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On the Generalization of Hilbert's 17th Problem and Pythagorean Fields

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ABSTRACT

The notion of preordering, which is a generalization of the notion of ordering, has been introduced by Serre. On the other hand, the notion of round quadratic forms has been introduced by Witt. Based on these ideas, it is here shown that 1) a field F is formally real *n*-pythagorean iff the *n*th radical, R_nF is a preordering (Theorem 2), and 2) a field F is *n*-pythagorean iff for any *n*-fold Pfister form ρ. There exists an odd integer $l(>1)$ such that $l\times\rho$ is a round quadratic form (Theorem 8). By considering upper bounds for the number of squares on Pfister's interpretation, these results finally lead to the main result (Theorem 10) such that the generalization of pythagorean fields coincides with the generalization of Hilbert's 17th Problem.

Keywords: Hilbert's 17th Problem; Preorderings; *n*th Radical; Pythagorean Fields; Round Quadratic Forms

1. Introduction

In the latter half of the twentieth century, a consideration for the generalization of pythagorean fields has been made by many researchers, e.g., Elman and Lam [1], Becker [2], Koziol, Szymiczek, and Yucas [3-6], Kijima and Nishi [7], and so on.

Throughout the paper, let F be a field of characteristic different from 2 and $\dot{F}$ be the multiplicative group of F. A field F is said to be *pythagorean* if $F^2+F^2=F^2$. For a quadratic form φ over F, we put $D_F(\varphi)=\{a\in\dot{F};\varphi \text{ represents } a\}$ and $G_F(\varphi)=\{a\in\dot{F};a\varphi\cong\varphi\}$. Witt [8] defined a *round* quadratic form φ as $D_F(\varphi)=G_F(\varphi)$. Recall that Pfister forms are round ([8], Satz 4. (c)).

The class of fields with the following property A_n has been proposed by Elman and Lam [1]:

A_n : *Any torsion n-fold Pfister form over F is hyperbolic.*

Furthermore, they made a *hypothesis* that if a field F satisfies the property A_n, then the ideal I^nF is torsion-free, where IF is the ideal of even dimensional forms in the Witt ring WF. Szymiczek [5] replaced this hypothesis with a problem of rigid elements that if $1\in R^nF$, then $1\in SupR^nF$, and had studied this problem for amenable fields, linked fields, abstract Witt rings of elementary type, and so on. When a field F satisfies the property A_n, it is clear that F also satisfies the property A_{n+1}.

We denote by P_nF the set of *n*-fold Pfister forms over F and by R_nF the *nth radical* of F, which is given by $\bigcap\{D_F(\phi);\phi\in P_nF\}$. This radical defined by Yucas [6] shows a generalization of Kaplansky's radical $R(F)=R_1F$ [9].

Later, Koziol [3] has proposed the class of *n-pythagorean* fields with the following property as *every n-fold Pfister form represents all sums of squares over* F, that is, $R_nF\supseteq D_F(\infty)$.

Pythagorean fields are 0-pythagorean and the class of 1-pythagorean fields is the same as the class of quasi-pythagorean fields defined by Kijima and Nishi [7]. In fact, the class of *n*-pythagorean fields is the same as the class of fields which satisfy the property A_{n+1} ([3], Proposition 2.3).

On the other hand, a generalization of Hilbert's 17th Problem has been accomplished by Artin [10]. Later, an interpretation of this generalization has been made by Pfister [11], who has proposed the class of U_n-fields with the following property as *for any* $\rho\in P_nF$, $D_K(\rho)=\dot{K}$ *holds, where*

$$K=F\left(\sqrt{-1}\right).$$

Furthermore, he showed implicitly in ([11], chapter 6, Theorem 3.5) that if a field F is a U_n-field, then F is *n*-pythagorean.

Unexplained notation and terminology refer to [12,13].

2. Preorderings and Round Quadratic Forms

Pfister [11] has derived upper bounds for the number of squares on Hilbert's 17th Problem. Hence, the following can be shown by results of Artin [10] and Pfister ([11], chapter 6, Corollary 3.4).

Theorem 1. *Let* $F := R(x_1, x_2, \cdots, x_n)$ *be the rational function field in n variables over a real closed field R and* $f = f(x_1, x_2, \cdots, x_n)$ *be an element of* $\dot{F}$. *Then the following statements are equivalent*:

1) $f(a) = f(a_1, a_2, \cdots, a_n) \geq 0$ *for all* $a = (a_1, a_2, \cdots, a_n) \in R^n$ *where* $f(a)$ *is defined.*

2) $f \in D_F(2^n \times \langle 1 \rangle)$ *holds.*

3) f *is a totally positive element.*

We shall prove some results by use of the notions of preorderings (Serre [14]) and round quadratic forms. By Proposition 2.3 in [3] and Lemma 3.1 in [15], the following can be shown.

Theorem 2. (([16], Proposition 1), ([17], Proposition 2.1)). *For a field F, the following statements are equivalent*:

1) *F is n-pythagorean.*

2) $D_F(2\rho) = D_F(\rho)$ *holds for all* $\rho \in P_n F$.

In particular, if *F* is formally real, these statements are further equivalent to the condition.

3) *The nth radical* $R_n F$ *is a preordering.*

If a field F is *n*-pythagorean, then $D_F(\infty) = R_n F = D_F(2^n \times \langle 1 \rangle)$. Thus, the following can be obtained.

Corollary 3. (cf. [13], Corollary 11.4.11). *For any formally real n-pythagorean field F, every totally positive element of F is a sum of* 2^n *squares.*

Remark 4. Corollary 3 shows a generalization of Hilbert's 17th Problem. The notion of preordering and *n*th radical play an important role for this Problem. A typical example of *n*-pythagorean field is a field of transcendence degree *n* over a real closed field. Many examples of *n*-pythagorean fields are known. For example, *n*-Hilbert fields are so in [4]. Also, Kijima [18] has constructed many such examples by use of some results of Kula [19].

Next, we shall discuss about the generalization of pythagorean fields. The following result is well-known.

Theorem 5. (cf. [8], Satz 3. (g)). *Let* F *be a field and* l (>1) *be an odd integer. Then the following statements are equivalent*:

1) The form $\varphi := l \times \langle 1 \rangle$ is round.

2) F is pythagorean.

In particular, if the form φ is anisotropic, then F is a formally real field.

Proposition 6. ([16, Proposition 3]). *Let* ρ *be an n-fold Pfister form over F and l* (>1) *be an odd integer. If* $l \times \rho$ *is a round quadratic form, then* $D_F(2\rho) = D_F(\rho)$ *holds.*

Proof. For any $a \in D_F(2\rho)$, it is sufficient to show that $a \in D_F(\rho)$. The round form $l \times \rho$ means that $a(l \times \rho) \cong l \times \rho$. Since l is an odd integer, we put $l = 2m + 1$ for some integer m. Hence it follows that $a(2m\rho \perp \rho) \cong 2m\rho \perp \rho$. On the other hand, since 2ρ is a Pfister form, $a(2\rho) \cong 2\rho$ holds and then $a(2m\rho) \cong 2m\rho$. Thus $a\rho \cong \rho$ follows from Witt's Cancellation Theorem. This implies that $a \in D_F(\rho)$.

Corollary 7. ([16, Proposition 3]). *If there exist an integer* m (≥ 0) *and an odd integer* l (>1) *such that the form* $2^m l \times \langle 1 \rangle$ *is anisotropic round, then F is formally real.*

Proof. Since the form $2^m l \times \langle 1 \rangle$ is round, it follows from Proposition 6 that $D_F(2^m \times \langle 1 \rangle) = D_F(\infty)$. If a field F is non-real, then $\dot{F} = D_F(\infty)$. This contradicts the assumption that $2^m l \times \langle 1 \rangle$ is anisotropic.

As a characterization of an *n*-pythagorean property, the following generalization of pythagorean fields can be presented.

Theorem 8. ([16], Proposition 3). *For a field F, the following statements are equivalent*:

1) *F is n-pythagorean*.

2) *For any* $\rho \in P_n F$, *there exists an odd integer* $l(>1)$ *such that* $l \times \rho$ *is a round quadratic form*.

Proof. (1) ⇒ (2): If a field F is *n*-pythagorean, then $l \times \rho$ is a round quadratic form for any $m \geq n$, any positive integer l and any $\rho \in P_m F$.

(2) ⇒ (1): This follows from Theorem 2 and Proposition 6.

Theorem 9. *The n-pythagorean field is the generalization of pythagorean field and the Pythagoras number of this field is at most* 2^n.

Proof. If a field F is *m*-pythagorean, then F is (*m* + 1)-pythagorean. Thus, it follows from Theorem 5 and Theorem 8.

Finally, the main result of this paper has been established as follows.

Theorem 10. *The generalization of pythagorean fields coincides with the generalization of Hilbert's 17th Problem.*

Proof. If a field F is non-real, then F has no ordering and moreover $\dot{F} = D_F(\infty)$ holds. Therefore, Hilbert's 17th Problem results in a problem that if a field F is non-real, then does an equality $D_F(\infty) = D_F(2^n \times \langle 1 \rangle)$ hold? Thus, the required result can be established by use of Corollary 3 and Theorem 9.

Incidentally, the notion of round quadratic forms is connected with the torsion-freeness of the ideal $I^n F$. We shall extend Proposition 2.3 in [7] to an *n*-pythagorean field.

Proposition 11. ([17, Proposition 3.1]). *Let* n *be an integer* ≥ 1. *If* F *is an n-pythagorean field, then the following statements hold.*

1) $2^{n-1}W_tF = \{2^{n-1} \times \langle 1,-a \rangle ; a \in R_nF\}$, *where* W_tF *is the maximal torsion subgroup of* WF.

2) $2^{n-1}I^2F$ *is torsion-free.*

Proof. 1) For any element p of $2^{n-1}W_tF$, there exists an element q of W_tF such that $p = 2^{n-1}q$. By ([20], Satz 22), we can find $b_i \in \dot{F}$ and $a_i \in D_F(\infty) = R_nF$ $(i = 1, \cdots, m)$ such that $q = \sum_{i=1}^{m} b_i \langle 1,-a_i \rangle$ in WF. Because of $a_i \in R_nF = D_F(2^n \times \langle 1 \rangle)$, the Pfister form $2^{n-1} \cdot \langle 1,-a_i \rangle$ is universal and round. Hence $b_i(2^{n-1} \cdot \langle 1,-a_i \rangle) \cong 2^{n-1} \cdot \langle 1,-a_i \rangle$ and $p = 2^{n-1} \cdot \langle 1,-a_1a_2 \cdots a_m \rangle$ in WF.

2) For any element p of $2^{n-1}I^2F \cap W_tF$, it is sufficient to show that $p = 0$. Now there exists an element q of I^2F such that $p = 2^{n-1}q$. Since $p \in W_tF$, it follows from 1) that $2^n p = 2(2^{n-1}p) = 2^{2n-1}q = 0$. Therefore $q \in W_tF$ and then $p = 2^{n-1}q = 2^{n-1} \times \langle 1,-a \rangle$ for some element $a \in D_F(2^n \times <1>)$ from 1). Since p is an element of $I^{n+1}F$, it follows from ([13], Hauptsatz 10.5.1) that $p = 0$.

Remark 12. In case of 1-pythagorean fields, statements 1) and 2) of Proposition 11 are equivalent (see Remark 2.4 in [7]). As a characterization of 1-pythagorean fields, Corollary 4.4 of Krawczyk [21] is beautiful and can be extended to n-pythagorean fields. This will be given in the forthcoming paper [22].

3. Concluding Remark

Becker [2] has used the terminology of *n-pythagorean* fields F as $F^{2n} + F^{2n} = F^{2n}$. Therefore, for the field with the property A_{n+1} defined by Elman and Lam [1], the following name shall be recommended as

Hilbert-Pythagoras field of level n.

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