*Advances in Pure Mathematics*, 2013, 3, 670-679
Published Online November 2013 (http://www.scirp.org/journal/apm)
http://dx.doi.org/10.4236/apm.2013.38090

# The Lattice of Fully Invariant Subgroups of the Cotorsion Hull

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Received September 9, 2013; revised September 18, 2013; accepted October 7, 2013



## ABSTRACT

The paper considers the lattice of fully invariant subgroups of the cotorsion hull $T^{\bullet}$ when a separable primary group $T$ is an arbitrary direct sum of torsion-complete groups.The investigation of this problem in the case of a cotorsion hull is important because endomorphisms in this class of groups are completely defined by their action on the torsion part and for mixed groups the ring of endomorphisms is isomorphic to the ring of endomorphisms of the torsion part if and only if the group is a fully invariant subgroup of the cotorsion hull of its torsion part. In the considered case, the cotorsion hull is not fully transitive and hence it is necessary to introduce a new function which differs from an indicator and assigns an infinite matrix to each element of the cotorsion hull. The relation $\leq$ difined on the set $\Omega$ of these matrices is different from the relation proposed by the autor in the countable case and better discribes the lower semilattice $\bar{\Omega}$. The use of the relation $\leq$ essentially simplifies the verification of the required properties. It is proved that the lattice of fully invariant subgroups of the group $T^{\bullet}$ is isomorphic to the lattice of filters of the lower semilattice $\bar{\Omega}$.

**Keywords:** Lattice of Fully Invariant Subgroups; Direct Sum of Torsion-Complete Groups; Cotorsion Hull

## 1. Introduction

We consider questions of the theory of abelian groups and throughout the paper the word "group" means an additively written abelian group. The notation and terminology used in the text are borrowed from the two-volume monograph [1,2].

The symbol $p$ denotes a fixed prime number. $Z$ and $Q$ are respectively the groups of integer and rational numbers. If $p^n$ is the order of an element $a$ of the group, then the exponent of an element $a$ is equal to $n$ and written as $e(a)=n$. A subgroup $B$ of the group $A$ is called fully invariant if for any endomorphism of the group $A$ this subgroup $B$ is mapped into $B$. The examples of such subgroups are $nA=\{na \mid a\in A\}$, $A[n]=\{a \mid na=0, a\in A\}$, $n>0$, $n\in Z$, the torsion part of the group $A$.

The study of the lattice of fully invariant subgroups of a group is an important problem of the theory of abelian groups. For sufficiently wide classes of $p$-groups this topic is treated in [3-7] and in other papers. The works [8-13] and others are dedicated to the investigation of this question in torsion-free and mixed groups.

A group $A$ is called a cotorsion group if its extension by means of any torsion-free group $C$ splits, *i.e.*, $\operatorname{Ext}(C,A)=0$. The importance of the class of cotorsion groups in the theory of abelian groups is due to two facts (see [1, items 54, 55]): for any groups $A$, $B$, the group $\operatorname{Ext}(A,B)$ is a cotorsion group; any reduced group $A$ is isomorphically embeddable into the group $A^{\bullet}=\operatorname{Ext}(Q/Z,A)$ called the cotorsion hull of a group $A$ and, in addition, $A^{\bullet}/A$ is a divisible torsion-free group. Any reduced cotorsion group $A$ can be represented as the direct sum $A=T^{\bullet}\oplus C$ where $T^{\bullet}\cong\operatorname{Ext}(Q/Z,T)$, $T=tA$ is the torsion part of a group $A$ and $C$ is a torsion-free, algebraically compact group (see [14]). If $T=\oplus_p T_p$ is represented as a direct sum of primary components, then

$$\operatorname{Ext}(Q/Z,T)\cong\prod_p \operatorname{Ext}\left(Z\left(p^{\infty}\right),T_p\right).$$

The construction of algebraically compact groups is well known (see [4, item 40]). Hence the study of cotorsion groups reduces the study of groups of the form

$\operatorname{Ext}\left(Z\left(p^{\infty}\right),T\right)$ where $T$ is a $p$-primary group to a considerable extent.

Though the notion of a cotorsion group and its generalizations are studied sufficiently well (see [15-18]), little is known about the lattice of fully invariant subgroups of a cotorsion group. The investigation of this problem in the case of a cotorsion hull is important because endomorphisms in this class of groups are completely defined by their action on the torsion part and, as shown in [19], for mixed groups the ring of endomorphisms is isomorphic to the ring of endomorphisms of the torsion part if and only if the group is a fully invariant subgroup of the cotorsion hull of its torsion part. The study of the lattice of fully invariant subgroups makes essential use of the notions of an indicator and a fully transitive group.

By the $p$-indicator of an element $a$ of the group $A$ we mean an increasing sequence of ordinal numbers

$$H_A(a)\equiv H(a)=\left(h(a),h(pa),\cdots,h\left(p^n a\right),\cdots\right),$$

where $h$ denotes the generalized $p$-height of the element $a$, *i.e.* $h(a)=\sigma$ if $a\in p^{\sigma}A\setminus p^{\sigma+1}A$ and $h(0)=\infty$ (certainly, if $h\left(p^n a\right)=h(0)=\infty$, then $h\left(p^{n+1}a\right)=\infty$). In the set of indicators we can introduce the order

$$H(a)\le H(b)\Leftrightarrow h\left(p^i a\right)\le h\left(p^i b\right),\quad i=0,1,\cdots.$$

A reduced $p$-group is called fully transitive if for its arbitrary elements $a$ and $b$, when $H(a)\le H(b)$ there exists an endomorphism $\varphi$ of the group such that $\varphi a=b$. In fully transitive groups, the lattice of fully invariant subgroups is studied by means of indicators (see [2, Theorem 67.1]).

A. Mader [11] showed that an algebraically compact group is fully transitive and described by means of indicators of the lattice of fully invariant subgroups of an algebraically compact group. Moreover, he indicated the generalized conditions the fulfillment of which gives a description of the lattice of fully invariant submodules.

**Theorem 1.1 (A. Mader). 1** *Let $A$ be a module over a commutative ring $R$, $\Delta$ be the lattice of its fully invariant submodules, $\Omega$ be some lower semilattice, and $\Phi: A\to\Omega$ be the mapping with the following properties*:

1) $\Phi$ *is surjective*;
2) $\Phi(fa)\ge\Phi(a)$ $\forall a\in A$ *and* $f\in\operatorname{End}A$;
3) $\Phi(a+b)\ge\Phi(a)\wedge\Phi(b)$;
4) *if* $\Phi(a)\ge\Phi(b)$, *then there exists an endomorphism* $f$ *of the module* $A$ *such that* $f(b)=a$;
5) *if* $C\in\Delta$, *then for any* $a,b\in C$ *there exists* $c\in C$ *such that* $\Phi(c)=\Phi(a)\wedge\Phi(b)$.

*Then the set* $\Omega^*$ *of all filters of* $\Omega$, *which is ordered with respect to the inclusion*, *is a lattice and the mapping* $\alpha:\Omega^*\to\Delta$ *defined by the rule* $\alpha(D)=\{a\in A\,|\,\Phi(a)\in D\}$ *is a lattice isomorphism.*

In the same way as we did in $p$-groups we define the notion of full transitivity in the group $T^{\bullet}=\operatorname{Ext}\left(Z\left(p^{\infty}\right),T\right)$. If $T$ is a torsion-complete group, then its cotorsion hull is an algebraically compact group (see [1, item 56] and, as has been mentioned above, is fully transitive. A. Moskalenko [13] proved that when $T$ is the direct sum of cyclic $p$-groups, then $T^{\bullet}$ is also fully transitive and all the conditions of Theorem 1.1 are fulfilled. Therefore in this case, too, the lattice $\Omega^*$ of indicator filters describes the lattice of fully invariant subgroups. The direct sum of torsion-complete groups is a natural generalization of the direct sum of cyclic $p$-groups and torsion-complete groups. The author has shown in [20] that in this class of groups, if the sum is infinite, the cotorsion hull is not fully transitive. Therefore, because of condition 4) of Theorem 1.1 we cannot use indicators to describe the lattice of fully invariant subgroups. The lattice of fully invariant subgroups of $T^{\bullet}$ was studied in [21] when $T$ is the countable direct sum of torsion-complete groups. In the present paper, $T$ is an arbitrary direct sum of torsion-complete groups and the lower semilattice is defined by a simpler new relation $\le$ (see Definition 2.2) which makes it easier to verify the properties of Theorem 1.1.

## 2. A Semilattice $\bar{\Omega}$

Let a $p$-group $T$ be the direct sum of torsion-complete $p$-groups

$$T=\bigoplus_{i\in\mathcal{J}}\bar{B}_i, \tag{2.1}$$

where $B_i$ is a basic subgroup of the group $\bar{B}_i$ and $B=\bigoplus_{i\in\mathcal{J}}B_i$ is the basic subgroup of $T$. Assume that $\mathcal{J}$ is a fully ordered set of indexes. For a separable $p$-group $T$, A. Moskalenko [13] represented elements of the cotorsion hull $T^{\bullet}$ as countable sequences

$$T^{\bullet}=\left\{\left(a_0,a_1+T,\cdots,a_i+T,\cdots\right)\middle|\,a_i\in\hat{T},\right.$$
$$\left.pa_{i+1}-a_i\in T,\,i=0,1,\cdots\right\}.$$

Writing the element in this form, it is easy to calculate their height and indicator (see [21], (1.2))

Let $B=\bigoplus_{\alpha\in J}\langle x_\alpha\rangle$ be a fixed basic subgroup of a separable $p$-group $T$. If $a\in T^{\bullet}$, $a=\left(a_0,a_1+T,\cdots\right)$, then in the group $B$ there exists a sequence $\left(b_i\right)$, $i=0,1,\cdots$, such that for any $i$

$$b_i=\sum_{j=1}^{s}m_{i_j}x_{i_j},\ 0\le m_{i_j}<p \text{ and } a_i=\lim_{n\to\infty}\left(\sum_{s=0}^{n}p^s b_{i+s}\right). \tag{2.2}$$

This representation of an element $a$ is called canonical. The sequence $(b_i)$ is said to correspond to the canonical representation of $a$. The statements given below are true (see [13]).

**Proposition 2.1.** *If* $H_{T^\bullet}(a)=(k_0,k_1,\cdots)$, $(b_i)$ *is the sequence corresponding to the canonical representation of an element* $a$, *and between* $k_i$ *and* $k_{i+1}$ *there is a jump, then in the expansion* $b_{k_i-i}$ *with respect to the basis* $\{x_\alpha | \alpha \in \mathcal{J}\}$ *there is an element* $x_\alpha$ *of order* $p^{k_i+1}$.

**Proposition 2.2.** *If* $H_{T^\bullet}(a)$ *is a sequence of nonnegative integers, then it has an infinite number of jumps.*

Let a group $T$ be of form (2.1) and $a\in T^\bullet$. Denote by $\pi_i$ the projection of the group $T$ on the direct summand $\overline{B}_i$ and consider the sequence $\pi_i(b_j)=(b_{ij})$, $j=0,1,\cdots$ (see (2.2)). For each $i\ge 1$ and fixed $k$, the sequence $b_{i0},b_{i1},\cdots$ defines the element

$$a_{ik}=\lim_{n\to\infty}\sum_{s=0}^{n}p^s b_{i,k+s}, \tag{2.3}$$

and the elements $a_{i0},a_{i1},\cdots$ of the group $\hat{B}_i$ define the element $a^{(i)}=(a_{i0},a_{i1}+T,a_{i2}+T,\cdots)$ of the group $T^\bullet$. It is obvious that

$$a_k=\lim_{n\to\infty}\sum_{i=1}^{n}a_{ik}. \tag{2.4}$$

(Here we have separated two indexes $i$ and $k+s$ by the comma and we will sometimes do so in the sequel in order to distinguish their order). Note that the elements $a^{(i)}$, $i=1,2,\cdots$, are uniquely defined by an element $a$ (see [13], item 1.2)

To every element $a\in T^\bullet$ we put into correspondence the matrix

$$\Phi(a)=\left\|H(a_{i0})\right\|_{i\ge 0}, \tag{2.5}$$

where $H(a_{00})=H_{T^\bullet}(a)$, $H(a_{i0})=H_{\hat{T}}(a_{i0})$ for $i\ge 1$ and the indicators are written in a column.

**Definition 2.1.** The matrix $\|k_{ij}\|$, $i\in\mathcal{J}$, $j=0,1,\cdots$, made up of ordinal numbers and symbols $\infty$ is called admissible with respect to the group $T$ if the following conditions are fulfilled:

1) The 0th row $k_{00},k_{01},\cdots$ is an increasing sequence of ordinal numbers so that $k_{0i}<\omega+\omega$ or $k_{0i}=\infty$. If $k_{0j}\ge\omega$, then $k_{0n+1}=k_{0n}+1$ for any $n\ge j$, whereas the other rows are increasing sequences of nonnegative integer numbers or symbols $\infty$ (Here $\omega$ is the smallest infinite ordinal number and it is assumed that $\infty+1=\infty$).

2) If $k_{0n}=\omega+m$ is the first infinite ordinal number and $m<n$, then infinitely many rows contain a nonnegative integer number and there exists a row $i_0$ such that $k_{i,n-m}=\infty$ for $i\ge i_0$. If $k_{0n}=\omega+m$, $m\ge n$, then starting from some $i_0$ all rows consist only of symbols $\infty$.

3) If all elements in a row are nonnegative integers, then this row contains infinitely many jumps.

4) If between $k_{ij}$ and $k_{i,j+1}$ there is a jump, then in the group $B_i$ there exists a bases element of order $p^{k_{ij}+1}$ (it is assumed that $B_0=B$).

5) In each column $k_{ij}\to\infty$ as $i\to\infty$ (*i.e.* $i$ runs through all values of any fully completely ordered set $\mathcal{J}$); also, if $k_{0j}\ne\omega+m$, then $k_{0j}=\min\{k_{1j},k_{2j},\cdots\}$, and if $k_{0j}=\omega+m$, then $k_{1j}=k_{2j}=\cdots=\infty$.

Taking into account equality (2.1) and Propositions 2.1, 2.2, we notice that the matrix $\Phi(a)$ satisfies the above conditions for any $a\in T^\bullet$.

From Definition 2.1 it follows that we deal with matrices of the following three types:

$$\text{I.}\quad \begin{bmatrix} k_{00} & k_{01} & \cdots \\ k_{10} & k_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

where $k_{ij}$ are nonnegative integer numbers or symbols $\infty$;

$$\text{II.}\quad \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0n-1} & \omega+m & \omega+m+1 & \cdots \\ k_{10} & k_{11} & \cdots & k_{1n-1} & \infty & \infty & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ k_{t0} & k_{t1} & \cdots & k_{tn-1} & \infty & \infty & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix},$$

where $m<n$ and $k_{ij}$ are nonnegative integer numbers (see the first sentence in item 2 of Definition 2.1);

III.

$$\begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0n-1} & \omega+n+k & \omega+n+k+1 & \cdots \\ k_{10} & k_{11} & \cdots & k_{1n-1} & \infty & \infty & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ k_{i-10} & k_{i-11} & \cdots & k_{i-1\,n-i} & \infty & \infty & \cdots \\ \infty & \infty & \cdots & \infty & \infty & \infty & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $k_{ij}$ are nonnegative integer numbers (see the second sentence in item 2 of Definition 2.1).

Note that if $a=(a_0,a_1+T,\cdots)\in T^\bullet$ is given and $(b_i)$ is the sequence corresponding to a canonical representation of an element $a$, then $a_0=b_0+pb_1+p^2b_2+\cdots$ and each $b_i$ is the sum of finitely many bases elements $x_\alpha$ (see (2.2)). Hence it follows that in any matrix $\Phi(a)$, at most a countable number of rows is different from $\infty,\infty,\cdots$. Moreover, by virtue of the fourth condition of Definition 2.1, from the given *i*th row of an admissible matrix it is easy to find an element $a_{i0}\in\hat{T}$ the indicator of which is equal to this given row. Let for example

$$\left(k_{10}, k_{11}, k_{12}, \cdots\right) = \left(1_{10}, 2_{11}, 3_{12}, 5_{13}, 6_{14}, 9_{15}, 10_{16}, 11_{17}, 12_{18}, 15_{19}, 17_{1,10}, 18_{1,11}, \cdots\right) \quad (2.6)$$

(it is obvious that here $1_{10} = 1$ and the indexes are marked in order to determine in which row and column an element lies). Jumps here occur at positions $\left(3_{12}, 5_{13}\right)$, $\left(6_{14}, 9_{15}\right)$, $\left(12_{18}, 15_{19}\right)$, $\left(15_{19}, 17_{1,10}\right)$. Then, by virtue of the fourth condition of Definition 2.1 (see also Proposition 2.1), in the canonical representation of $a_{10}$ there is the element $b_{1,3_{12}-2} = b_{11}$ containing a basis element of order $p^{3+1} = p^4$. Just in the same way $b_{1,6_{14}-4} = b_{12}$ contains a basic element of order $p^{6+1} = p^7$; $b_{1,12_{18}-8} = b_{14}$ contains an element of order $p^{12+1} = p^{13}$; $b_{1,15_{19}-9} = b_{16}$ an element of order $p^{15+1} = p^{16}$, $b_{1,18_{1,11}-11} = b_{17}$ an element of order $p^{18+1} = p^{19}$ and so on. Therefore

$$a_{10} = pb_{11} + p^2 b_{12} + p^4 b_{14} + p^6 b_{16} + p^7 b_{17} + \cdots.$$

The indicator $H\left(a_{10}\right)$ is obviously equal to sequence (2.6) and $a_{i0}$ is not the only element the indicator of which is equal to (2.6).

Denote by $\Omega$ the set of admissible (with respect to $T$) matrices and define, on the set $\Omega$ the following relation $\le$ different from the relation given in [21, Definition 1.2].

**Definition 2.2.** Let $K = \left\|k_{ij}\right\|$, $K' = \left\|k'_{ij}\right\|$, $i \in \mathcal{J}$, $j = 0,1,\cdots$, be the elements of the set $\Omega$. We say that $K \le K'$ if $k_{0i} \le k'_{0i}$, $i = 0,1,\cdots$, and to each element $k'_{ij}$ $(i \ge 1)$ where there occurs a jump we can put into correspondence the element $k_{mn}$ $(m \ge 1)$ where there also occurs a jump so that $k_{mn} - n \le k'_{ij} - j$ and $n \ge j$. Then the following two conditions are fulfilled:

1) Each element $k_{mn}$ where there is a jump has finitely many (possibly none) pre-images.

2) If $k_{ij_0} k_{ij_1}, \cdots$, are infinitely many elements of the $i$th row of the matrix $K$ which are different from the symbol $\infty$ and $k'_{m_0 n_0}, k'_{m_1 n_1}, \cdots$ are respectively their pre-images such that the sequence of the numbers of rows $m_0, m_1, \cdots$ infinitely increases, then $j_k - n_k \to \infty$ as $k \to \infty$.

It can be easily verified that the relation $\le$ on the set $\Omega$ is reflexive and transitive. However, as seen from the next simple example, it is not anti-symmetric. Indeed, let $U = \left\|u_{ij}\right\|$, $V = \left\|v_{ij}\right\|$, $i \in \mathcal{J}$, $j = 0,1,\cdots$, be admissible matrices all rows of which, except for the third one, are identical. Consider the table

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_{0j} = v_{0j}$ | 1 | 2 | 3 | 5 | 7 | 8 | 9 | 10 | 12 | 14 | 15 |
| $u_{1j} = v_{1j}$ | 1 | 2 | 3 | 6 | 7 | 9 | 10 | 12 | 13 | 14 | 15 |
| $u_{2j} = v_{2j}$ | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 | 14 | 15 |
| $u_{3j}$ | 4 | 5 | 6 | 8 | 9 | 10 | 12 | 13 | 14 | 16 | 17 |
| $v_{3j}$ | 5 | 6 | 9 | 10 | 12 | 13 | 17 | 18 | 19 | 21 | 22 |

| $j$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $u_{0j} = v_{0j}$ | 17 | 19 | 20 | 25 | 26 | 27 | 29 | 30 | $\cdots$ |
| $u_{1j} = v_{1j}$ | 18 | 19 | 20 | 25 | 26 | 27 | 29 | 30 | $\cdots$ |
| $u_{2j} = v_{2j}$ | 17 | 20 | 21 | 26 | 27 | 28 | 30 | 35 | $\cdots$ |
| $u_{3j}$ | 18 | 23 | 24 | 27 | 28 | 30 | 31 | 36 | $\cdots$ |
| $v_{3j}$ | 23 | 27 | 28 | 29 | 31 | 32 | 37 | 38 | $\cdots$ |

Let us assume that the elements lying at the positions of dots in the matrices $U$ and $V$ are identical. To the elements of the third row of the matrix $U$, where there are jumps, we put into correspondence the elements of the second row of the matrix $V$. Thus

$$6_{32} \to 5_{23},\ 10_{35} \to 10_{27},\ 14_{38} \to 12_{28},\ 18_{3,11} \to 17_{2,11},$$
$$24_{3,13} \to 21_{2,13},\ 28_{3,15} \to 30_{2,17},\ 31_{3,17} \to 30_{2,17}.$$

To the elements of the third row of the matrix $V$ where there are jumps there correspond the elements of the third row of the matrix $U$ in the following manner:

$$6_{31} \to 6_{32},\ 10_{33} \to 10_{35},\ 13_{35} \to 10_{35},\ 19_{38} \to 18_{3,11},$$
$$23_{3,11} \to 24_{3,13},\ 29_{3,14} \to 31_{3,17},\ 32_{3,16} \to 31_{3,17}.$$

It is obvious that $U \le V$ and $V \le U$, whereas $U \ne V$. Therefore the relation $\le$ on the set $\Omega$ is not anti-symmetric. Then

$$U \rho V \underset{def}{=} \left[U \le V \text{ and } V \le U\right]$$

is the relation of equivalence on the set $\Omega$, whereas

$$\bar{U} \le \bar{V} \underset{def}{=} U \le V$$

defined on the factor set $\bar{\Omega} = \Omega/\rho$ is the relation of order.

Let $U = \left\|u_{ij}\right\|$, $V = \left\|v_{ij}\right\|$, $i \in \mathcal{J}$, $j = 0,1,\cdots$, be admissible matrices. We denote $W = U \wedge V = \left\|\min\left(u_{ij}, v_{ij}\right)\right\| = \left\|w_{ij}\right\|$ and will show that $W$ is also an admissible matrix. Let

$$\left(u_{i0}, u_{i1}, \cdots\right), \quad (2.7)$$

$$\left(v_{i0}, v_{i1}, \cdots\right) \quad (2.8)$$

be respectively the $i$th rows of the matrices $U$ and $V$ where there occur infinitely many jumps. Let us show that then

$$\left(\min\left(u_{i0},v_{i0}\right),\min\left(u_{i1},v_{i1}\right),\cdots\right) \quad (2.9)$$

is also an increasing sequence of nonnegative integer numbers where there are infinitely many jumps. Assume the contrary: let, starting from some number $s$ in (2.9), there are no jumps and $w_{is}=u_{is}\le v_{is}$. Assume that $\left(u_{ij},u_{i,j+1}\right)$ is the first jump to the right from $u_{is}$ in (2.7), and $\left(v_{im},v_{i,m+1}\right)$ is the first jump to the right from $v_{is}$ in sequence (2.8) and $j\ge m$. Then $w_{i,s+1}=u_{i,s+1},\cdots,w_{ij}=u_{ij}$, $w_{i,j+1}<u_{i,j+1}$ and, obviously, $w_{i,j+1}<v_{i,j+1}$. If $j<m$, then $w_{i,m+1}<u_{i,m+1}$ and $w_{i,m+1}<v_{i,m+1}$, which contradicts the definition of $W$. Therefore the third condition of Definition 2.1 is fulfilled.

Let $\left(w_{ij},w_{i,j+1}\right)$ be a jump in (2.9) and $w_{ij}=u_{ij}$. Then $u_{i,j+1}\ge w_{i,j+1}>w_{ij}+1=u_{ij}+1$ and $\left(u_{ij},u_{i,j+1}\right)$ is a jump in (2.7). Then $B_{i,u_{ij}+1}=B_{i,w_{ij}+1}\ne 0$, *i.e.* the fourth requirement of Definition 2.1 is also fulfilled. The fulfillment of the remaining conditions of Definition 2.1 is obvious. Therefore $W$ is an admissible matrix.

It is not difficult to verify that $W\le U$ and $W\le V$. Moreover, if $K=\left\|k_{ij}\right\|\le U$ and $K\le V$, then $K\le W$. Let now $\bar{U},\bar{V}\in\bar{\Omega}$, where $U$ and $V$ are admissible matrices. Let us define the exact lower bound of $\bar{U}$ and $\bar{V}$ as follows: $\inf\left(\bar{U},\bar{V}\right)=\bar{U}\wedge\bar{V}=\bar{W}$ where $W=\left\|\min\left(u_{ij},v_{ij}\right)\right\|$, $i\in\mathcal{J}$, $j=0,1,\cdots$. If $U'=\left\|u'_{ij}\right\|\in\bar{U}$ and $V'=\left\|v'_{ij}\right\|\in\bar{V}$, then, by virtue of the above properties, $W'=\left\|\min\left(u'_{ij},v'_{ij}\right)\right\|\le U'\le U$ and $W'\le V'\le V$. Hence $W'\le W$ and, by symmetry, $W\le W'$, *i.e.* $\bar{W}=\bar{W}'$, which shows that the definition of $\bar{W}=\bar{U}\wedge\bar{V}$ is reasonable. Since $W\le U$, $W\le V$, we have $\bar{W}\le\bar{U}$, $\bar{W}\le\bar{V}$, and if $\bar{K}\le\bar{U}$, $\bar{K}\le\bar{V}$, we have $K\le U$, $K\le V$, $K\le W$, *i.e.* $\bar{K}\le\bar{W}$. Thus all conditions of the definition of the exact lower bound are fulfilled. Therefore the set $\bar{\Omega}$ with relation $\le$ is the lower semilattice.

## 3. The Lattice of Fully Invariant Subgroups of the Group $T^{\bullet}$

Let us show that the function $\bar{\Phi}:T^{\bullet}\to\bar{\Omega}$, $\bar{\Phi}(a)=\bar{\Phi(a)}$, where $T$ has form (2.1) and $\Omega$ is the set of all admissible matrices with respect to $T$, satisfies all conditions of Theorem 1.1.

**Condition 1.** $\bar{\Phi}$ *is surjective.*

*Proof.* Let $\bar{K}\in\bar{\Omega}$, $K=\left\|k_{ij}\right\|$, $i\in\mathcal{J}$, $j=0,1,\cdots$, and the 0th row consist of nonnegative integer numbers. For any $i\ge 1$ th row, denote all jumps by $\left(k_{ii_s},k_{ii_s+1}\right)$, $s=1,2,\cdots$. By virtue of the admissibility of the matrix $K$, for each jump of this kind we can choose in the basis $\left\{x_{i\alpha}\middle|\alpha\in\mathcal{J}_i\right\}$ of the group $B_i$ an element $x_{is}$ of order $p^{k_{ii_s}+1}$. Let $b_{ii_s}-i_s=\lambda_{is}$. Denote $c_{is}=p^{\lambda_{i1}}x_{i1}+p^{\lambda_{i2}}x_{i2}+\cdots+p^{\lambda_{is}}x_{is}$, then $a_{i0}=\lim_{s\to\infty}c_{is}$; $a_{i0}=\hat{B}_i$. Taking into account that $\lambda_{i1}=k_{ii_1}-i_1=k_{i0}$ and for every $s$, $\lambda_{is+1}=k_{ii_s+1}-i_{s+1}=k_{ii_s+1}-\left(i_s+1\right)$, we see that $H_{\hat{B}_i}\left(a_{i0}\right)=\left(k_{i0},k_{i1},\cdots\right)$. Further, since $\lambda_{i1}<\lambda_{i2}<\cdots$ and the matrix is admissible, for each fixed $t$ we have $\lambda_{it}=k_{ii_t}-i_t\to\infty$ as $i\to\infty$. Now we can define the element $a_0=\lim_{s\to\infty}\sum_{i=1}^{s}a_{i0}\in\hat{T}$, $H_{\hat{T}}\left(a_0\right)=\left(k_{00},k_{01},\cdots\right)$ and $\mathcal{O}\left(a_0\right)=\infty$. Since $\hat{T}/T$ is a divisible group, there are elements $a_1,a_2,\cdots\in\hat{T}$ such that for any $i=0,1,\cdots$ we have $pa_{i+1}+T=a_i+T$. Let $a=\left(a_0,a_1+T,\cdots\right)$, then $a\in T^{\bullet}$ and $\Phi(a)=K$, $\bar{\Phi}(a)=\bar{K}$.

Now assume that the matrix $K$ is of type II. Then its 0th row has the form

$$\left(k_{00},k_{01},\cdots,k_{0n-1},\omega+m,\omega+m+1,\cdots\right)$$

where $m<n$ and there exists an index $i_0$ such that $k_{in-m}=\infty$ for every $i\ge i_0$. Like in the preceding case, for each $i\ge 1$ -th row there exists an element $a_{i0}=p^{\lambda_{i1}}x_{i1}+p^{\lambda_{i2}}x_{i2}+\cdots+p^{\lambda_{is_i}}x_{is_i}\in B_i$. Here the number of summands is finite since the row has finitely many jumps. Now, taking into account that $\lambda_{i1}<\lambda_{i2}<\cdots<\lambda_{is_i}$ and the matrix $K$ is admissible, we obtain $\lambda_{it}=k_{ii_t}-i_t\to\infty$ as $i\to\infty$, $i_t\le n-1$. Then we can define the element

$$a_0=\lim_{k\to\infty}\sum_{j=1}^{k}\left(p^{\lambda_{j1}}x_{j1}+p^{\lambda_{j2}}x_{j2}+\cdots+p^{\lambda_{js_j}}x_{js_j}\right)\in\hat{T}$$

$s_j\le n-1$ and $H_{\hat{B}}\left(a_0\right)=\left(k_{00},k_{01},\cdots,k_{0n-1},\infty,\cdots\right)$. Denote

$$a_k=\lim_{t\to\infty}\sum_{j=1}^{t}\left(p^{\lambda_{j1}-k}x_{j1}+p^{\lambda_{j2}-k}x_{j2}+\cdots+p^{\lambda_{js_j}-k}x_{js_j}\right)$$

assuming that that $p^{\lambda_{jr}-k}x_{jr}=0$ when $l>\lambda_{jr}$. Then it is obvious that $pa_{k+1}-a_k\in T$, $k=0,1,\cdots$. Consider an element $a=\left(a_0,a_1+T,\cdots\right)\in T^{\bullet}$. It easily follows that $\Phi(a)=K$. Therefore $\bar{\Phi}(a)=\bar{K}$.

If $K$ is a matrix of form III, then, starting from some $t$-th row, every row consists only of symbols $\infty$. We choose a row $\left(k_{i0},k_{i1},\cdots\right)$, $1\le i<t$, and, just in the same way as in the preceding case, find $a_{i0}=p^{\lambda_{i1}}x_{i1}+\cdots+p^{\lambda_{is_i}}x_{is_i}$, $x_{ik}\in B_i$, $k=1,2,\cdots,s_i$, and $H_{\bar{B}_i}\left(a_{i0}\right)=\left(k_{i0},k_{i1},\cdots,k_{in-1},\infty,\cdots\right)$. Let us consider an

element

$$a_0=\sum_{j=1}^{t-1}\left(p^{\lambda_{j1}}x_{j1}+\cdots+p^{\lambda_{js_j}}x_{js_j}\right)\in T\ ,$$

$$\mathcal{O}(a_0)=p^n,\ \ s_j\le n-1\,.$$

Then $H_T(a_0)=(k_{00},k_{01},\cdots,k_{0n-1},\infty,\cdots)$. Since $\hat{T}/T$ is a divisible group and $T\ne t\hat{T}$, there exists in $\hat{T}/T$ a quasi-cyclic divisible subgroup. Let $\{g_i+T\,|\,i=1,2,\cdots\}$ be its system of generators such that $pg_1\in T$ and for every $i$, $pg_{i+1}-g_i\in T$. Since $T$ is a pure subgroup in $\hat{T}$, it can be assumed that $\mathcal{O}(g_i)=p^i$ for each $i=1,2,\cdots$. Now let $a_1=a_2=\cdots=a_k=0,\quad a_{k+i}=g_i$ and consider the element $a=(a_0,a_1+T,\cdots,a_k+T,a_{k+1}+T,\cdots)$. It is obvious that $\Phi(a)=K$, *i.e.* $\bar{\Phi}(a)=\bar{K}$. Condition 1 is proved.

**Condition 2.** *If* $a\in T^\bullet$ *and* $f^\bullet\in \mathrm{End}T^\bullet$, *then* $\bar{\Phi}(a)\le\bar{\Phi}(f^\bullet a)$.

*Proof.* Let $a=(a_0,a_1+T,\cdots)$, $c=(c_0,c_1+T,\cdots)\in T^\bullet$ and there exist an endomorphism $f^\bullet$ of the group $T^\bullet$ such that $f^\bullet a=c$. As is known (see [22, item 1.5]), $f^\bullet$ is induced by the endomorphism $f$ of the group $T$ which in its turn induces an endomorphism $\hat{f}$ of the algebraically compact group $\hat{T}$ such that

$$f^\bullet a=\left(\hat{f}a_0,\hat{f}a_1+T,\cdots\right)=(c_0,c_1+T,\cdots)=c\,.$$

Let $(b_i)$ and $(d_i)$ be respectively the sequences corresponding to canonical representations of the elements $a$ and $c$. Then (see (2.3), (2.4))

$$a_0=\sum_{i\in\mathcal{J}}a_{i0},\quad a_{i0}=\lim_{n\to\infty}\sum_{s=0}^{n}p^s b_{is},$$

where $i$ runs through at most a countable set of increasing indexes. Since $\hat{T}$ is an algebraically compact group, we have

$$\hat{f}a_0=\sum_{i\in\mathcal{J}}\hat{f}a_{i0}=c_0=\sum_{j\in\mathcal{J}}c_{j0}. \tag{3.1}$$

Let $\Phi(a)=\|k_{ij}\|,\ \ \Phi(c)=\|k'_{ij}\|,\ \ i\in\mathcal{J},\ \ j=0,1,\cdots$, and assume that there is a jump at the position $k'_{ij}$, $i\ge1$. Then in a component $c_{i0}$ of the element $c_0$ there exists an element $p^{k'_{ij}-j}d_{i,k'_{ij}-j}$ of the exponent $j+1$ which, by virtue of equality (3.1), is obtained by mapping the sum of a finite number of summands of the element $a_0$ under the homomorphism $\hat{f}$ and by projecting this mapping on a basis element of the subgroup $B_i$ that has the exponent $k'_{ij}+1$. Denoting this projection by $\pi_{i,k'_{ij}+1}$, we have

$$\begin{aligned}&\pi_{i,k'_{ij}+1}\hat{f}\left(p^{k_{i_1j_1}-j_1}b_{i_1,k_{i_1j_1}-j_1}+\cdots+p^{k_{i_sj_s}-j_s}b_{i_s,k_{i_sj_s}-j_s}\right)\\&=p^{k'_{ij}-j}d_{i,k'_{ij}-j},\end{aligned} \tag{3.2}$$

where the above-mentioned sum of a finite number of summands is enclosed in the brackets. It is obvious that the height of such a summand is less than or equal to $k'_{ij}-j$, whereas the exponent is greater than or equal to $k+1$. Hence without loss of generality we put into correspondence to $k'_{ij}$ an element $k_{i_sj_s}$ of the largest exponent. Thus, to each element $k'_{ij}$ of the matrix $K'$ where there is a jump we put into correspondence an element of the matrix $K$.

Let $k_{ij_0},k_{ij_1},\cdots$ be elements $i\ge1$ of a row of the matrix $K$ where there are jumps, and $k'_{m_0n_0},k'_{m_1n_1},\cdots$ be respectively their preimages for the above-mentioned correspondence so that a sequence of numbers of the rows $m_0,m_1,\cdots$ of the matrix $K'$ increases infinitely. Taking into account that $\hat{T}$ is an algebraically compact group and its torsion part is a torsion-complete group, by virtue of equality (3.2) we assume without loss of generality that

$$\pi_{m_t,k'_{m_tn_t}+1}\hat{f}p^{k_{ij_t}-j_t}b_{i,k_{ij_t}-j_t}=p^{k'_{m_tn_t}-n_t}d_{m_t,k'_{m_tn_t}-n_t},\quad t=0,1,\cdots.$$

We have

$$\begin{aligned}&\left(\cdots+\pi_{m_1,k'_{m_1n_1}+1}+\pi_{m_0,k'_{m_0n_0}+1}\right)\\&\times\hat{f}\left(p^{k_{ij_0}-j_0}b_{i,k_{ij_0}-j_0}+p^{k_{ij_1}-j_1}b_{i,k_{ij_1}-j_1}+\cdots\right)\\&=p^{k'_{m_0n_0}-n_0}d_{m_0,k'_{m_0n_0}-n_0}+p^{k'_{m_1n_1}-n_1}d_{m_1,k'_{m_1n_1}-n_1}+\cdots.\end{aligned} \tag{3.3}$$

Since $\hat{f}$ induces an endomorphism on $T$, and $\pi_{m_t,k'_{m_tn_t}+1}$ are projections,

$$\hat{\psi}=\left(\cdots+\pi_{m_1,k'_{m_1n_1}+1}+\pi_{m_0,k'_{m_0n_0}+1}\right)\hat{f}$$

induces an endomorphism on the subgroup $T$. Let us fix a positive integer number $m$ and consider an element of order $p^m$ from $T$:

$$a_0^*=p^{k_{ij_0}-m+1}b_{i,k_{ij_0}-j_0}+p^{k_{ij_1}-m+1}b_{i,k_{ij_1}-j_1}+\cdots.$$

If here the initial summands $k_{ij_s}-m+1<0$, then we assume that these summands are zero. Then, in view of (3.3),

$$\begin{aligned}\hat{\psi}a_0^*&=\hat{\psi}\left(p^{k_{ij_0}-j_0+j_0-m+1}b_{i,k_{ij_0}-j_0}+p^{k_{ij_1}-j_1+j_1-m+1}b_{i,k_{ij_1}-j_1}+\cdots\right)\\&=p^{k'_{m_0n_0}-n_0+j_0-m+1}d_{m_0,k'_{m_0n_0}-n_0}+p^{k'_{m_1n_1}-n_1+j_1-m+1}d_{m_1,k'_{m_1n_1}-n_1}+\cdots.\end{aligned} \tag{3.4}$$

Since, by condition, a sequence of numbers of the rows $m_0,m_1,\cdots$ of the matrix $K'$ increases infinitely, only a finite number of summands on the right-hand part of equality (3.4) must differ from zero; otherwise the element $\hat{\psi}a_0^*$ does not belong to $T$. Therefore for each concrete positive integer $m$, starting from some $t$ we have $j_t-n_t-m\ge0$, $j_t-n_t\ge m$, *i.e.* $j_t-n_t\to\infty$ as $t\to\infty$. Thus the second condition of Definition 2.2 is fulfilled too. Therefore $\Phi(a)\le\Phi(f^\bullet a)$ and

$\bar{\Phi}(a) \le \bar{\Phi}(f^{\bullet} a)$. Condition 2 is proved.

**Condition 3.** *For any* $a, c \in T^{\bullet}$,

$$\bar{\Phi}(a+c) \ge \bar{\Phi}(a) \wedge \bar{\Phi}(c)$$

*Proof.* Let $a = (a_0, a_1 + T, \cdots)$, $c = (c_0, c_1 + T, \cdots)$ be the elements of the group $T^{\bullet}$. Then (see [13, item 1]) $a + c = (a_0 + c_0, a_1 + c_1 + T, \cdots)$. Denote (see (2.5))

$$\Phi(a) = \|H(a_{i0})\|, \quad \Phi(c) = \|H(c_{i0})\|,$$

$$\Phi(a) \wedge \Phi(c) = \|k_{ij}\|, \quad \Phi(a+c) = \|H(a_{i0} + c_{i0})\| = \|k'_{ij}\|,$$

$i \in \mathcal{J}$, $j = 0, 1, \cdots$. By virtue of the properties of the indicator (see [1, item 37]),

$$(k'_{i0}, k'_{i1}, \cdots) = H(a_{i0} + c_{i0}) \ge H(a_{i0}) \wedge H(c_{i0}) = (k_{i0}, k_{i1}, \cdots)$$

for any $i \in \mathcal{J}$. Let $i \ge 1$ and at $k'_{is}$ there be a jump, then $k'_{is} \ge k_{is}$. If to the right from $k_{is}$ the first jump occurs at the position $k_{is+t}$, then $k_{is+t} - (s+t) \le k'_{is} - s$ and $s + t \ge s$. To the element $k'_{is}$ we put into correspondence the element $k_{is+t}$. For this correspondence, if $k_{is_1}, k_{is_2}, \cdots$ are the elements of the *i*th row where there occur jumps, then their pre-images lie in the same *i*th row of the matrix $\|k'_{ij}\|$ and therefore the second condition of Definition 2.2 will be fulfilled, too, *i.e.* $\Phi(a+c) \ge \Phi(a) \wedge \Phi(c)$ or $\bar{\Phi}(a+c) \ge \bar{\Phi}(a) \wedge \bar{\Phi}(c)$. Condition 3 is proved.

**Condition 4.** *If* $a, c \in T^{\bullet}$ *and* $\bar{\Phi}(a) \le \bar{\Phi}(c)$, *then there exists an endomorphism* $\gamma^{\bullet}$ *of the group* $T^{\bullet}$ *such that* $\gamma^{\bullet} a = c$.

*Proof.* Let $a = (a_0, a_1 + T, \cdots)$, $c = (c_0, c_1 + T, \cdots)$ and

$$a_i = \lim_{n \to \infty} \sum_{s=0}^{n} p^s b_{i+s}, \; c_i = \lim_{n \to \infty} \sum_{s=0}^{n} p^s d_{i+s}, \; i = 0, 1, \cdots,$$

where $(b_i)$, $(d_i)$ are respectively the sequences corresponding to their canonical representations.

1) Denote $\Phi(a) = \|k_{ij}\|$, $\Phi(c) = \|k'_{ij}\|$, $i \in \mathcal{J}$, $j = 0, 1, \cdots$, and assume that the sequence $(k_{00}, k_{01}, \cdots)$ contains only nonnegative integer numbers. Let $k_{ij_1}, k_{ij_2}, \cdots$ be the elements of the $i$-row which have pre-images in the matrix $\Phi(c)$. Denote the pre-images of the element $k_{ij_s}$ as follows

$$k'_{i_1^{(s)} n_1^{(s)}}, k'_{i_2^{(s)} n_2^{(s)}}, \cdots, k'_{i_{m(s)}^{(s)} n_{m(s)}^{(s)}}, \quad s = 1, 2, \cdots. \tag{3.5}$$

Then the element $a_{i0}$ has the summand $p^{k_{ij_s} - j_s} b_{i, k_{ij_s} - j_s}$, where $b_{i, k_{ij_s} - j_s}$ contains, as a summand, the basis element $x_{s(i)} = x_s$ of the exponent $k_{ij_s} + 1$. Denote

$$A_i = \left\{ x_{s(i)} \middle| i \in \mathcal{J}, \; s = 1, 2, \cdots \right\},$$

$$\lambda_s = k_{ij_s} - j_s, \; \lambda_m^{(s)} = k'_{i_m^{(s)} n_m^{(s)}} - n_m^{(s)}.$$

We will show that for each $(i, j_s)$ there exists an endomorphism $\gamma_i$ of the group $T$ such that

$$\gamma_i \left( b_{i\lambda_s} + p b_{i, \lambda_s + 1} + \cdots + p^{\lambda_{s+1} - \lambda_s - 1} b_{i, \lambda_{s+1} - 1} \right) = p^{\lambda_1^{(s)} - \lambda_s} d_{i_1^{(s)} \lambda_1^{(s)}} + p^{\lambda_2^{(s)} - \lambda_s} d_{i_2^{(s)} \lambda_2^{(s)}} + \cdots + p^{\lambda_{m(s)}^{(s)} - \lambda_s} d_{i_{m(s)}^{(s)} \lambda_{m(s)}^{(s)}} \tag{3.6}$$

and $\gamma_i(x) = 0$ for any basic element $x$ contained in the sequence $(b_i)$ of the canonical representation of $a$ when $x \notin A_i$.

Note that when $s = 1$, the summands of the element

$$b_{i\lambda_1} + p b_{i, \lambda_1 + 1} + \cdots + p^{\lambda_2 - \lambda_1 - 1} b_{i, \lambda_2 - 1}$$

do not contain the basis element $x_s$ when $s \ge 2$. Indeed, when $t < \lambda_2 - \lambda_1$, we have $\lambda_1 + t < \lambda_2 = k_{ij_2} - j_2 = k_{ij_1 + 1} - (j_1 + 1)$. Hence

$$\lambda_1 + t + j_1 + 1 < k_{i, j_1 + 1}. \tag{3.7}$$

On the other hand, $h_{\hat{T}}(p^{j_1 + 1} a_{i0}) = k_{i, j_1 + 1}$ and, if we take into account the definition of the height and the representation of the element $a_{i0}$, then for each $n < k_{i, j_1 + 1}$ we will have $p^m b_{i, m - (j_1 + 1)} = 0$. Therefore, by (3.7),

$$p^{\lambda_1 + t + j_1 + 1} b_{i, \lambda_1 + t + j_1 + 1 - (j_1 + 1)} = p^{\lambda_1 + t + j_1 + 1} b_{i, \lambda_1 + t} = 0,$$

but for each $s > 1$

$$\lambda_1 + t + j_1 + 1 < \lambda_2 + j_1 + 1 = k_{ij_2} - j_2 + j_1 + 1 = k_{ij_2} + 1 - (j_2 - j_1) < k_{ij_s} + 1 = e(x_s).$$

Denote by $m_s^{(t)}$ the coefficient with which $x_s$ is contained in the expansion of the element $b_t$. Recall that $0 \le m_s^{(t)} < p$. The condition

$$\gamma_i \left( b_{i\lambda_1} + p b_{i, \lambda_1 + 1} + \cdots + p^{\lambda_2 - \lambda_1 - 1} b_{i, \lambda_2 - 1} \right) = \left( m_1^{(\lambda_1)} + p m_1^{(\lambda_1 + 1)} + \cdots + p^{\lambda_2 - \lambda_1 - 1} m_1^{(\lambda_2 - 1)} \right) \gamma_i x_1 = p^{\lambda_1^{(1)} - \lambda_1} d_{i_1^{(1)} \lambda_1^{(1)}} + p^{\lambda_2^{(1)} - \lambda_1} d_{i_2^{(1)} \lambda_2^{(1)}} + \cdots + p^{\lambda_{m(1)}^{(1)} - \lambda_1} d_{i_{m(1)}^{(1)} \lambda_{m(1)}^{(1)}} \tag{3.8}$$

must be fulfilled for the sought homomorphism $\gamma_i$. Since $x_1$ participates in the expansion $b_{i\lambda_1}$ with respect to the basis, we have $\left( m_1^{(\lambda_1)}, p \right) = 1$. Therefore $\gamma_i x_1$ can be uniquely defined in the subgroup $B$ from (3.8) if

$$e(\gamma_i x_1) \le k_{ij_1} + 1. \tag{3.9}$$

But this inequality holds true because the pre-images of $k_{ij_1}$ are (3.5) (for $s = 1$) and, by the definition of the relation $\le$ between the matrices $\Phi(a)$ and $\Phi(c)$, for each $t = 1, 2, \cdots, m(1)$ we have

$$p^{k_{ij_1} + 1} \left( p^{\lambda_t^{(1)} - \lambda_1} d_{i_t^{(1)} \lambda_t^{(1)}} \right) = p^{k_{ij_1} + 1 + k'_{i_t^{(1)} n_t^{(1)}} - n_t^{(1)} - k_{ij_1} + j_1} d_{i_t^{(1)} \lambda_t^{(1)}} = p^{k'_{i_t^{(1)} n_t^{(1)}} + 1 + j_1 - n_t^{(1)}} d_{i_t^{(1)} \lambda_t^{(1)}} = 0,$$

since $j_1 > n_t^{(1)}$, which proves the validity of inequality (3.9).

Just in the same way as in the case $s=1$ we will show that in the expansion of elements $b_{i\lambda_2}, b_{i,\lambda_2+1}, \cdots, b_{i,\lambda_3-1}$ there is no $x_s$ when $s>2$. Therefore the condition

$$\gamma_i\left(b_{i\lambda_2} + pb_{i,\lambda_2+1} + \cdots + p^{\lambda_3-\lambda_2-1}b_{i,\lambda_3-1}\right)$$
$$=\left(m_1^{(\lambda_2)} + pm_1^{(\lambda_2+1)} + \cdots + p^{\lambda_3-\lambda_2-1}m_1^{(\lambda_3-1)}\right)\gamma_i x_1$$
$$+\left(m_2^{(\lambda_2)} + pm_2^{(\lambda_2+1)} + \cdots + p^{\lambda_3-\lambda_2-1}m_2^{(\lambda_3-1)}\right)\gamma_i x_2$$
$$= p^{\lambda_1^{(2)}-\lambda_2}d_{i_1^{(2)}\lambda_1^{(2)}} + p^{\lambda_2^{(2)}-\lambda_2}d_{i_2^{(2)}\lambda_2^{(2)}} + \cdots + p^{\lambda_{m(2)}^{(2)}-\lambda_2}d_{i_{m(2)}^{(2)}\lambda_{m(2)}^{(2)}}$$

is fulfilled for the endomorphism $\gamma_i$. Since $\left(m_2^{(\lambda_2)}, p\right)=1$, according to this condition we define $\gamma_i x_2$ uniquely. Just like for $\gamma_i x_1$ we can verify that $e\left(\gamma_i x_2\right) \le k_{ij_2}+1$ and so on.

The endomorphism $\gamma_i$ is likewise defined uniquely by giving the images of basis elements and it is obvious that $\gamma_i$ maps the basic subgroup into. The endomorphism $\gamma_i$ uniquely continues up to the endomorphism $\hat{\gamma}_i$ of the group $\hat{T}$. Let us show that $\hat{\gamma}_i$ induces an endomorphism on the group $T$.

Let $t \in T$. Since the group $T$ has form (2.1), it suffices to show that $\hat{\gamma}_i t \in T$ when $t \in \overline{B}_i$. Since for the basis elements $x$ we have $\hat{\gamma}_i x = 0$ when $x \notin A_i$, using (3.6) let, without loss of generality, $t$ be an element of order $p^m$,

$$t = p^{k_{ij_1}+1-m}b_{i\lambda_1} + p^{n_1^{(1)}}b_{i\lambda_1+1} + \cdots + p^{n_{\lambda_2-\lambda_1-1}^{(1)}}b_{i\lambda_2-1} + \cdots$$
$$+ p^{k_{ij_s}+1-m}b_{i\lambda_s} + p^{n_1^{(s)}}b_{i\lambda_s+1} + \cdots + p^{n_{\lambda_{s+1}-\lambda_s-1}^{(1)}}b_{i\lambda_{s+1}-1}.$$

(Here it is assumed that $e\left(p^{n_j^{(s)}}b_{i,\lambda_s+j}\right)=m$, and if in several summands $k_{ij_s}+1-m<0$, then these summands are equated to zero.) Then

$$\gamma_i t = p^{k_{ij_1}+1-m}\gamma_i\left(b_{i\lambda_1} + p^{n_1^{(1)}-k_{ij_1}-1+m}b_{i,\lambda_1+1} + \cdots \right.$$
$$\left. + p^{n_{\lambda_2-\lambda_1-1}^{(1)}-k_{ij_1}-1+m}b_{i,\lambda_2+1}\right) + \cdots + p^{k_{ij_s}+1-m}\gamma_i\left(b_{i\lambda_s}\right.$$
$$\left. + p^{n_1^{(s)}-k_{ij_s}-1+m}b_{i,\lambda_s+1} + \cdots + p^{n_{\lambda_{s+1}-\lambda_s-1}^{(1)}-k_{ij_s}-1+m}b_{i,\lambda_{s+1}-1}\right) + \cdots$$
$$= q_1 p^{k_{ij_1}+1-m}\left(p^{\lambda_1^{(1)}-\lambda_1}d_{i_1^{(1)},\lambda_1^{(1)}} + p^{\lambda_2^{(1)}-\lambda_1}d_{i_2^{(1)},\lambda_2^{(1)}} + \cdots \right.$$
$$\left. + p^{\lambda_{m(1)}^{(1)}-\lambda_1}d_{i_{m(1)}^{(1)},\lambda_{m(1)}^{(1)}}\right) + \cdots + q_s p^{k_{ij_s}+1-m}\left(p^{\lambda_1^{(s)}-\lambda_s}d_{i_1^{(s)},\lambda_1^{(s)}}\right.$$
$$\left. + p^{\lambda_2^{(s)}-\lambda_s}d_{i_2^{(s)},\lambda_2^{(s)}} + \cdots + p^{\lambda_{m(s)}^{(s)}-\lambda_s}d_{i_{m(s)}^{(s)},\lambda_{m(s)}^{(s)}}\right) + \cdots, \quad (3.10)$$

where $\left(q_i, p\right)=0$. If in equality (3.10) the numbers of rows $i_1^{(s)}, i_2^{(s)}, \cdots, i_{m(s)}^{(s)}$, $s=1,2,\cdots$, of the matrix $\Phi(c)$ infinitely increase, by virtue of Definition 2.2 (see also (3.5)) $j_s - n_{m(s)}^{(j)} \to \infty$ when $s$ infinitely increases. But then

$$k_{ij_s} + 1 - m + \lambda_{m(s)}^{(s)} - \lambda_s$$
$$= k_{ij_s} + 1 - m + k'_{i_{m(s)}^{(s)} n_{m(s)}^{(s)}} - n_{m(s)}^{(s)} - k_{ij_s} + j_s$$
$$= k'_{i_{m(s)}^{(s)} n_{m(s)}^{(s)}} + 1 + j_s - m - n_{m(s)}^{(s)}$$
$$> k'_{i_{m(s)}^{(s)} n_{m(s)}^{(s)}} + 1 = e\left(d_{i_{m(s)}^{(s)} \lambda_{m(s)}^{(s)}}\right).$$

This means that starting from some $s$ all summands on the right-hand side of equality (3.10) are equal to zero. Therefore $\hat{\gamma}_i t \in T$.

The sum of endomorphisms $\sum_{i\in\mathcal{J}}\hat{\gamma}_i = \hat{\gamma}$, which, on the algebraically compact group $\hat{T}$, is induced by the endomorphism $\gamma_i$ on the $i$th component of the group $T = \bigoplus_{i\in\mathcal{J}}\overline{B}_i$, is the endomorphism of the group $\hat{T}$ which maps the subgroup $T$ into. It can be easily verified that $\hat{\gamma}a_0 = c_0$. $\hat{\gamma}$ in turn defines the endomorphism $\gamma^{\bullet}$ of the group $T^{\bullet}$ for which

$$\gamma^{\bullet}a = \left(\hat{\gamma}a_0, \hat{\gamma}a_1 + T, \cdots\right) = \left(c_0, c_1 + T, \cdots\right) = c$$

At the beginning of the proof we have assumed that the 0th row $H_{T^{\bullet}}(a) = \left(k_{00}, k_{01}, \cdots\right)$ consists of non-negative integer numbers and in the $i$th row there are an infinite number of jumps. It is obvious that this reasoning is also true when the $i$th row contains a finite number of jumps (at least one jump) or when the 0th row of the matrix $\Phi(a)$ has the form

$$H_{T^{\bullet}}(a) = \left(k_{00}, k_{01}, \cdots, k_{0n-1}, \omega+m, \omega+m+1, \cdots\right)$$

and $c = \left(c_0, T, T, \cdots\right)$, $c_0 \in T$.

2) Let us separately consider the case where $H_{T^{\bullet}}(a) = \left(\omega+m, \omega+m+1, \cdots\right)$ or, since $T$ in $\hat{T}$ is pure, assume that in the notation $a = \left(a_0, a_1 + T, \cdots\right)$ we have $a_0 = a_1 = \cdots = a_m = 0$ and $a_{m+1} \notin T$, $\mathcal{O}\left(a_{m+1}\right) = p$. Then in the representation $a_{m+1} = \lim_{s\to\infty}\sum_{i=0}^{s}p^i b_{m+1+i}$, $b_n$ contains a basis element $x_n \in B_{j_n}$ of arbitrarily large order and $n$ takes values from an infinite set of indexes. Let us fix a sequence $t_1 < t_2 < \cdots < t_n < \cdots$ of positive integer numbers such that the expansion of $b_{t_n}$ contains a basis element $x_n \in B_{j_n}$ such that $e\left(x_n\right) = t_n - m$. Let $A = \left\{x_n \middle| n = 1,2,\cdots\right\}$. We assume that $\gamma x_\alpha = 0$ for $x_\alpha \notin A$ and, analogously to part 1 of the proof, for any $n$ we define $\gamma x_n$ such that the equalities

$$\gamma\left(b_{t_n} + pb_{t_n+1} + \cdots p^{t_{n+1}-t_n-1} b_{t_{n+1}-1}\right) = d_{t_n} + pd_{t_{n+1}} + \cdots + p^{t_{n+1}-t_n-1} d_{t_{n+1}-1} \quad (3.11)$$

are fulfilled. It can be assumed that $t_1 = m+1$. Note that $x_{n+k}$, $k = 1,2,\cdots$, does not participate in the expansion $b_{t_n+s}$ with respect to the basis when $s = 0,1,\cdots,t_{n+1}-t_n-1$. Indeed, if it were not so, then the exponent of the element $p^{t_n-m-1}b_{t_n} + p^{t_n-m}b_{t_n+1} + \cdots + p^{t_{n+1}-m-2}b_{t_{n+1}-1}$ would be larger than 1, which is not so, $e(a_{m+1}) = 1$ $(t_{n+1}-m-2 < t_{n+1}-m)$. Moreover, $e(\gamma x_n) \le e(x_n)$. Indeed, when $n=1$, a coprime number with respect to $p$ serves as the coefficient $\gamma x_1$. Therefore it suffices to show that $p^{t_1-m}\left(d_{t_1} + pd_{t_1+1} + \cdots + p^{t_2-t_1-1}d_{t_2-1}\right) = 0$. We have

$$c_{m+1} = d_{m+1} + pd_{m+2} + \cdots + p^{t_1-m-1}d_{t_1} + \cdots + p^{t_2-m-2}d_{t_2-1} + \cdots,$$

but

$$pc_{m+1} = 0 = pd_{m+1} + p^2 d_{m+2} + \cdots p^{t_1-m}d_{t_1} + \cdots + p^{t_2-m-1}d_{t_2-1} + \cdots.$$

Hence, since $d_{m+i}$ is not divisible by $p$, all summands must be equal to zero, *i.e.* $e(\gamma x_1) \le \gamma(x_1)$. Analogously, $e(\gamma x_2) \le e(x_2)$ and so on. It is obvious that by virtue of (3.11), $\gamma x_n \in T$ and, by our construction, $x_n$ belongs to various $B_{j_n}$ and therefore $\gamma$ induces the endomorphism on the group $\hat{T}$ and maps the subgroup $T$ into. For the induced endomorphism $\gamma^\bullet$ on the group $\hat{T}$ we have

$$\gamma^\bullet a = \gamma^\bullet\left(0, 0+T, \cdots, a_{m+1}+T\right) = \left(0, T, \cdots, \hat{\gamma} a_{m+1} + T, \cdots\right) = \left(0, T, \cdots, c_{m+1}+T, \cdots\right) = c.$$

3) Let $H_{T^\bullet}(a) = \left(k_{00}, k_{01}, \cdots, k_{0n}, \omega+m, \omega+m+1, \cdots\right)$. Then $H_{T^\bullet}\left(p^n a\right) = \left(\omega+m, \omega+m+1, \cdots\right) \le H_{T^\bullet}\left(p^n c\right)$ and, as shown in part 2 of the proof, there exists an endomorphism $f^\bullet$ of the group $T^\bullet$ which induces the endomorphism on the $T$ so that $f^\bullet\left(p^n a\right) = p^n c$. Hence $p^n\left(f^\bullet a - c\right) = 0$, *i.e.* $f^\bullet a - c = t \in T$ and $H_{T^\bullet}(t) = H_{T^\bullet}\left(f^\bullet a - c\right) \ge H_{T^\bullet}(a)$. Therefore, by virtue the last sentence at the end of part 1 of the proof, there exists an endomorphism $\varphi^\bullet$ of the group $T^\bullet$ which induces the endomorphism on the subgroup $T$ so that $\varphi^\bullet a = t$. Then $f^\bullet a - c = \varphi^\bullet a$ or $\left(f^\bullet - \varphi^\bullet\right)a = c$. Obviously, $f^\bullet - \varphi^\bullet = \gamma^\bullet$ induces the endomorphism on $T$ and $\gamma^\bullet a = c$. Condition 4 is proved.

**Condition 5.** *If $C$ is a fully invariant subgroup of the group $T^\bullet$ and $a, b \in C$, then there exists $c \in C$ such that $\bar{\Phi}(c) = \bar{\Phi}(a) \wedge \bar{\Phi}(b)$.*

*Proof.* Let $\alpha = (k_0, k_1, \cdots)$ and $\beta = (l_0, l_1, \cdots)$ be respectively the *i*th rows of the matrices $\Phi(a)$ and $\Phi(b)$, $i \ge 1$. Let $j$ be a smallest index such that $k_j = l_j$. In these sequences, the nearest jump is to the right from $j$. Let this jump occur at the position $(k_s, k_{s+1})$ in the sequence $\alpha$. In the latter sequence, to the left from $j$ there is the preceding jump. Let this jump occur at the position $(k_t, k_{t+1})$, then, from $k_{t+1}$ to $k_s$ inclusive, we add $m$ to each element so that between $k_s + m$ and $k_{s+1}$ there would be no jump. Obviously,

$$k_{t+1} + m, \cdots, k_s + m \quad (3.12)$$

exceeds respectively $l_{t+1}, \cdots, l_s$ and, to the left from the index $s$, each $k_i \ne l_i$. The obtained sequences $\alpha_1$ and $\beta_1$, where $\beta_1 = \beta$ and $\alpha$ differ from $\alpha_1$ by the elements of (3.12), obviously satisfy the conditions of a row of the admissible matrix and $\alpha \wedge \beta = \alpha_1 \wedge \beta_1$.

Now let us assume that in the sequences $\alpha_1$ and $\beta_1$ the equality of elements takes place at the number $n$, $k_n = l_n$, where $n > s > j$. Then, in these sequences we have, to the right from $n$, a jump and repeat the previous reasoning. If in the sequences there are infinitely many jumps and at each stage the first jump occurs in one and the same sequence $\beta_r$, then not to violate condition 3 of Definition 2.1 we proceed as follows: let in the sequences $\alpha_r$ and $\beta_r$, at the position $n$, $k_n = l_n$ and to the right from $n$ there occur a jump between $(l_m, l_{m+1})$. Then in the first sequence $\alpha_r$, where there are infinitely many jumps, there exists a jump $(k_s, k_{s+1})$ such that $k_s = l_t < l_n$. On the right from $l_t$, we increase the numbers $l_{t+1}, \cdots, l_m$ so that there would be no jump between the numbers $l_m$, $l_{m+1}$, *i.e.* in the sequence $\beta_r$ we ourselves have intentionally created a jump between the numbers $l_t$ and $l_{t+1}$. Note that at this position the condition of admissibility of a row has not been violated since $l_t = k_s$, and there exists a jump between $k_s$ and $k_{s+1}$. We have $\alpha \le \alpha_1 \le \alpha_2 \le \cdots$; $\beta \le \beta_1 \le \beta_2 \le \cdots$. Denote $\alpha^* = \vee\alpha_i$, $\beta^* = \vee\beta_i$, $i = 1,2,\cdots$. The rows $\alpha^*$ and $\beta^*$ are admissible and $\alpha^* \wedge \beta^* = \alpha \wedge \beta$, where each element of $\alpha^*$ differs from the corresponding element of $\beta^*$. Now, if in every row of the matrices $\Phi(a)$ and $\Phi(b)$ we perform such transformations, then we obtain admissible matrices $U$ and $V$ the corresponding elements of which differ from one another and $\Phi(a) \le U$, $\Phi(b) \le V$, $\Phi(a) \wedge \Phi(b) = U \wedge V$. It is not difficult to verify that this reasoning holds for all type of matrices $\Phi(a)$ and $\Phi(b)$. Since $U$ and $V$ are admissible matrices, there exists $x, y \in T^\bullet$ such that $\Phi(x) = U$ and $\Phi(y) = V$. Then, by virtue of condition 4, there exists $f, \varphi \in \mathrm{End}T^\bullet$ such that $fa = x$, $\varphi b = y$. Hence $x, y \in C$, $\Phi(x+y) = \Phi(a) \wedge \Phi(b)$, $x + y = c \in C$. This means that $\Phi(c) = \Phi(a) \wedge \Phi(b)$. Therefore $\bar{\Phi}(c) = \bar{\Phi}(a) \wedge \bar{\Phi}(b)$. Condition 5 is proved.

We have obtained that the function $\bar{\Phi}: T^{\bullet} \to \bar{\Omega}$, where $T$ has form (2.1) and the set $\Omega$ of all admissible (with respect to $T$) matrices satisfies the conditions of Theorem 1.1. Hence the following statement is true.

**Theorem 3.1.** *The lattice of fully invariant subgroups of the cotorsion hull of a direct sum of torsion-complete $p$-groups is isomorphic to the lattice of filters of the semilattice $\bar{\Omega}$.*

## 4. Acknowledgements

This study was supported by the grant (ATSU-2013/44) of Akaki Tsereteli State University.

## REFERENCES

[1] L. Fuchs, "Infinite Abelian Groups. I," Academic Press, New York, London, 1970.

[2] L. Fuchs, "Infinite Abelian Groups. II," Academic Press, New York, London, 1973.

[3] R. Baer, "Type of Elements and Characteristic Subgroups of Abelian Groups," *Proceedings London Mathematical Society*, Vol. s2-39, No. 1, 1935, pp. 481-514. http://dx.doi.org/10.1112/plms/s2-39.1.481

[4] I. Kaplansky, "Infinite Abelian Groups," The University of Michigan Press, Ann Arbor, 1969.

[5] R. S. Linton, "On Fully Invariant Subgroups of Primary Abelian Groups," *Michigan Mathematical Journal*, Vol. 22, No. 3, 1976, pp. 281-284. http://dx.doi.org/10.1307/mmj/1029001528

[6] J. D. Moore and E. J. Hewett, "On Fully Invariant Subgroups of Abelian *p*-Groups," *Commentarii Mathematici Universitatis Sancti Pauli*, Vol. 20, 1971-1972, pp. 97-106.

[7] R. S. Pierce, "Homomorphisms of Primary Abelian Groups," In: *Topics in Abelian Groups* (*Proc. Sympos*., *New Mexico State Univ.*, 1962), Scott, Foresman and Co., Chicago, 1963, pp. 215-310.

[8] A. R. Chekhlov, "On Projective Invariant Subgroups of Abelian Groups," *Vestnik Tomskogo Gosudarstvennogo Universiteta Matematika i Mekhanika*, Vol. 2009, No. 1, 2009, pp. 31-36 (in Russian).

[9] R. Göbel, "The Characteristic Subgroups of the Baer-Specker Group," *Mathematische Zeitschrift*, Vol. 140, No. 3, 1974, pp. 289-292. http://dx.doi.org/10.1007/BF01214169

[10] S. Ya. Grinshpon and P. A. Krylov, "Fully invariant Subgroups, Full Transitivity, and Homomorphism Groups of Abelian Groups. Algebra," *Journal of Mathematical Sciences* (*New York*), Vol. 128, No. 3, 2005, pp. 2894-2997. http://dx.doi.org/10.1007/s10958-005-0245-5

[11] A. Mader, "The Fully Invariant Subgroups of Reduced Algebraically Compact Groups," *Publicationes Mathematicae Debrecen*, Vol. 17, 1970, pp. 299-306.

[12] V. M. Misyakov, "On Full Transitivity of Reduced Abelian Groups," In: *Abelian Groups and Modules*, No. 11, 12 (Russian), Tomsk State University, Tomsk, 1994, pp. 134-156 (in Russian).

[13] A. I. Moskalenko, "Cotorsion Hull of a Separable Group," *Algebra i Logika*, Vol. 28, No. 2, 1989, pp. 207-226 (in Russian); Translation in *Algebra and Logic*, Vol. 28, No. 2, 1989, pp. 139-151 (1990).

[14] D. K. Harrison, "Infinite Abelian Groups and Homological Methods," *Annals of Mathematics*, Vol. 69, No. 2, 1959, pp. 366-391. http://dx.doi.org/10.2307/1970188

[15] S. Bazzoni and L. Salce, "An Independence Result on Cotorsion Theories over Valuation Domains," *Journal of Algebra*, Vol. 243, No. 1, 2001, pp. 294-320. http://dx.doi.org/10.1006/jabr.2001.8800

[16] S. Bazzoni and J. Štóviček, "Sigma-Cotorsion Modules over Valuation Domains," *Forum Mathematicum*, Vol. 21, No. 5, 2009, pp. 893-920. http://dx.doi.org/10.1515/FORUM.2009.044

[17] R. Göbel, S. Shelah and S. L. Wallutis, "On the Lattice of Cotorsion Theories," *Journal of Algebra*, Vol. 238, No, 1, 2001, pp. 292-313. http://dx.doi.org/10.1006/jabr.2000.8619

[18] M. Hovey, "Cotorsion Pairs, Model Category Structures, and Representation Theory," *Mathematische Zeitschrift*, Vol. 241, No. 3, 2002, pp. 553-592. http://dx.doi.org/10.1007/s00209-002-0431-9

[19] W. May and E. Toubassi, "Endomorphisms of Abelian Groups and the Theorem of Baer and Kaplansky," *Journal of Algebra*, Vol. 43, No. 1, 1976, pp. 1-13. http://dx.doi.org/10.1016/0021-8693(76)90139-3

[20] T. Kemoklidze, "On the Full Transitivity of a Cotorsion Hull," *Georgian Mathematical Journal*, Vol. 13, No. 1, 2006, pp.79-84.

[21] T. Kemoklidze, "The Lattice of Fully Invariant Subgroups of a Cotorsion Hull," *Georgian Mathematical Journal*, Vol. 16, No. 1, 2009, pp. 89-104.

[22] T. Kemoklidze, "On the Full Transitivity and Fully Invariant Subgroups of Cotorsion Hulls of Separable *p*-Groups," *Journal of Mathematical Sciences* (*New York*), Vol. 155, No. 5, 2008, pp. 748-786. http://dx.doi.org/10.1007/s10958-008-9240-y