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Primes in Arithmetic Progressions to Moduli with a Large Power Factor

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ABSTRACT

Recently Elliott studied the distribution of primes in arithmetic progressions whose moduli can be divisible by high-powers of a given integer and showed that for integer $a \ge 2$ and real number $A > 0$. There is a $B = B(A) > 0$ such that

$$\sum_{\substack{d \le x^{\frac{1}{2}} q^{-1} L^{-B} \\ (d,q)=1}} \max_{y \le x} \max_{(r,qd)=1} \left| \pi(y;qd,r) - \frac{Li(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q) L^A},$$

holds uniformly for moduli $q \le x^{\frac{1}{3}} \exp\left(-(\log\log x)^3\right)$ that are powers of a. In this paper we are able to improve his result.

Keywords: Primes; Arithmetic Progressions; Riemann Hypothesis

1. Introduction and Main Results

Let p denote a prime number. For integer a, q with $(a,q)=1$, we introduce

$$\pi(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a (\bmod q)}} 1$$

to count the number of primes in the arithmetic progression $a (\bmod q)$ not exceeding x. For fixed q, we have

$$\pi(x;q,a) \sim \frac{1}{\phi(q)} \pi(x)$$

as x tends to infinity. However the most important thing in this context is the range uniformity for the moduli q in terms of x. The Siegel-Walfisz Theorem, see for example [1], shows that this estimate is true only if $q \le L^A$, where and throughout this paper we denote $\log x$ by L. The Generalized Riemann Hypothesis for Dirichlet L-functions could give a much better result: non-trivial estimate holds for $q \le x^{\frac{1}{2}} L^{-2}$. Unfortunately the Generalized Riemann Hypothesis has withstood the attack of several generations of researchers and it is still out of reach. However number theorists still want to live a better life without the Generalized Riemann Hypothesis. Therefore they try to find a satisfactory substitute. In this direction the famous Bombieri-Vinogradov theorem [2, 3], states that

Theorem A. For any $A > 0$ there exists a constant $B = B(A) > 0$ such that

$$\sum_{q \le Q} \max_{y \le x} \max_{(a,q)=1} \left| \pi(y;q,a) - \frac{Li(y)}{\phi(q)} \right| \ll x L^{-A},$$

where $\phi(q)$ is the Euler totient function, $Q = x^{\frac{1}{2}} L^{-B}$, and $Li(y) = \int_2^y \frac{\mathrm{d}u}{\log u}$.

Recently in order to study the arithmetic functions on shifted primes, Elliott [4] studied the distribution of primes in arithmetic progressions whose moduli can be divisible by high-powers of a given integer. More precisely, he showed that

Theorem B. Let a be an integer, $a \ge 2$. If $A > 0$, then there is a $B = B(A) > 0$ such that

$$\sum_{\substack{d \le x^{\frac{1}{2}} q^{-1} L^{-B} \\ (d,q)=1}} \max_{y \le x} \max_{(r,qd)=1} \left| \pi(y;qd,r) - \frac{Li(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q) L^A},$$

holds uniformly for moduli $q \le x^{\frac{1}{3}} \exp\left(-(\log\log x)^3\right)$ that are powers of a.

When $q=1$, his result recovers the Bombieri-Vinogradov theorem. And obviously his result gives a deep insight into the distribution of primes in arithmetic progressions.

The most important thing Elliott concerned in [4] is that in Theorem B the parameter q may reach a fixed power of x. However we want to purse the widest uniformity in q by using some new techniques established in the study of Waring-Goldbach problems.

We shall prove the following result.

Theorem 1.1. Let a be an integer, $a \ge 2$. If $A>0$, then there is a $B=B(A)>0$ such that

$$\sum_{\substack{d \le x^{\frac{1}{2}} q^{-1} L^{-B} \\ (d,q)=1}} \max_{y \le x} \max_{(r,qd)=1} \left| \pi(y;qd,r) - \frac{Li(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q) L^A},$$

holds uniformly for moduli $q \le x^{\frac{2}{5}} \exp\left(-(\log\log x)^3\right)$ that are powers of a.

When $d=1$ and a an odd prime, our result gives that for these particular moduli q with the form $q=p^n, (n=1,2,3,\cdots)$

$$\pi(x;q,r) = \left\{1+O\left(L^{-A}\right)\right\} \frac{Li(x)}{\phi(q)},$$

holds uniformly for moduli $q \le x^{\frac{2}{5}} \exp\left(-(\log\log x)^3\right)$. Then the special case of our result shows that the least prime $P_{\min}(q,r)$ in these special progressions $n \equiv r \pmod q$ satisfies

$$P_{\min}(q,r) \ll q^{5/2+\epsilon}.$$

This result improves a former result given by Barban, Linnik and Tshudakov [5],

$$P_{\min}(q,r) \ll q^{8/3+\epsilon}$$

where $q=p^n, (n=1,2,3,\cdots)$.

If we focus our attention on the least prime in arithmetic progressions with special moduli, we can prove the following result.

Theorem 1.2. Let a be an integer, $a \ge 2$. If $A>0$, then there is a $B=B(A)>0$ such that

$$\sum_{\substack{d \le x^{\frac{9}{20}} q^{-1} L^{-B} \\ (d,q)=1}} \max_{y \le x} \max_{(r,qd)=1} \left| \pi(y;qd,r) - \frac{Li(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q) L^A},$$

holds uniformly for moduli $q \le x^{\frac{5}{12}} \exp\left(-(\log\log x)^3\right)$ that are powers of a.

Then our result shows that the least prime $P_{\min}(q,r)$ in these special progressions $n \equiv r \pmod q$ satisfies

$$P_{\min}(q,r) \ll q^{12/5+\epsilon}.$$

It should be remarked that the Generalized Riemann Hypothesis for Dirichlet L-functions would allow $qd \le x^{\frac{1}{2}} L^{-A-1}$ with no further restriction upon the nature of q. Therefore our Theorems 1.1 and 1.2 can be compared with the result under the Generalized Riemann Hypothesis.

2. Preliminary Reduction

Let $\Lambda(n)$ denote von Mangoldt's function, and for mutually prime integers w and r, let

$$\psi(y;w,r) = \sum_{\substack{n \le y \\ n \equiv r \pmod w}} \Lambda(n).$$

For $2 \le w \le x^{3/4}$ and an integer $q \ge 1$, define

$$G(w) = \sum_{\substack{d \le w \\ (d,q)=1}} \max_{(r,qd)=1} \max_{y \le x} \left| \psi(y;qd,r) - \frac{1}{\phi(d)} \psi(y;q,r) \right|.$$

Then

Lemma 2.1. For any $K>0, 1/4<\delta \le 1/2$, we have

$$G\left(x^{\delta} q^{-1} L^{-K}\right) \ll G\left(\exp\left(\frac{1}{2}(\log\log x)^3\right)\right) \log x + \tau(q) q^{-1} x (\log x)^{6-K}. \tag{1}$$

uniformly for positive integers $q \le x^{\theta} \exp\left(-(\log\log x)^3\right), x \ge 3$ where $\theta = 2/5$, if $9/20 < \delta \le 1/2$ and $\theta = 5/12$, if $1/4 < \delta \le 9/20$. Here $\tau(q) = \sum_{n|q} 1$.

For Dirichlet characters χ and real $y>0$ define

$$\psi(y,\chi) = \sum_{n \le y} \chi(n) \Lambda(n). \tag{2}$$

Lemma 2.2. Let $\psi(y,\chi)$ defined as in (2). Then

$$\sum_{d \le Q} \sum_{\chi \pmod{Dd}}^{*} \max_{y<x} \left|\psi(y,\chi)\right| \ll \left(x + x^{1/2} Q^2 D + x^{4/5} Q D^{1/2}\right) L^c, \tag{3}$$

holds uniformly for all integers $D \ge 1$ and real numbers $x \ge 2, Q \ge 1$.

Lemma 2.3. Let $\psi(y,\chi)$ defined as in (2). Then

$$\sum_{d \le Q} \sum_{\chi \pmod{Dd}}^{*} \max_{y<x} \left|\psi(y,\chi)\right| \ll \left(x + x^{11/20} Q^2 D\right) L^c. \tag{4}$$

holds uniformly for all integers $D \ge 1$ and real numbers $x \ge 2, Q \ge 1$. Here the inner sum is taken over all primitive Dirchlet characters $\left(\bmod Dd\right)$.

3. Proof of Lemma 2.2

Let

$$X^{\frac{2}{5}} < Y \le X$$

and $M_1,\cdots,M_{10}$ be positive real numbers such that

$$Y \le M_1\cdots M_{10} < X \quad \text{and} \quad 2M_6,\cdots,2M_{10} \le X^{\frac{1}{5}}. \tag{5}$$

For $j=1,\cdots,10$ define

$$a_j(m)=\begin{cases}\log m, & \text{if } j=1,\\ 1, & \text{if } j=2,\cdots,5,\\ \mu(m), & \text{if } j=6,\cdots,10,\end{cases} \tag{6}$$

where $\mu(n)$ is the Möbius function. Then we define the functions

$$f_j(s,\chi)=\sum_{m\sim M_j}\frac{a_j(m)\chi(m)}{m^s},$$

and

$$F(s,\chi)=f_1(s,\chi)\cdots f_{10}(s,\chi), \tag{7}$$

where χ is a Dirchlet character, s a complex variable.

Lemma 3.1. Let $F(s,\chi)$ be as in (7), and $A\ge 1$ arbitrary. Then for any $1\le R\le X^{2A}$ and $0<T\ll X^A$,

$$\sum_{\substack{r\sim R\\ d|r}}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\, r)}\int_{-T}^{T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\mathrm{d}t \ll\left(\frac{R^2}{d}T+\frac{R}{d^{1/2}}T^{\frac{1}{2}}X^{\frac{3}{10}}+X^{\frac{1}{2}}\right)\log^c X, \tag{8}$$

where $c>0$ is an absolute constant independent of A, but the constant implied in $\ll$ depends on A.

Proof of Lemma 3.1. This lemma with $d=1$ was established in [6], and in this general form [7]. We mention that in general the exponent 3/10 to X in the second term on the right-hand side is the best possible on considering the lack of sixth power mean value of Dirchlet L-functions.

Now we complete the proof of Lemma 2.2.

Proof of Lemma 2.2. In (5), we take

$$Y=x^{\frac{2}{5}},\ X=x.$$

Define $a_j(m), f_j(s,\chi)$ and $F(s,\chi)$ as above. To go further, we first recall Heath-Brown's identity [8], which states that for any $n<2z^k$ with $z\ge 1$ and $k\ge 1$,

$$\Lambda(n)=\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\sum_{\substack{n_1n_2\cdots n_{2j}=n\\ n_{j+1}\cdots n_{2j}\le z}}\cdots\sum(\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}).$$

Then for

$$2Y=2x^{\frac{2}{5}}<y\le X=x,$$

$\psi(y,\chi)$ is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\mathfrak{S}(\mathbb{M}):=\sum_{\substack{m_1\sim M_1,\cdots,m_{10}\sim M_{10}\\ y/2<m_1\cdots m_{10}\le y}}\cdots\sum a_1(m_1)\chi(m_1)\cdots a_{10}(m_{10})\chi(m_{10}),$$

where $\mathbb{M}$ denotes the vector $(M_1,M_2,\cdots,M_{10})$ with M_j as in (5). Obviously some of the intervals $(M_j,2M_j]$ may contain only integer 1. By using Perron's summation formula with $T=y$ (see Proposition 5.5 in [1]), and then shifting the contour to the left, we have

$$\mathfrak{S}(\mathbb{M})=\frac{1}{2\pi i}\int_{1+1/L-iy}^{1+1/L+iy}F(s,\chi)\frac{y^s-(y/2)^s}{s}\mathrm{d}s+O(L^2)$$
$$=\frac{1}{2\pi i}\left\{\int_{1+1/L-iy}^{1/2-iy}+\int_{1/2-iy}^{1/2+iy}+\int_{1/2+iy}^{1+1/L+iy}\right\}+O(L^2).$$

On using the trivial estimate

$$F(\sigma\pm iy,\chi)\ll\left|f_1(\sigma\pm iy,\chi)\right|\cdots\left|f_{10}(\sigma\pm iy,\chi)\right| \ll\left(M_1^{1-\sigma}L\right)M_2^{1-\sigma}\cdots M_{10}^{1-\sigma}\ll x^{1-\sigma}L,$$

the integral on the two horizontal segments above can be estimated as

$$\ll\max_{1/2\le\sigma\le 1+1/L}\left|F(\sigma\pm iy,\chi)\right|\frac{y^\sigma}{y}$$
$$\ll\max_{1/2\le\sigma\le 1+1/L}x^{1-\sigma}L\frac{y^\sigma}{y}\ll x^{\frac{1}{2}}y^{-\frac{1}{2}}L\ll x^{\frac{3}{10}}L.$$

Then we have

$$\mathfrak{S}(\mathbb{M})$$
$$=\frac{1}{2\pi}\int_{-y}^{y}F\left(\frac{1}{2}+it,\chi\right)\frac{y^{\frac{1}{2}+it}-(y/2)^{\frac{1}{2}+it}}{\frac{1}{2}+it}\mathrm{d}t+O\left(x^{\frac{3}{10}}L\right)$$
$$\ll y^{\frac{1}{2}}\int_{-y}^{y}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{\mathrm{d}t}{|t|+1}+x^{\frac{3}{10}}L.$$

Noting that $F(s,\chi)$ does not depend on y, we have

$$\max_{2Y<y\le x}\left|\psi(y,\chi)\right|\ll L^{10}x^{\frac{1}{2}}\int_{-x}^{x}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{\mathrm{d}t}{|t|+1}+x^{\frac{3}{10}}L^{11}. \tag{9}$$

On the other hand we have

$$\max_{y\le 2Y}\left|\psi(y,\chi)\right|\ll Y. \tag{10}$$

From (9) and (10), we have

$$\sum_{d\le Q}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\, Dd)}\max_{y\le x}\left|\psi(y,\chi)\right|$$
$$\ll\sum_{d\le Q}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\, Dd)}\max_{2Y<y\le x}\left|\psi(y,\chi)\right|+\sum_{d\le Q}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\, Dd)}\max_{y\le 2Y}\left|\psi(y,\chi)\right|$$
$$\ll L^{10}x^{\frac{1}{2}}\sum_{d\le Q}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\, Dd)}\int_{-x}^{x}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{\mathrm{d}t}{|t|+1}+Q^2Dx^{\frac{2}{5}}.$$

Further let $q = Dd$ and then we obtain

$$\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le x}\left|\psi(y,\chi)\right|$$

$$\ll L^{12}x^{\frac{1}{2}}\max_{0<T\le x}\max_{1\le R\le QD}\frac{1}{T+1}\sum_{\substack{q\sim R\\ D|q}}\sum_{\chi(\bmod q)}^{*}\int_T^{2T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\mathrm{d}t$$

$$+Q^2Dx^{\frac{2}{5}}.$$

From Lemma 3.1, we have

$$\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le x}\left|\psi(y,\chi)\right|$$

$$\ll L^c x^{\frac{1}{2}}\max_{0<T\le x}\max_{1\le R\le QD}\frac{1}{T+1}\left\{\frac{R^2}{D}T+\frac{R}{D^{1/2}}T^{\frac{1}{2}}x^{\frac{3}{10}}+x^{\frac{1}{2}}\right\}$$

$$+Q^2Dx^{\frac{2}{5}}$$

$$\ll L^c x^{\frac{1}{2}}\left\{\frac{(QD)^2}{D}+\frac{QD}{D^{1/2}}(T+1)^{-\frac{1}{2}}x^{\frac{3}{10}}+x^{\frac{1}{2}}(T+1)^{-1}\right\}$$

$$+Q^2Dx^{\frac{2}{5}}$$

$$\ll\left(x+x^{\frac{1}{2}}Q^2D+x^{\frac{4}{5}}QD^{\frac{1}{2}}\right)L^c.$$

This completes the proof of Lemma 2.2.

4. Proof of Lemma 2.3

Firstly we recall one result of Choi and Kumchev [9] about mean value of Dirichlet polynomials. Let $m\ge1, r\ge1,$ and $Q\ge r,$ Let $\mathcal{H}(m,r,Q)$ denote the set of character $\chi=\xi\psi$ modulo mq, where ξ is a character modulo m and ψ is a primitive character modulo q with $r\le q\le Q$, $r|q$ and $(q,m)=1$. Then the result of Choi and Kumchev states as follows.

Lemma 4.1. Let $m\ge1, r\ge1, T\ge2, N\ge2,$ and $\mathcal{H}(m,r,Q)$ be a set of characters as described as above, Then

$$\sum_{\chi\in\mathcal{H}(m,r,Q)}\int_{-T}^{T}\left|\sum_{N<n\le2N}\Lambda(n)\chi(n)n^{-it}\right|\mathrm{d}t\ll\left(N+HN^{\frac{11}{20}}\right)L^c,$$

where c is an absolute constant, $H=mr^{-1}Q^2T$ and $L=\log HN.$ Now we complete the proof of Lemma 2.3.

Proof of Lemma 2.3. Let $Y=x^{\frac{1}{2}}$ and $X=x$. We define

$$F(s,\chi)=\sum_{Y<n\le X}\Lambda(n)\chi(n)n^{-s}.$$

If y satisfies

$$Y<y\le X, \tag{11}$$

we apply Perron's summation formula with $T=y$ (see Proposition 5.5 in [1]), and then obtain

$$\psi(y,\chi)=\frac{1}{2\pi i}\int_{b-iy}^{b+iy}F(s,\chi)\frac{y^s-(y/2)^s}{s}\mathrm{d}s+O\left(xy^{-1}L^2\right)$$

$$=\frac{1}{2\pi i}\int_{b-iy}^{b+iy}F(s,\chi)\frac{y^s-(y/2)^s}{s}\mathrm{d}s+O\left(x^{\frac{1}{2}}L^2\right),$$

where $0<b<L^{-1}$. If we let $b\to0$, we have

$$\psi(y,\chi)\ll\int_{-y}^{y}F(it,\chi)\frac{1}{|t|+1}\mathrm{d}t+O\left(xy^{-1}L^2\right).$$

Noting that $F(s,\chi)$ does not depend on y, we have

$$\max_{Y<y\le x}\left|\psi(y,\chi)\right|\ll\int_{-x}^{x}\left|F(it,\chi)\right|\frac{\mathrm{d}t}{|t|+1}+O\left(x^{\frac{1}{2}}L^2\right). \tag{12}$$

On the other hand we have

$$\max_{y\le2Y}\left|\psi(y,\chi)\right|\ll Y=x^{\frac{1}{2}}. \tag{13}$$

From (12) and (13), we have

$$\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le x}\left|\psi(y,\chi)\right|$$

$$\ll\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{Y<y\le x}\left|\psi(y,\chi)\right|+\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le2Y}\left|\psi(y,\chi)\right|$$

$$\ll\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\int_{-x}^{x}F(it,\chi)\frac{\mathrm{d}t}{|t|+1}+Q^2Dx^{\frac{1}{2}}L^2.$$

Further let $q=Dd$ and then we obtain

$$\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le x}\left|\psi(y,\chi)\right|$$

$$\ll\max_{0<T\le x}\max_{1\le R\le QD}\frac{1}{T+1}\sum_{\substack{q\sim R\\ D|q}}\sum_{\chi(\bmod Dd)}^{*}\int_T^{2T}\left|F(it,\chi)\right|\mathrm{d}t+Q^2Dx^{\frac{1}{2}}L^2.$$

Lemma 4.1 with $m = 1$ gives that

$$\sum_{\substack{q\sim R\\ D|q}}\sum_{\chi(\bmod Dd)}^{*}\int_T^{2T}\left|F(it,\chi)\right|\mathrm{d}t\ll\left(x+\frac{R^2T}{D}x^{\frac{11}{20}}\right)L^c. \tag{14}$$

From (14), we have

$$\sum_{d\le Q}\sum_{\chi(\bmod Dd)}^{*}\max_{y\le x}\left|\psi(y,\chi)\right|$$

$$\ll\max_{0<T\le x}\max_{1\le R\le Q}\frac{1}{T+1}\left\{x+\frac{R^2T}{D}x^{\frac{11}{20}}\right\}L^c+Q^2Dx^{\frac{1}{2}}L^2$$

$$\ll L^c\left\{\frac{(QD)^2}{D}x^{\frac{11}{20}}+x(T+1)^{-1}\right\}+Q^2Dx^{\frac{1}{2}}L^2$$

$$\ll\left(x+x^{\frac{11}{20}}Q^2D\right)L^c.$$

This completes the proof of Lemma 2.3.

5. Proof of Lemma 2.1

We partition the moduli qd as qd_1d_2, where the prime factor of d_2 not exceed L^K and those of d_1 do not. If $\omega(n)$ denotes the number of distinct prime divisors of the integer n, and $t = 2K\log\log x/\log\log\log x$, with estimate $\psi(x;q,r) \ll x/\phi(q)$, for $1/4 < \delta \le 1/2$, we have

$$\sum_{\substack{d \le q^{-1}x^{\delta} \\ \omega(d_1)>t}} \psi(x;qd,r) \ll \frac{x}{\phi(q)} \sum_{\substack{d_1 \le x \\ \omega(d_1)>t}} \frac{1}{\phi(d_1)} \sum_{d_2 \le x} \frac{1}{\phi(d_2)}$$

$$\ll \frac{x\log x}{\phi(q)} \sum_{k>t} \frac{1}{k!} \left(\sum_{p \le L^K} \sum_{m=1}^{\infty} \frac{1}{p^m} \right)^k$$

$$\ll \frac{x\log x}{\phi(q)} \sum_{k>t} \left(\frac{e\log(K\log L) + O(1)}{k} \right)^k$$

$$\ll \frac{x\log x}{\phi(q)} \exp\left(-t(1+o(1))\log\log\log x\right),$$

which is $O\left(\phi(q)^{-1} xL^{-K}\right)$.

Moreover the corresponding sum, taken over those moduli d for which d_1 is divisible by the v^{th} power of some prime, $v \ge 8$, is

$$\ll \frac{x}{\phi(q)} \sum_{d_2 \le x} \frac{1}{\phi(d_2)} \sum_{p \le L^K} \frac{1}{\phi(p^v)} \sum_{m \le x^{\delta}} \frac{1}{\phi(m)}$$

$$\ll \phi(q)^{-1} 2^{-v/2} x(\log x)^2.$$

With $v = \left[4(K+3)\log\log x\right]$, this is $O\left(\phi(q)^{-1} xL^{-K}\right)$, too.

We denote $\exp\left(\frac{1}{2}(\log\log x)^3\right)$ by Δ. Arguing similarly for $\phi(d)^{-1}\psi(x;q,r)$, we have

$$\sum_{\substack{d \le q^{-1}x^{\delta} \\ d_1 \ge \Delta}} \max_{(r,qd)=1} \max_{y \le x} \left| \psi(y;qd,r) - \frac{1}{\phi(d)}\psi(y;q,r) \right|$$

$$\ll \frac{x}{\phi(q)L^K}.$$

We collect together those moduli qd with a fixed value of d_1 not exceeding Δ and set $D = qd_1$. Noting that

$$\psi(y;D,r) = \sum_{\substack{n \le y, (n,d_2)=1 \\ n \equiv r \pmod D}} \Lambda(n) + O(\log yd_2),$$

we see from the orthogonality of Dirichlet characters that

$$\psi(y;qd,r) - \frac{1}{\phi(d)}\psi(y;q,r)$$

$$-\frac{1}{\phi(d_2)}\left\{\psi(y;qd_1,r) - \frac{1}{\phi(d_1)}\psi(y;q,r)\right\}$$

$$= \psi(y;Dd_2,r) - \frac{1}{\phi(d_2)}\psi(y;D,r)$$

$$= \frac{1}{\phi(Dd_2)} \sideset{}{'}\sum_{\chi \pmod{Dd_2}} \bar{\chi}(r)\psi(y,\chi) + O\left(\frac{\log d_2 y}{\phi(d_2)}\right).$$

where $'$ denotes that if we factorise χ as $\chi_1\chi_2$ defined $(\bmod D)$, χ_2 defined $(\bmod d_2)$, then the character χ_2 is not principal.

In order to establish Lemma 2.1 it will therefore suffice to prove that the sum S given by

$$\sum_{d_1 \le \Delta} \sum_{d_2 \le L^{-K}(qd_1)^{-1}x^{\delta}} \sideset{}{'}\sum_{\chi \pmod{Dd_2}} \max_{y \le x} \left|\psi(y,\chi)\right| \frac{1}{\phi(qd_1d_2)},$$

is $\ll \tau(q)q^{-1}x(\log x)^{6-K}$. For a fixed value of $D(=qd_1)$, we collect together those terms involving the characters χ induced by a particular primitive character $\chi^*(\bmod D_1\rho)$, where $D_1|D$ and $\rho|d_2$. Since χ and χ^* differ on at most the integers n for which $(n, D_1\rho) = 1$ but $(n, Dd_2) > 1$,

$$\psi(y,\chi) = \psi(y,\chi^*) + O(\log Dd_2 y).$$

Interchanging summations,

$$\sum_{d_2 \le L^{-K}(qd_1)^{-1}x^{\delta}} \sideset{}{'}\sum_{\chi \pmod{Dd_2}} \max_{y \le x} \left|\psi(y,\chi^*)\right| \frac{1}{\phi(Dd_2)}$$

$$\ll \sum_{D_1|D} \sum_{\rho} \sideset{}{^*}\sum_{\chi \pmod{D_1\rho}} \max_{y \le x} |\phi(y,\chi)| \sum_{d_2 \equiv 0 \pmod \rho} \frac{1}{\phi(Dd_2)}.$$

Here $\rho \le L^{-K}D^{-1}x^{\delta}$, and the innermost bounding sum is $\ll \phi(D\rho)^{-1}\log x$. We cover the range of ρ with adjoining intervals $U < \rho \le 2U$, subject to $L^K \le U \ll L^{-K}D^{-1}x^{\delta}$. When $\delta = 1/2$, by Lemma 2.2 a typical interval contributes

$$\ll \left(\frac{x}{U} + x^{\frac{1}{2}}UD_1 + x^{\frac{4}{5}}D_1^{\frac{1}{2}}\right)\frac{\log^5 x \log\log x}{D}.$$

Since

$D_1^{1/2} \le D^{1/2} = (qd_1)^{1/2} \le x^{1/5}\exp\left(-\frac{1}{4}(\log\log x)^3\right)$, the whole sum over ρ is

$$\ll D^{-1}x(\log x)^{5-K}\log\log x.$$

Arguing similarly for $\delta = 9/20$, by Lemma 2.3 the whole sum over ρ is also $\ll D^{-1}x(\log x)^{5-K}\log\log x$. Noting that

$$\sum_{d_1\le\Delta}\frac{\tau(D)}{D}\ll\tau(q)q^{-1}\sum_{d_1\le\Delta}\frac{\tau(d_1)}{d_1}$$

$$\ll\tau(q)q^{-1}\prod_{p\le\Delta}\left(1+\frac{\tau(p)}{p}+\frac{\tau(p^2)}{p^2}+\cdots\right)$$

$$=\tau(q)q^{-1}\prod_{p\le\Delta}\left(1-\frac{1}{p}\right)^{-2}\ll\tau(q)q^{-1}(\log\Delta)^2$$

$$\ll\tau(q)q^{-1}(\log\log x)^6.$$

summation over d_1 delivers the desired bound on S.

This completes the proof of Lemma 2.1.

6. Zeros of Dirichlet L-Functions

Lemma 4.1. Let $L(s,\chi), s=\sigma+it$, denote an L-function formed with a Dirichlet character $\chi(\operatorname{mod} q), q\ge 3, h=\prod_{p|q}p.$ With $l=\log q(|t|+3)$, define

$$\theta^{-1}=4.10^4\left(\log h+(l\log 2l)^{3/4}\right).$$

Then there can be at most one non-principal character $(\operatorname{mod} q)$ for which the corresponding L-function has a zero in the region $\sigma>1-\theta.$ Moreover such a character would be real and the zero would be real and simple.

Proof of Lemma 4.1. This is Theorem 2 of Iwaniec, [10].

Lemma 4.2. Let $\chi_j(\operatorname{mod} D_j), j=1,2$ be distinct primitive real characters. There is a positive real c_1 so that at most one of the functions $L(s,\chi)$ formed with these characters can vanish on the line segment

$$1-c_1(\log D_1D_2)^{-1}\le\sigma\le 1,\ t=0. \qquad (15)$$

Proof of Lemma 4.2. This is result of Landau, which can be found at Satz 6.4, p. 127, of Prachar [11].

Lemma 4.3. For any modulus $D, 0<\alpha\le 1, T\ge 0,$ let $N(\alpha,T,D)$ denote the number of zeros, counted with multiplicity, of all functions $L(s,\chi)$ formed with a character $\chi(\operatorname{mod} D)$, that lie in the rectangle $\alpha\le\operatorname{Re}s\le 1,|\operatorname{Im}s|\le T.$ Then we have

$$N(\alpha,T,D)\ll(DT)^{\frac{12}{5}(1-\alpha)},$$

uniformly for $0\le\alpha\le 1, T\ge 2.$

Proof of Lemma 4.3. This is Theorem of Heath-Brown [12], on p. 249.

7. Proof of Theorems 1.1 and 1.2

We shall first provide a version of the theorem with $\psi(y;qd,r)$ in place of $\pi(y;qd,r)$. After Lemma 2.1 it will suffice to establish the bound

$$G(\Delta)\ll x\left(\phi(q)(\log x)^A\right)^{-1},$$

for any fixed positive A.

We employ the representation

$$\sum_{n\le y}\chi(n)\Lambda(n)=E_\chi y-\sum_{|\gamma|\le T}\frac{y^\rho}{\rho}+O\left(\frac{y(\log Dy)^2}{T}+y^{1/4}\log Dy\right),$$

valid for all characters $(\operatorname{mod} D)$, where $y\ge T\ge 2$; E_χ is 1 if χ is principal, zero otherwise; $\rho=\beta+i\gamma$ runs through all the zeros of $L(s,\chi)$ in the rectangle $0\le\operatorname{Re}(s)<1,|\operatorname{Im}(s)|\le T$ with a half disc $|s|\le c_4(\log D)^{-1}>0,\operatorname{Re}(s)\ge 0$ removed. This representation is a slightly modified version of that given in Satz 4.6, pp. 232-234 of Prachar [11].

Since $L(s,\chi)$ has $\ll\log DT$ zeros in the strip $0\le\operatorname{Re}(s)<1, T<|\operatorname{Im}(s)|\le T+1,$ cf. Prachar [11], Satz 3.3, p. 220,

$$\sum_{\substack{\beta>1/2\\|\gamma|\le T}}\frac{y^\rho}{\rho}\ll y^{1/2}\left(\log 2D+\sum_{m\le T}\sum_{m<\gamma\le m+1}\frac{1}{|\rho|}\right)$$

$$\ll y^{1/2}(\log Dy)\sum_{m\le T}m^{-1}\ll y^{1/2}(\log Dy)^2,$$

and at the expense of raising $y^{1/4}\log Dy$ to $y^{1/2}(\log Dy)^2$ we may confine the zeros ρ to the half-plane $\operatorname{Re}(s)>1/2$.

From the orthogonality of Dirichlet characters

$$\phi(D)\psi(y;D,r)-y$$

$$=\sum_{\chi(\operatorname{mod} D)}\bar\chi(r)\left(\sum_{n\le y}\chi(n)\Lambda(n)-E_\chi y\right)$$

$$\ll\sum_\chi\sum_{\substack{\beta>1/2\\|\gamma|\le T}}\frac{y^\rho}{\rho}+\left(\frac{y}{T}+y^{1/2}\right)D(\log Dy)^2,$$

where it is understood that the $\rho(=\beta+i\gamma)$ are the zeros of the L-function formed with the character χ of the outer summation.

We replace y by z and average over the interval $y\le z\le y+w$ with $w=y(\log y)^{-A-2}$ to obtain

$$\frac{1}{w}\int_y^{y+w}\left(\phi(D)\psi(z;D,r)-z\right)\mathrm{d}z$$

$$\ll\frac{y}{w}\sum_\chi\sum_{\substack{\beta>1/2\\|\gamma|\le T}}\frac{y^\beta}{|\rho|^2}+\left(\frac{y}{T}+y^{1/2}\right)(\log Dy)^2.$$

Replacing z in the integrand by y introduces an error of

$$\ll w+\phi(D)\sum_{\substack{y<n\le y+w\\n\equiv r(\operatorname{mod} D)}}\log y\ll w+\phi(D)\left(\frac{w}{D}+1\right)\log y,$$

and we may remove the integral averaging:

$$\phi(D)\psi(y;D,r)-y\ll(\log y)^{A+2}\sum_{\chi}\sum_{\substack{\beta>1/2\\|\gamma|\le T}}\frac{y^{\beta}}{|\rho|^2}+\left(\frac{y}{T}+y^{1/2}\right)D(\log Dy)^2+\frac{y}{(\log y)^{A+1}}.$$

This bound will be satisfactory for $y > x(\log x)^{-A-2}$. Otherwise, we shall employ the crude bound

$$\psi(y;D,r)-\frac{y}{\phi(D)}\ll\frac{y\log(y+2)}{D}+1,$$

which is valid for all positive y. With these bounds

$$R_D=\max_{y\le x}\phi(D)\left|\psi(y;D,r)-\frac{y}{\phi(D)}\right|$$

$$\ll\sum_{\chi(\operatorname{mod}D)}\sum_{\substack{\beta>1/2\\|\gamma|\le T}}\frac{x^{\beta}(\log x)^{A+2}}{|\beta+i\gamma|^2}+\left(\frac{x}{T}+x^{1/2}\right)D(\log x)^2+\frac{x}{(\log x)^{A+1}},$$

holds uniformly for $2\le T\le x(\log x)^{-A-2}, D\le x^{3/4}$. We set $T=x^{1/2}$.

The double-sum does not exceed

$$4\sum_{1\le 2^k\le T}2^{-2k}\sum_{\chi(\operatorname{mod}D)}\sum_{\substack{\beta>1/2\\|\gamma|\le 2^{k+1}}}x^{\beta}$$

$$=-4\sum_{1\le 2^k\le T}2^{-2k}\int_{1/2+}^{1-\theta+}x^u\,\mathrm{d}N\left(u,2^{k+1},D\right)$$

where $1-\theta$ is the largest value of β taken over all the zeros $\beta+i\gamma$ in the rectangle $0<\operatorname{Re}(s)<1,|\operatorname{Im}(s)|\le 2T$.

Supposing for the moment that $D=qd$ and that there is no zero that is exceptional in the sense of Lemma 4.1, then we may take

$$\theta=c\left(\log d+\left(\log 2q(T+3)\log\log 2q(T+3)\right)^{3/4}\right)^{-1}.$$

In view of Lemma 4.3, typically

$$-\int_{1/2+}^{1-\theta+}x^u\,\mathrm{d}N(u,\tau,D)$$

$$=-x^uN(u,\tau,D)\Big|_{1/2+}^{1-\theta+}+\int_{1/2}^{1-\theta}N(u,\tau,D)x^u\log x\,\mathrm{d}u$$

$$\ll x^{1/2}N(1/2,\tau,D)+c_2\int_{1/2}^{1-\theta}(D\tau)^{\frac{12}{5}(1-u)}x^u\log x\,\mathrm{d}u.$$

with restriction $q\le x^{5/12}\Delta^{-2}$ we have $D^{12/5}\le x\exp\left(-(\log x)^{7/8}\right)$, then the integral is

$$\ll x\exp\left(-\theta(\log x)^{7/8}\right)\tau^{3/2}\ll x\exp\left(-(\log x)^{1/9}\right)\tau^{3/2},$$

uniformly for $\tau\le 2T$ and $d\le\Delta$. Moreover, $N(1/2,\tau,D)\ll D(\tau+2)\log D(\tau+2)$, Prachar [11], Satz 3.3, p. 220, as earlier. Altogether

$$R_{qd}\ll x(\log x)^{-A-1}$$

with the same uniformity in d.

If there is an exceptional zero $(\operatorname{mod}qd)$, for which $\beta>1-c_1(2\log 4a\Delta)^{-1}$, and the corresponding function $L(s,\chi)$ is attached to a real character induced by a primitive character $\chi'(\operatorname{mod}D')$, then D' is a divisor of some $4ad$ with $d\le\Delta$, and an application of Lemma 4.2 shows that there is no further L-function formed with a real character $(\operatorname{mod}D), D\le 4a\Delta$, that has a real zero on the line-segment $1-c_1(2\log 4a\Delta)^{-1}\le\operatorname{Re}(s)<1,\operatorname{Im}(s)=0$ unless that character is also induced by $\chi'(\operatorname{mod}D')$. In particular, D will be divisible by D'. For those moduli qd for which $4ad$ is not a multiple of D' we may choose the same θ as before and recover the above estimate for R_{qd}.

Hence

$$\sum_{d\le\Delta}{}''\max_{y\le x}\left|\psi(y;qd,r)-\frac{y}{\phi(qd)}\right|\ll\frac{x}{(\log x)^{A+1}}\sum_{d\le\Delta}\frac{1}{\phi(qd)}\ll\frac{x}{\phi(q)(\log x)^{A}},$$

where $''$ indicates that the moduli are not divisible by the (possibly non-existent) modulus D'.

A theorem of Siegel shows that for any $\epsilon>0$ there is a positive constant $c(\epsilon)$ so that an L-function fromed with a real character $(\operatorname{mod}D)$ has no zero on the line-segment $1-c(\epsilon)D^{-\epsilon}\le\operatorname{Re}(\epsilon)<1,\operatorname{Im}(s)=0;$ cf. Prachar [11], Satz 8.2, p.144. Unless $D'\ge\left(c(\epsilon)(\log x)^{1/2}\right)^{1/\epsilon}$, this again allows the argument to proceed. We may therefore assume that $D'>(\log x)^{A+2}$ and remove the restriction $''$ from the above summation over d at an expense of

$$\ll\sum_{\substack{d\le\Delta\\4ad\equiv 0(\operatorname{mod}D')}}\frac{x\log x}{qd}\ll\frac{x(\log x)^2}{qD'}\ll\frac{x}{q(\log x)^{A}}.$$

A modified version of this argument delivers the bound

$$\max_{y\le x}\left|\psi(y;q,r)-\frac{y}{\phi(q)}\right|\ll\frac{x}{\phi(q)(\log x)^{A+1}},$$

and in this case there is no exceptional zero.

By substraction we see that $G(\Delta)\ll x\left(\phi(q)(\log x)^{A}\right)^{-1}$ indeed holds for every fixed $A>0$.

Since $\tau(q)\ll\log q$, an application of Lemma 2.1 shows that with $B=A+6$,

$$\sum_{qd \le x^{\delta}(\log x)^{-B}} \max_{(r,qd)=1} \max_{y \le x} \left| \psi(y;qd,r) - \frac{y}{\phi(qd)} \right| \ll \frac{x}{\phi(q)(\log x)^{A-1}},$$

uniformly for moduli $q \le x^{\theta} \exp\left(-(\log\log x)^3\right)$ that are powers of a where $\theta = 2/5$, if $\delta = 1/2$ and $\theta = 5/12$, if $\delta = 9/20$.

Replacing $\psi(y;qd,r)$ in this bound by

$$\theta(y;qd,r) = \sum_{\substack{p \le y \\ p \equiv r (\bmod qd)}} \log p$$

introduces an error

$$\ll \sum_{qd \le x^{1/2}(\log x)^{-B}} \sum_{2 \le m \ll \log x} \sum_{\substack{p \le x^{1/m} \\ p^m \equiv r (\bmod qd)}} \log p$$

$$\ll \sum_{qd \le x^{1/2}(\log x)^{-B}} x^{1/2} \ll xq^{-1}(\log x)^{-A-6},$$

the congruence condition $p^m \equiv r(\bmod qd)$ having been ignored.

Employing the Brun-Titchmarsh bound $\pi(y;D,r) \ll y(\phi(D)\log y)^{-1}$, valid unifromly for $1 \le D \le y^{3/4}, (r,D) = 1$, We see that the contribution to the sum in the theorem that arises from maxima that occur in the range $0 < y \le y_0 = x(\log x)^{-A}$ is

$$\ll \sum_{d \le x^{1/2}q^{-1}} y_0 \left(\phi(qd)\log y_0\right)^{-1} \ll y_0\phi(q)^{-1}$$

$$\ll x\left(\phi(q)(\log x)^{A}\right)^{-1}.$$

We may therefore confine our attention to maxima over the range $y_0 \le y \le x$.

Integration by parts shows that

$$\max_{y_0 \le y \le x} \left| \pi(y;D,r) - \frac{Li(y)}{\phi(D)} \right|$$

$$\ll \left| \pi(y_0;D,r) - \frac{Li(y)}{\phi(D)} \right| + \frac{1}{\log x} \max_{y_0 \le y \le x} \left| \theta(y;D,r) - \frac{y}{\phi(D)} \right|.$$

The theorems hold with $B = A + 6$.

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