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Strong Convergence Results for Hierarchical Circularly Iterative Method about Hierarchical Circularly Optimization

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ABSTRACT

An hierarchical circularly iterative method is introduced for solving a system of variational circularly inequalities with set of fixed points of strongly quasi-nonexpansive mapping problems in this paper. Under some suitable conditions, strong convergence results for the hierarchical circularly iterative sequence are proved in the setting of Hilbert spaces. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé.

Keywords: Hierarchical Optimization Problems; Circularly Variational Inequalities; Fixed Point; Hierarchical Circularly Iterative Sequence; Strongly Quasi-Nonexpansive Mapping

1. Introduction

For a given nonlinear operator $T:H\to H$, the following classical variational inequality problem is formulated as finding a point $p^*\in D$ such that

$$\langle Tp^*, x-p^*\rangle \ge 0,\quad x\in D.$$

Variational inequalities were initially studied by Stampacchia [1] and ever since have been widely studied, since they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance. On the other hand, a number of mathematical programs and iterative algorithms have been developed to resolve complex real world problems.

The concept of variational inequalities plays an important role in structural analysis, mechanics and economics. Recently, the hierarchical variational inequalities and hierarchical iterative sequence problems have attached many authors' attention (see [2-11]).

2. Preliminaries and Lemma

It is well-known that, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$P_C(x)=\inf_{y\in H}\|x-y\|,\ \forall x\in H.$$

Such a mapping P_C from H onto C is called the metric projection.

Lemma 2.1 (*see* [12]) *The metric projection* $P_C:H\to C$ *has the following basic properties*:

1) P_C is firmly nonexpansive, *i.e.*,

$$\langle P_C(x)-P_C(y), x-y\rangle \ge \|P_C(x)-P_C(y)\|^2\ (\forall x,y\in H),$$

and so P_C is nonexpansive.

2) $\langle x-P_Cx, y-P_Cx\rangle\le 0$, for all $x\in H$ and $y\in C$.

Definition 2.2

1) A mapping $T:H\to H$ is said to be α-inverse-strongly monotone if there exists $\alpha>0$ such that

$$\langle Tx-Ty, x-y\rangle\ge\alpha\|Tx-Ty\|,\ \forall x,y\in H.$$

2) A mapping $T:H\to H$ is said to be α-Lipschitzian if

$$\|Tx-Ty\|\le\alpha\|x-y\|,\ \forall x,y\in H.$$

3) A mapping $T:H\to H$ is said to be quasi-nonexpansive if $Fix(T)\ne\Phi$ and

$$\|Tx-p\|\le\|x-p\|,\ \forall x\in H,\ p\in Fix(T).$$

4) A mapping $T:H\to H$ is said to be strongly quasi-nonexpansive if T is quasi-nonexpansive and $x_n-Tx_n\to 0$, whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n-p\|-\|Tx_n-p\|\to 0$ for some $p\in Fix(T)$.

5) (see [13]) A mapping $T:H\to H$ is said to be

ω -demicontractive if $Fix(T) \neq \Phi$ and

$$\langle x-Tx, x-p\rangle \geq \frac{1-\omega}{2}\|x-Tx\|^2, \forall x \in Hp \in Fix(T).$$

Obviously, the above inequality is equivalent to

$$\|Tx-p\|^2 \leq \|x-p\|^2 + \omega\|x-Tx\|^2,$$

and it is clear from the preceding definitions that every quasi-nonexpansive mapping is 0-demicontractive.

Lemma 2.3 (*see* [14]) *For* $x, y \in H$ *and* $\omega \in [0,1]$, *we have the following statements*:

a) $|\langle x, y\rangle| \leq \|x\|\|y\|$;

b) $\|x+y\| \leq \|x\|^2 + 2\langle y, y+x\rangle$;

c) $\|(1-\omega)x+\omega y\|^2$
$= (1-\omega)\|x\|^2 + \omega\|y\|^2 - \omega(1-\omega)\|x-y\|^2$.

For prove our result, we give the following lemma.

Lemma 2.4 ([11]) *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\alpha_{n_i} \leq \alpha_{n_i+1}$ *for all* $i \in N$. *Then there exists a nondecreasing* $\{m_k\} \subset N$, *such that* $m_k \to \infty$ *and the following properties are satisfied for all* (*sufficiently large*) *numbers sequence* $k \subset N$:

$$\alpha_{m_k} \leq \alpha_{m_k+1} \text{ and } \alpha_k \leq \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j \leq \alpha_{j+1}\}$.

Lemma 2.5 ([11]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that* (*a*) $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$, (*b*) $\limsup_{n\to\infty}\delta_n \leq 0$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.6 ([11]) *Let* $\{a_n\} \subset [0,\infty)$,

$$\{\alpha_n\} \subset [0,1], \{b_n\} \subset (-\infty, +\infty)$$

and $\lambda \in [0,1]$, *such that*

- $\{a_n\}$ is a bounded sequence;
- $a_{n+1} \leq (1-\alpha_n)^2 a_n + 2\alpha_n\lambda\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, for all $n \in N$;
- whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, it follows that

$$\limsup_{k\to\infty} b_{n_k} \leq 0;$$

- $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

In [11], the existence and uniqueness of solutions of some related hierarchical optimization problems had been discussed.

Inspired by these results in the literature, a circularly iterative method in this paper is introduced for solving a system of variational inequalities with fixed-point set constraints. Under suitable conditions, strong convergence results are proved in the setting of Hilbert spaces. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé. The results presented in the paper improve and extend the corresponding results in [11] and other.

3. Main Results

First, we discuss the existence and uniqueness of solutions of some related hierarchical optimization problems.

Theorem 3.1 *Let* $S_i : H \to H$ *be quasi-nonexpansive mappings and* $f_i : H \to H$ *be contractions* $(i = 1, 2, \cdots, m)$. *Then there exists a unique element* $p_i \in Fix(S_i)$ *such that the following inequalities,*

$$\begin{cases} \langle p_i - f_i(p_{i+1}), u_i - p_i\rangle \geq 0, \\ \forall u_i \in Fix(S_i), i = 1, 2, \cdots, m-1, \\ \langle p_m - f_m(p_1), u_m - p_m\rangle \geq 0, \\ \forall u_m \in Fix(S_m). \end{cases} \quad (1)$$

Proof. The proof is a consequence of the well-known Banach's contraction principle but it is given here for the sake of completeness. It is known that both sets $Fix(S_i)$ $(i = 1, 2, \cdots, m)$ are closed and convex, and hence the projections $P_{Fix(S_i)}$ $(i = 1, 2, \cdots, m)$ are well defined. It is clear that the mapping

$$P_{Fix(S_1)} \cdot f_1 \cdot P_{Fix(S_2)} \cdot f_2 \cdots P_{Fix(S_m)} \cdot f_m$$

is a contraction. Hence, there exists a unique element $p_1 \in H$ such that

$$p_1 = \left(P_{Fix(S_1)} \cdot f_1 \cdot P_{Fix(S_2)} \cdot f_2 \cdots P_{Fix(S_m)} \cdot f_m\right) p_1.$$

Put $p_m = P_{Fix(S_m)} f_m p_1$ and

$$p_i = P_{Fix(S_1)} f_1 p_{i+1} \ (i = 1, 2, \cdots, m-1).$$

Then $p_2 \in P_{Fix(S_2)}$, $p_3 \in P_{Fix(S_3)}$ and $p_1 = P_{Fix(S_1)} f_1 p_2$. Suppose that there is an element

$$p_i^* \in Fix(S_i) \ (i = 1, 2, \cdots, m),$$

such that the following inequalities,

$$\begin{cases} \langle p_i^* - f_i(p_{i+1}^*), u_i - p_i^*\rangle \geq 0, \\ \forall u_i \in Fix(S_i), i = 1, 2, \cdots, m-1, \\ \langle p_m^* - f_m(p_1^*), u_m - p_m^*\rangle \geq 0, \\ \forall u_m \in Fix(S_m). \end{cases}$$

Then $p_m^* = P_{Fix(S_m)} f_m p_1^*$ and

$$p_i^* = P_{Fix(S_1)} f_1 p_{i+1}^* \ (i = 1, 2, \cdots, m-1).$$

Hence,

$$p_1^* = \left(P_{Fix(S_1)} \cdot f_1 \cdot P_{Fix(S_2)} \cdot f_2 \cdots P_{Fix(S_m)} \cdot f_m\right) p_1^*.$$

This implies that $p_1 = p_1^*$ and hence

$$p_2 = p_2^*, \cdots, p_m = p_m^*.$$

This completes the proof. □

For mappings $S_i, f_i : H \to H$, suppose $i = 1, 2, \cdots, m$, we define the iterative sequences $\left\{x_n^{(i)}\right\}$ by

$$\begin{cases} x_0^{(i)} \in H, \\ x_{n+1}^{(1)} = (1-\alpha_n) S_1 x_n^{(1)} + \alpha_n f_1\left(S_2 x_n^{(2)}\right), \\ x_{n+1}^{(2)} = (1-\alpha_n) S_2 x_n^{(2)} + \alpha_n f_2\left(S_3 x_n^{(3)}\right), \\ \quad \vdots \\ x_{n+1}^{(m-1)} = (1-\alpha_n) S_{m-1} x_n^{(m-1)} + \alpha_n f_{m-1}\left(S_m x_n^{(m)}\right), \\ x_{n+1}^{(m)} = (1-\alpha_n) S_m x_n^{(m)} + \alpha_n f_m\left(S_1 x_n^{(1)}\right), \end{cases} \tag{2}$$

where $\alpha_n \in [0,1]$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Theorem 3.2 *For every $i \in \{1, 2, \cdots, m\}$, let $S_i : H \to H$ be strongly quasi-nonexpansive mappings such that $I - S_i$ are demiclosed at zero and let f_i be contractions with the coefficient $\hat{\alpha}$. Then the iterative sequences $\left\{x_n^{(i)}\right\}$ by* (2) *strong converge to p_i, respectively, where p_i is the unique element in $Fix(S_i)$ verifying* (1).

Recall that a mapping $T : H \to H$ is demiclosed at zero iff $Tx = 0$ whenever $x_n \rightharpoonup x$ and $Tx_n \to 0$.

We split the proof of Theorem 3.2 into the following lemmas.

Lemma 3.3 *The sequences $\left\{x_n^{(i)}\right\} (i = 1, 2, \cdots, m)$ are bounded.*

Proof. Since $S_i : H \to H$ be strongly quasi-nonexpansive mappings, f_i be contractions with the coefficient $\hat{\alpha}$. Then we have

$$\begin{aligned} \left\|x_{n+1}^{(1)} - p_1\right\| &\le (1-\alpha_n)\left\|S_1 x_n^{(1)} - p_1\right\| + \alpha_n \left\|f_1\left(S_2 x_n^{(2)}\right) - p_1\right\| \\ &\le (1-\alpha_n)\left\|x_n^{(1)} - p_1\right\| + \alpha_n \left\|f_1\left(S_2 x_n^{((2))}\right) - f_1(p_2)\right\| + \alpha_n \left\|f_1(p_2) - p_1\right\| \\ &\le (1-\alpha_n)\left\|x_n^{(1)} - p_1\right\| + \alpha_n \hat{\alpha} \left\|S_2 x_n^{(2)} - p_2\right\| + \alpha_n \left\|f_1(p_2) - p_1\right\| \\ &\le (1-\alpha_n)\left\|x_n^{(1)} - p_1\right\| + \alpha_n \hat{\alpha} \left\|x_n^{(2)} - p_2\right\| + \alpha_n \left\|f_1(p_2) - p_1\right\|. \end{aligned}$$

Similarly, we also have

$$\begin{cases} \left\|x_{n+1}^{(2)} - p_2\right\| \le (1-\alpha_n)\left\|x_n^{(2)} - p_2\right\| + \alpha_n \hat{\alpha}\left\|x_n^{(3)} - p_3\right\| + \alpha_n \left\|f_2(p_3) - p_2\right\|, \\ \qquad \vdots \\ \left\|x_{n+1}^{(m-1)} - p_{m-1}\right\| \le (1-\alpha_n)\left\|x_n^{(m-1)} - p_{m-1}\right\| + \alpha_n \hat{\alpha}\left\|x_n^{(m)} - p_m\right\| + \alpha_n \left\|f_{m-1}(p_m) - p_{m-1}\right\|, \\ \left\|x_{n+1}^{(m)} - p_m\right\| \le (1-\alpha_n)\left\|x_n^{(m)} - p_m\right\| + \alpha_n \hat{\alpha}\left\|x_n^{(1)} - p_1\right\| + \alpha_n \left\|f_m(p_1) - p_m\right\|. \end{cases}$$

It implies that

$$\begin{aligned} &\sum_{i=1}^m \left\|x_{n+1}^{(i)} - p_i\right\| \le \left[1 - (1-\hat{\alpha})\alpha_n\right] \sum_{i=1}^m \left\|x_n^{(i)} - p_i\right\| \\ &+\alpha_n \left(\left\|f_1(p_2) - p_1\right\| + \left\|f_2(p_3) - p_2\right\| + \cdots + \left\|f_{m-1}(p_m) - p_{m-1}\right\| + \left\|f_m(p_1) - p_m\right\|\right) \\ &\le \max\left\{\sum_{i=1}^m \left\|x_n^{(i)} - p_i\right\|, \frac{\left\|f_1(p_2) - p_1\right\| + \left\|f_2(p_3) - p_2\right\| + \cdots + \left\|f_{m-1}(p_m) - p_{m-1}\right\| + \left\|f_m(p_1) - p_m\right\|}{1-\hat{\alpha}}\right\}. \end{aligned}$$

By induction, we have

$$\sum_{i=1}^m \left\|x_{n+1}^{(i)} - p_i\right\| \le \max\left\{\sum_{i=1}^m \left\|x_0^{(i)} - p_i\right\|, \frac{\left\|f_1(p_2) - p_1\right\| + \left\|f_2(p_3) - p_2\right\| + \cdots + \left\|f_{m-1}(p_m) - p_{m-1}\right\| + \left\|f_m(p_1) - p_m\right\|}{1-\hat{\alpha}}\right\},$$

for all $n \in N$. In particular, sequences $\left\{x_n^{(i)}\right\}$ are bounded. Consequently, the sequences $\left\{S_i x_n^{(i)}\right\}$ are also bounded.

Lemma 3.4 *For each* $n \in N$*, the following inequality holds*:

$$\begin{cases} \left\|x_{n+1}^{(1)} - p_1\right\|^2 \le (1-\alpha_n)^2 \left\|\left(x_n^{(1)} - p_1\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(2)} - p_2\right\| \left\|x_{n+1}^{(1)} - p_1\right\| + 2\alpha_n \left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle, \\ \quad \vdots \\ \left\|x_{n+1}^{(m-1)} - p_{m-1}\right\|^2 \le (1-\alpha_n)^2 \left\|\left(x_n^{(m-1)} - p_{m-1}\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(m)} - p_m\right\| \left\|x_{n+1}^{(m-1)} - p_{m-1}\right\| \\ \qquad + 2\alpha_n \left\langle f_{m-1}(p_m) - p_{m-1}, x_{n+1}^{(m-1)} - p_{m-1} \right\rangle, \\ \left\|x_{n+1}^{m} - p_m\right\|^2 \le (1-\alpha_n)^2 \left\|x_n^{m} - p_m\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(1)} - p_1\right\| \left\|x_{n+1}^{m} - p_m\right\| + 2\alpha_n \left\langle f_1(p_1) - p_m, x_{n+1}^{m} - p_m \right\rangle. \end{cases} \tag{3}$$

Proof. Since

$$\begin{aligned} &\left\|x_{n+1}^{(1)} - p_1\right\|^2 = \left\|(1-\alpha_n)\left(S_1 x_n^{(1)} - p_1\right) + \alpha_n \left(f_1\left(S_2 x_n^{(2)}\right) - p_1\right)\right\|^2 \\ &\le \left\|(1-\alpha_n)\left(S_1 x_n^{(1)} - p_1\right)\right\|^2 + 2\left\langle \alpha_n \left(f_1\left(S_2 x_n^{(2)}\right) - p_1\right), x_{n+1}^{(1)} - p_1 \right\rangle \\ &\le (1-\alpha_n)^2 \left\|\left(S_1 x_n^{(1)} - p_1\right)\right\|^2 + 2\alpha_n \left\langle f_1\left(S_2 x_n^{(2)}\right) - f_1(p_2), x_{n+1}^{(1)} - p_1 \right\rangle + 2\alpha_n \left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle \\ &\le (1-\alpha_n)^2 \left\|\left(S_1 x_n^{(1)} - p_1\right)\right\|^2 + 2\alpha_n \left\|f_1\left(S_2 x_n^{(2)}\right) - f_1(p_2)\right\| \left\|x_{n+1}^{(1)} - p_1\right\| \\ &\le (1-\alpha_n)^2 \left\|\left(x_n^{(1)} - p_1\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|S_2 x_n^{(2)} - p_2\right\| \left\|x_{n+1}^{(1)} - p_1\right\| \\ &\quad + 2\alpha_n \left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle + 2\alpha_n \left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle \\ &\le (1-\alpha_n)^2 \left\|\left(x_n^{(1)} - p_1\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(2)} - p_2\right\| \left\|x_{n+1}^{(1)} - p_1\right\| + 2\alpha_n \left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle. \end{aligned}$$

Similarly, we also have

$$\begin{cases} \left\|x_{n+1}^{(2)} - p_2\right\|^2 \le (1-\alpha_n)^2 \left\|\left(x_n^{(2)} - p_2\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(3)} - p_3\right\| \left\|x_{n+1}^{(2)} - p_2\right\| + 2\alpha_n \left\langle f_2(p_3) - p_2, x_{n+1}^{(2)} - p_2 \right\rangle, \\ \quad \vdots \\ \left\|x_{n+1}^{(m-1)} - p_{m-1}\right\|^2 \le (1-\alpha_n)^2 \left\|\left(x_n^{(m-1)} - p_{m-1}\right)\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(m)} - p_m\right\| \left\|x_{n+1}^{(m-1)} - p_{m-1}\right\| \\ \qquad + 2\alpha_n \left\langle f_{m-1}(p_m) - p_{m-1}, x_{n+1}^{(m-1)} - p_{m-1} \right\rangle, \\ \left\|x_{n+1}^{m} - p_m\right\|^2 \le (1-\alpha_n)^2 \left\|x_n^{m} - p_m\right\|^2 + 2\alpha_n \hat{\alpha} \left\|x_n^{(1)} - p_1\right\| \left\|x_{n+1}^{m} - p_m\right\| + 2\alpha_n \left\langle f_1(p_1) - p_m, x_{n+1}^{m} - p_m \right\rangle. \end{cases}$$

By Lemma 3.3, we give following result,

$$\begin{aligned} &\left\|x_{n+1}^{(1)} - p_1\right\|^2 + \left\|x_{n+1}^{(2)} - p_2\right\|^2 + \cdots + \left\|x_{n+1}^{(m)} - p_m\right\|^2 \le (1-\alpha_n)^2 \left(\left\|\left(x_n^{(1)} - p_1\right)\right\|^2 + \left\|\left(x_n^{(2)} - p_2\right)\right\|^2 + \cdots + \left\|\left(x^{(m)} - p_m\right)\right\|^2\right) \\ &+2\alpha_n \hat{\alpha} \left(\left\|x_n^{(2)} - p_2\right\| \left\|x_{n+1}^{(1)} - p_1\right\| + \left\|x_n^{(2)} - p_2\right\| \left\|x_{n+1}^{(2)} - p_2\right\| + \cdots + \left\|x_n^{(1)} - p_1\right\| \left\|x_{n+1}^{(m)} - p_m\right\|\right) \\ &+2\alpha_n \left(\left\langle f_1(p_2) - p_1, x_{n+1}^{(1)} - p_1 \right\rangle + \left\langle f_2(p_3) - p_2, x_{n+1}^{(2)} - p_2 \right\rangle + \cdots + \left\langle f_m(p_1) - p_m, x_{n+1}^{(m)} - p_m \right\rangle\right). \end{aligned} \tag{4}$$

Lemma 3.5 *If there exists a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$\liminf_{k\to\infty}\left(\sum_{i=1}^{m}\left\|x_{n_k+1}^{(i)}-p_i\right\|^2-\sum_{i=1}^{m}\left\|x_{n_k}^{(i)}-p_i\right\|^2\right)\geq 0,$$

then

$$\limsup_{k\to\infty}\left(\sum_{i=1}^{m-1}\left\langle f_i\left(p_{i+1}\right)-p_i, x_{n_k+1}^{(i)}-p_i\right\rangle+\left\langle f_m\left(p_1\right)-p_m, x_{n_k+1}^{(m)}-p_m\right\rangle\right)\leq 0. \quad (5)$$

Proof. In fact, we first consider the following assertion:

$$0\leq\liminf_{k\to\infty}\left(\sum_{i=1}^{m}\left\|x_{n_k+1}^{(i)}-p_i\right\|^2-\sum_{i=1}^{m}\left\|x_{n_k}^{(i)}-p_i\right\|^2\right)\leq\liminf_{k\to\infty}\left[\sum_{i=1}^{m-1}\left(1-\alpha_{n_k}\right)\left\|S_i x_{n_k}^{(i)}-p_i\right\|^2+\alpha_{n_k}\left\|f_i\left(S_{i+1}x_{n_k}^{(i+1)}\right)-p_{i+1}\right\|^2\right.$$
$$\left.+\left(1-\alpha_{n_k}\right)\left\|S_m x_{n_k}^{(m)}-p_m\right\|^2+\alpha_{n_k}\left\|f_m\left(S_1 x_{n_k}^{(m)}\right)-p_1\right\|^2-\sum_{i=1}^{m}\left\|x_{n_k}^{(i)}-p_i\right\|^2\right]$$
$$=\liminf_{k\to\infty}\sum_{i=1}^{m}\left(\left\|S_i x_{n_k}^{(i)}-p_i\right\|^2-\left\|x_{n_k}^{(i)}-p_i\right\|^2\right)\leq\limsup_{k\to\infty}\sum_{i=1}^{m}\left(\left\|S_i x_{n_k}^{(i)}-p_i\right\|^2-\left\|x_{n_k}^{(i)}-p_i\right\|^2\right)\leq 0.$$

By Lemma 3.3, the sequences

$$\left\{\left\|S_1 x_{n_k}^{(i)}-p_i\right\|+\left\|x_{n_k}^{(i)}-p\right\|\right\}\left(i=1,2,\cdots,m\right)$$

are bounded. So we have

$$\lim_{k\to\infty}\left(\left\|S_i x_{n_k}^{(i)}-p_i\right\|^2-\left\|x_{n_k}^{(i)}-p_i\right\|^2\right)=0,$$
$$\left(i=1,2,\cdots,m\right).$$

Since $S_i\left(i=1,2,3\right)$ are strongly quasi-nonexpansive,

$$\lim_{k\to\infty}\left(S_i x_{n_k}^{(i)}-x_{n_k}^{(i)}\right)=0,\left(i=1,2,\cdots,m\right).$$

by the iteration scheme (1), we have

$$\lim_{k\to\infty}\left(x_{n_k}^{(i)}-x_{n_k+1}^{(i)}\right)=0,\left(i=1,2,\cdots,m\right).$$

It follows from the boundedness of $\left\{x_{n_k}^{(1)}\right\}$ that there exists a subsequence $\left\{x_{n_{k_l}}^{(1)}\right\}$ of $\left\{x_{n_k}^{(1)}\right\}$ such that $\left\{x_{n_{k_l}}^{(1)}\right\}\rightharpoonup x$ and

$$\lim_{l\to\infty}\left\langle f_1\left(p_2\right)-p_1, x_{n_{k_l}}^{(1)}-p_1\right\rangle$$
$$=\limsup_{k\to\infty}\left\langle f_1\left(p_2\right)-p_1, x_{n_k}^{(1)}-p_1\right\rangle$$
$$=\limsup_{k\to\infty}\left\langle f_1\left(p_2\right)-p_1, x_{n_k+1}^{(1)}-p_1\right\rangle.$$

Since $I-S_1$ is demiclosed at zero, it follows that $x\in Fix\left(S_1\right)$. It follows from (1), we get

$$\lim_{l\to\infty}\left\langle f_1\left(p_2\right)-p_1, x_{n_{k_l}}^{(1)}-p_1\right\rangle$$
$$=\left\langle f_1\left(p_2\right)-p_1, x-p_1\right\rangle\leq 0.$$

Consequently,

$$\limsup_{k\to\infty}\left\langle f_1\left(p_2\right)-p_1, x_{n_k+1}^{(1)}-p_1\right\rangle\leq 0.$$

By using the same argument, we have

$$\begin{cases}\limsup_{k\to\infty}\left\langle f_2\left(p_3\right)-p_2, x_{n_k+1}^{(2)}-p_2\right\rangle\leq 0,\\ \limsup_{k\to\infty}\left\langle f_3\left(p_4\right)-p_3, x_{n_k+1}^{(3)}-p_3\right\rangle\leq 0,\\ \quad\vdots\\ \limsup_{k\to\infty}\left\langle f_m\left(p_1\right)-p_m, x_{n_k+1}^{(m)}-p_m\right\rangle\leq 0.\end{cases}$$

Therefore, we obtain the desired inequality (4).

Next, we prove Theorem 3.2. Denote

$$a_n:=\sum_{i=1}^{m}\left\|x_n^{(i)}-p_i\right\|^2$$

$$b_n:=2\left(\left\langle f_1\left(p_2\right)-p_1, x_{n+1}^{(1)}-p_1\right\rangle\right.$$
$$+\left\langle f_2\left(p_3\right)-p_2, x_{n+1}^{(2)}-p_2\right\rangle+\cdots$$
$$\left.+\left\langle f_m\left(p_1\right)-p_m, x_{n+1}^{(m)}-p_m\right\rangle\right).$$

Since

$$\left\|x_n^{(2)}-p_2\right\|\left\|x_{n+1}^{(1)}-p_1\right\|+\left\|x_n^{(3)}-p_3\right\|\left\|x_{n+1}^{(2)}-p_2\right\|+\cdots$$
$$+\left\|x_n^{(1)}-p_1\right\|\left\|x_{n+1}^{(m)}-p_m\right\|$$
$$\leq\left(\sum_{i=1}^{m}\left\|x_n^{(i)}-p_i\right\|^2\right)^{\frac{1}{2}}\times\left(\sum_{i=1}^{m}\left\|x_{n+1}^{(i)}-p_i\right\|^2\right)^{\frac{1}{2}},$$

We have the following statements from Lemma (3.3), Lemma(3.4) and Lemma(3.5):

- $\left\{a_n\right\}$ is a bounded sequence;
- $a_{n+1}\leq\left(1-\alpha_n\right)^2 a_n+2\alpha_n\lambda\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_n b_n$, for all $n\in N$;
- whenever $\left\{a_{n_k}\right\}$ is a subsequence of $\left\{a_n\right\}$ satisfying $\liminf_{k\to\infty}\left(a_{n_k+1}-a_{n_k}\right)\geq 0$, it follows that

$\limsup_{k\to\infty} b_{n_k} \le 0$.

Hence, it follows from Lemma 2.6 that $a_n \to 0$, It implies that

$$\lim_{n\to\infty} \sum_{i=1}^{m} \left\| x_n^{(i)} - p_i \right\|^2 = 0.$$

This means that

$$\lim_{n\to\infty} x^{(i)} = p_i \ (i = 1, 2, \cdots, m).$$

The proof of Theorem 3.2 is completed. □

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