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On the Behavior of Connectedness Properties in Isotonic Spaces under Perfect Mappings

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ABSTRACT

The topological study of connectedness is heavily geometric or visual. Connectedness and connectedness-like properties play an important role in most topological characterization theorems, as well as in the study of obstructions to the extension of functions. In this paper, the behaviour of these properties in the realm of closure spaces is investigated using the class of perfect mappings. A perfect mapping is a type of map under which the image generally inherits the properties of the mapped space. It turns out that the general behaviour of connectedness properties in topological spaces extends to the class of isotone space.

Keywords: Closure Operator; Closure Axiom; Isotonic Space; Perfect Mapping; Connectedness

1. Introduction

The concept of a topological space is generally introduced and studied in terms of the axioms of open sets. However, alternate methods of describing a topology are often used: neighborhood systems, family of closed sets, closure operator and interior operator. Of these, the closure operator—records [1], was axiomated by Kuratowski.

The class of continuous functions forms a very broad spectrum of mappings comprising of different subclasses with varying properties. In the spectrum of continuous functions is the class of perfect mappings which, although weaker than homeomorphisms, provides a general yet satisfactory means of investigating topological invariants and hence the equivalence of topological spaces.

In an attempt to extend the boundaries of topology, [2] has shown that topological spaces do not constitute a natural boundary for the validity of theorems and results in topology. Many results therefore, can be extended to closure spaces where some of the basic axioms in this space can be dropped. Many properties which hold in basic topological spaces hold in spaces possessing the isotonic property.

2. Literature Review

2.1. Closure Operator and Generalized Closure Space

A closure operator is an arbitrary set-valued, set-function $cl:\mathcal{P}(X)\to\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of a non-void set X, [3]. Let $A,B\subset\mathcal{P}(X)$.

1) Grounded: $cl(\varnothing)=\varnothing$
2) Expansive: $A\subset cl(A)$
3) Sub-additive: $cl(A\cup B)\subset cl(A)\cup cl(B)$. This axiom implies the Isotony axiom: $A\subset B$ implies $cl(A)\subset cl(B)$
4) Idempotent: $cl(cl(A))=cl(A)$

The structure (X,cl), where cl satisfies the first three axioms is called a closure space. If in addition the idempotent axiom is satisfied, then the structure is a topological space.

2.2. Isotonic Space

A closure space (X,cl) satisfying only the grounded and the isotony closure axioms is called an isotonic space [4].

In the dual formulation, a space (X,cl) is isotonic if and only if the interior function $int:\mathcal{P}(X)\to\mathcal{P}(X)$ satisfies;

1) $int(X) = X$.
2) $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

2.3. C-Compactness

Let (X, cl) be a closure space. A family $\{A_i : i \in I\}$ of subsets of X is called a *c*-cover of X if $\{intA_i : i \in I\}$ covers X. A closure space is *c*-compact if every c-cover of X has a finite subcover, [3].

2.4. Connectedness

Various characterizations of connectedness exist in the realm of topological spaces. For instance, [5] described a space X to be connected if and only if for every decomposition of X into two non-void closed sets A and B, the condition $A \cap B \neq \varnothing$ is satisfied. In other words, X is connected if and only if it cannot be represented as a union of two non-empty, disjoint closed sets.

According to [4] an isotonic space (X, cl) is connected if and only if for all T_1-isotonic doubleton spaces $Y = \{0,1\}$, any continuous function $f : X \to Y$ is a constant. This characterization of connectedness is an extension of an equivalent definition of connectedness in topological spaces that was given by [2] where Y is a discrete space.

In topological spaces, connectedness is defined by means of open and closed sets. That definition can only be extended to closure spaces which have the expanding closure axiom; $A \subseteq cl(A)$. This is why the above definition is more suitable for isotonic spaces since it doesn't involve open or closed sets.

2.4.1. Total Disconnectedness

The component $C(x)$ of $x \in X$ is the union of all connected subsets of X containing *x*, [5]. It is clear, from the fact that the union of any family of connected subsets having at least one point in common is also connected, that $C(x)$ is connected.

An isotonic space (X, cl) is said to be totally disconnected if for every $x \in X$, the component $C(x) = \{x\}$, [4]. Since the components of a space X are closed, then every totally disconnected space is T_1.

2.4.2. Z-Connectedness

The concept of Z-connectedness, according to [6], is obtained by replacing the discrete space {0,1} in the characterization of connectedness, by some other space Z.

Let (Z, cl) be an isotonic space with more than one element. An isotonic space (X, cl) is called Z-connected if and only if any continuous function $f : X \to Z$ is constant, [5].

2.4.3. Strongly Connected

An isotonic space (X, cl) is strongly connected if there is no countable collection of pair-wise semi-separated sets $\{A_i\}$ such that $X = \bigcup A_i$. It follows from the definition that a strongly connected isotonic space is connected.

2.5. Perfect Mappings

The class of perfect mappings is descended from the broader class of bi-quotient maps. A map $f : X \to Y$ is called bi-quotient if whenever $y \in Y$ and $\mathcal{U}$ is a covering of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$ with $U \in \mathcal{U}$ cover some neighbourhood of y in Y, [7].

Let X and Y be topological spaces and $f : X \to Y$ be a mapping. From [8],

1) f is called a compact mapping if $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

2) f is called a perfect mapping if it is a closed and compact mapping.

3) f is called an open perfect mapping if f is an open and perfect mapping.

4) Every perfect mapping is a quotient map, where $f : X \to Y$ is a quotient map if $V \subset Y$ is open in Y whenever $f^{-1}(V)$ is open in X.

The class of perfect mappings may be used in substitution of homeomorphisms to investigate the invariance of topological properties. By investigating topological properties in isotonic spaces under the class of perfect mappings, which is a more general class of functions than homeomorphisms, the results and theorems obtained become a pointer of how general the concepts of topology can get.

3. Main Results

This section summarizes the results of this work.

3.1. Perfect Mappings on Isotonic Spaces

Let X and Y be isotonic spaces and $f : X \to Y$ be a continuous surjective mapping. f is called a c-compact mapping if $f^{-1}(y)$ is a c-compact subset of $X \, \forall y \in Y$.

A continuous surjective mapping between two isotonic spaces is said to be perfect if it is closed and $\forall y \in Y$, $f^{-1}(y)$ is a c-compact subset of X.

These two definitions closely correlate with their definitions in general topological spaces, except for the fact that a different approach is employed while defining the foregoing topological notions.

3.2. Invariance of Connectedness Conditions

There are different forms of connectedness and disconnectedness that have been defined, both in topological spaces and in closure spaces. These definitions can be found under Section 2.3 of this paper. The next two theorems describe the behavior of the different forms of connectedness with respect to perfect mappings.

Theorem: Connectedness, Z-connectedness and strong connectedness are invariants of perfect mappings.

Proof: Let $f: X \to Y$ be a perfect mapping of a connected isotonic space X onto an isotonic space Y. Since f is a surjection, then $f(X) \subseteq Y$. Let $g: f(X) \to \{0,1\}$, be a perfect mapping, where $\{0,1\}$ is a T_1 -doubleton isotonic space. Therefore, $gof: X \to \{0,1\}$ is perfect. Since X is connected, gof is a constant and hence g is also a constant function. This implies that $f(X)$ is connected.

Let (X, cl) be a Z-connected isotonic space and (Y, cl) be an isotonic space. Let $f: X \to Y$ be a perfect mapping (*f* is a continuous surjection). If $g: Y \to Z$ is a perfect mapping, where (Z, cl) is an isotonic space with more than one element, then by Z-connectedness of X, the composition $gof: X \to Z$ is a constant. It then follows that g is also a constant and hence Y is Z-connected.

Let $f: X \to Y$ be a perfect mapping between isotonic spaces. Suppose X is strongly connected and $f(X) \subseteq Y$ is not strongly connected. Then, there exists a countable collection $\{A_i\}$ of pair-wise semi-separated sets such that $f(X) = \bigcup A_i$. The countable collection $\{f^{-1}(A_i)\}$, is such that

$$f^{-1}(A_i) \cap cl_X\left(f^{-1}(A_j)\right) \subseteq f^{-1}(A_i) \cap f^{-1}\left(cl_Y(A_j)\right) = \varnothing$$

for all $i \neq j$. Moreover, $X = \bigcup f^{-1}(A_i)$. That is, X is not strongly connected which is a contradiction. The contrary, Y is strongly connected, is true.

Theorem: Total disconnectedness and extremal disconnectedness are not invariants of perfect mapping.

Proof: Let $f: \{0,1\} \to Y$ be a perfect mapping from the two-point isotonic space $\{0,1\}$ to any isotonic space Y. Clearly, $\{0,1\}$ is totally disconnected while Y doesn't have to be. Total disconnectedness is not invariant with respect to perfect mappings.

Since every extremally disconnected space is totally disconnected, then it follows that extremal disconnectedness is not an invariant of perfect mappings.

3.3. Local Connectedness

In this section, we localize the concept of connectedness. We say that a topological property holds locally at a point $x \in X$ if there exists a neighborhood $N \in \mathcal{N}(x)$ which has that property, whether the whole space X has that property or not.

Definition

An isotonic space X is said to be locally connected at a point $x \in X$ if for every neighborhood $N \in \mathcal{N}(x)$, there exists a connected neighborhood $M \in \mathcal{N}(x)$ such that $M \subset N$. X is said to be lo- cally connected if it is locally connected at each of its points. This definition follows directly from general topological spaces.

Theorem: Let $f: X \to Y$ be a perfect map of a locally connected isotonic space onto an isotonic space Y. Then Y is locally connected as well.

Proof: If $f: X \to Y$ is perfect, then for every $y \in Y$ we have $f^{-1}(y) \in X$ and since X is locally connected, then for every $N \in \mathcal{N}\left(f^{-1}(y)\right)$, there exists a connected neighborhood $M \in \mathcal{N}\left(f^{-1}(y)\right)$ such that $M \subset N$. Further

$f(N) \in f\left(\mathcal{N}\left(f^{-1}(y)\right)\right) = \mathcal{N}\left(f\left(f^{-1}(y)\right)\right) = \mathcal{N}(y)$.

Similarly

$f(M) \in f\left(\mathcal{N}\left(f^{-1}(y)\right)\right) = \mathcal{N}\left(f\left(f^{-1}(y)\right)\right) = \mathcal{N}(y)$.

Thus we have for every $f(N) \in \mathcal{N}(y)$ there exists $f(M) \in \mathcal{N}(y)$ such that $f(M) \subset f(N)$. Therefore Y is locally connected.

This result shows that local connectedness is an in- variant of perfect mappings, despite the fact that the property is not an invariant of continuous functions.

4. Conclusion

The results obtained in this paper show that the invariance of connectedness properties in isotonic spaces under continuous functions extends to perfect mappings.

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