I_r\left(F\right)\left(\omega\right)\right)\\&=d_H\left(I_T\left(\mathbf{1}_{[0,r]}F\right)\left(\omega\right)+I_T\left(\mathbf{1}_{[r,t]}F\right)\left(\omega\right),I_T\left(\mathbf{1}_{[0,r]}\left(F\right)\left(\omega\right)\right)\right)\\&\le d_H\left(I_T\left(\mathbf{1}_{[r,t]}F\left(\omega\right),\{0\}\right)\right)\\&=\sup_{f\in S_T(F)(\omega)}\left\|\int_0^t f\left(s\right)\mathrm{d}A_s\right\|\\&\le\int_r^t\left\|F\left(s,\omega\right)\right\|_k\left|\mathrm{d}A_s\right|<\infty\end{aligned}\tag{10}$$

since for each $f\in S_T\left(F\right)\left(\omega\right)$, $\left\|\int_0^t f\left(s\right)\mathrm{d}A_s\right\|\le\int_r^t\left\|F\left(s,\omega\right)\right\|_{\mathcal{K}}\left|\mathrm{d}A_s\right|$. *Hence,* $\lim_{r\uparrow t}\int_r^t\left\|F\left(s,\omega\right)\right\|_{\mathcal{K}}\mathrm{d}A_s\left(\omega\right)=0$. *So* $I_t\left(F\right)\left(\omega\right)$ *is left-continuous in* $t\in\left[0,T\right]$ *for all a.s. In a similar way, we see that* $I_t\left(F\right)\left(\omega\right)$ *is right-continuous in* $t\in\left[0,T\right]$ *a.s.*

Similar to the proof of Theorem 3.15 in [8], we have the following theorem:

Theorem 5. *Let* $F,G\in M^2\left(\Omega\times\left[0,T\right],\Sigma,\nu_A;\mathcal{K}_{kc}\left(R^d\right)\right)$, *for any* $t\in\left[0,T\right]$, *we have*

$$E\left[d_H\left(I_t\left(F\right),I_t\left(G\right)\right)\right]\le\int_{\Omega\times[0,T]}d_H\left(F_s,G_s\right)\nu_A\left(\mathrm{d}s\right)$$

and

$$E\left[d_H^2\left(I_t\left(F\right),I_t\left(G\right)\right)\right]\le2\int_{\Omega\times[0,T]}d_H^2\left(F_s,G_s\right)\nu_A\left(\mathrm{d}s\right).$$

4. Conclusion

When the integrand takes values in compact and convex subsets of R^d, we defined the integral with respect to real-valued variation processes. And then we proved some properties of this kind of integral such as measurability, L^2-boundedness and continuity under the Hausdorff metric.

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