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On Some Integral Inequalities of Hardy-Type Operators

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ABSTRACT

In recent time, hardy integral inequalities have received attentions of many researchers. The aim of this paper is to obtain new integral inequalities of hardy-type which complement some recent results.

Keywords: Hardy's Inequality; Measurable; Weight Functions & Hardy-Type Operators

1. Introduction

The classical hardy integral inequality reads:

Theorem 1 *Let* $f(x)$ *be a non-negative p-integrable function defined on* $(0,\infty)$, *and* $p>1$. *Then,* f *is integrable over the interval* $(0,x)$ *for each* x *and the following inequality*:

$$\int_0^\infty \left[\frac{1}{x}\left(\int_0^x f(y)\,\mathrm{d}y\right)\right]^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,\mathrm{d}x \qquad (1)$$

holds, where $\left(\frac{p}{p-1}\right)^p$ is the best possible constant (see [1]).

This inequality can be found in many standard books (see [2-7]). Inequality (1) has found much interest from a number of researchers and there are numerous new proofs, as well as, extensions, refinements and variants which is refer to as Hardy type inequalities.

In the recent paper [8], the author proved the following generalization which is an extension of [9].

Theorem 2 *Let* $f(x)\in L^p(X)$, $g(x)\in L^q(X)$ *and* $fg\in L^p(X)$ *be finite, non-negative measurable functions on* $(0,\infty)$, $0<t<a<b<\infty$ *and* $\frac{1}{p}+\frac{1}{q}+1=\frac{1}{r}$ *with* $1<p\le q<\infty$ *such that* $a<x<b$. *Then, the following inequality holds*:

$$\left[\int_a^b\left(\frac{1}{x^q}\left(T(fg)^q\right)\mathrm{d}x\right)\right]^{\frac{r}{q}} \le C\left[\left(\int_a^b t^{(p-1)}\left|f(t)\right|^p \mathrm{d}t\right)\left(\int_a^b t^{(p-1)}\left|g(t)\right|^p \mathrm{d}t\right)\right]^r \qquad (2)$$

where,

$$C=\frac{(b-t)^{1-r}}{1-r}\left[\ln\left|\frac{(b-t)}{a}\right|^{\frac{1}{p^2}}+\left[\frac{1}{p^2(1-r)}\right]\left(\sum_{k=0}^\infty\sum_{n=1}^\infty(-1)^{k+1}(n-1)-(k-1)\left(p^2+1\right)\right)\ln\left[\frac{(b-t)}{a}\right]^R\right]$$

and

$$R=\frac{1}{p^2}\sum_{k=0}^\infty\sum_{n=1}^\infty\left(n-k\left(p^2+1\right)\right) \quad \forall k(1)n.$$

[10] also proved the following integral inequality of Hardy-type mainly by Jensen's Inequality:

Theorem 3 *Let* g *be continuous and nondecreasing on* $[a,b]$, $0\le a\le b<\infty$, *with* $g(x)>0$ *for* $x>0$. *Let* $q\ge p\ge 1$ *and* $f(x)$ *be nonnegative and Lebesgue-Stieltjes integrable with respect to* $g(x)$ *on* $[a,b]$. *Suppose* δ *is a real number such that* $\frac{-p}{q}<\delta<0$, *then*

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)\,\mathrm{d}g(t)\right)^q \mathrm{d}g(x)\right]^{\frac{1}{q}} \le C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p\,\mathrm{d}g(x)\right]^{\frac{1}{p}} \qquad (3)$$

where,

$$C(a,b,p,q,\delta)=(-\delta)^{\frac{q(1-p)}{p}}\left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}}$$

$$\cdot g(b)^{p+\delta q}\left(g(b)^{-\delta}-g(a)^{-\delta}\right)^{\frac{q}{p}(p-1)}>0.$$

Other recent developments of the Hardy-type inequalities can be seen in the papers [11-16]. In this article, we point out some other Hardy-type inequalities which will complement the above results (2) and (3).

2. Main Results

The following lemma is of particular interest (see also [8]).

Lemma. *Let* $1<b<\infty$, $1<p$, $\frac{1}{p}+\frac{1}{q}=1$, *and let* $f(x)$ *be a non-negative measurable function such that* $0\le\int_a^b f^p(t)\,\mathrm{d}t<\infty$. *Then the following inequality holds*:

$$\left(\int_x^b f(t)^q\,\mathrm{d}t\right)^{\frac{1}{q}}<\left(\sqrt[p^2]{\left|\ln\frac{b}{x}\right|}\right)^{(p-1)^2}\left(\int_x^b t^{p-1}f(t)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^{\frac{1}{p}} \quad (4)$$

Proof

Let

$$I=\left(\int_x^b f(t)^q\,\mathrm{d}t\right)^{\frac{1}{q}},$$

then,

$$I=\left[\int_x^b t^{\frac{1}{q}}f(t)^q\,t^{-\frac{1}{q}}\mathrm{d}t\right]^{\frac{1}{q}}$$

by Holder's inequality, we have,

$$I\le\left(\int_x^b t^{\frac{p}{q}}f(t)^{pq}\,\mathrm{d}t\right)^{\frac{1}{pq}}\left(\int_x^b t^{-1}\mathrm{d}t\right)^{\frac{1}{q^2}}$$

$$=\left(\sqrt[p^2]{\left|\ln\frac{b}{x}\right|}\right)^{(p-1)^2}\left(\int_x^b t^{p-1}f(t)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^{\frac{1}{p}}$$

We need to show that there exists $x_0\in(a,b)$ such that for any $x\in(a,x_0)$, equality in (4) does not hold. If otherwise, there exist a decreasing sequence $(x_n)_{n\in\mathbf{N}}$ in (a,b), $x_n\searrow a$ such that for $n\in\mathbf{N}$ the inequality (4), written $x=x_n$, becomes an equality. Then, to every $n\in\mathbf{N}$ there correspond real constants c_n and $d_n\ge0$ not both zero, such that $c_n\left[t^{\frac{1}{q}}f(t)\right]^p=d_n\left[t^{-\frac{1}{q}}\right]^q$ almost everywhere in (x_n,b).

There exists positive integer N such that for $n>N$, $f(t)\ne0$ almost everywhere in (x,b). Hence, $c_n=c\ne0$ and $d_n=d\ne0$ for $n>N$, and also

$$\int_a^b f^p(t)\,\mathrm{d}t=\lim_{n\to\infty}\int_{x_n}^b f^p(t)\,\mathrm{d}t$$

$$=\frac{c}{1-p}\left(b^{1-p}-x_n^{1-p}\right)=\infty$$

This contradicts the facts that $0<\int_a^b f^p(t)\,\mathrm{d}t<\infty$. The lemma is proved.

Theorem 4 *Let* $f(x)\in L^p(X)$, $g(x)\in L^q(X)$ *be finite non-negative measurable functions on* $(0,\infty)$, $0<a<t<b<\infty$ *and* $\frac{1}{p}+\frac{1}{q}+1=\frac{1}{r}$ *with* $1<p\le q\le\infty$ *such that* $a<x<b$, *then the following inequality holds*:

$$\left[\int_a^b\frac{1}{x^q}\left(\int_x^b(fg)^q\,\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{r}{q}}\le C\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^r \quad (5)$$

where

$$C=\frac{(t-a)^{1-r}}{1-r}\left[\ln\left|\frac{b}{(t-a)}\right|^{\frac{2}{p-1}}\right.$$

$$\left.+\frac{2}{(1-r)(p-1)}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(n-1)-(k-1)p\right)\ln\left|\frac{b}{t-a}\right|^R\right]$$

and

$$R=\frac{1}{p-1}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[(n+1)-(k-1)p\right]\quad\forall k(1)n.$$

Proof

$$\left[\int_a^b\frac{1}{x^q}\left(\int_x^b(fg)^q\,\mathrm{d}t\right)\mathrm{d}t\right]^{\frac{r}{q}}$$

$$\le\left[\int_a^b\frac{1}{x^q}\left(\int_x^b|f|^q\,\mathrm{d}t\right)\left(\int_x^b|g|^q\,\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{r}{q}}$$

$$\le\left[\int_a^b\frac{1}{x^q}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\left(\int_x^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^{\frac{1}{p}}\mathrm{d}x\right]^{\frac{r}{q}}$$

$$=\left[\int_a^b x^{-q}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\left(\int_a^t t^{p-1}(fg)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^{\frac{1}{p}}\mathrm{d}x\right]^{\frac{r}{q}}$$

$$\le\int_a^b x^{-r}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\mathrm{d}x\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^r$$

$$=C\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\,\mathrm{d}t\right)^r$$

where C is as stated in the statement of the theorem and this proves the theorem.

The next results are on convex functions as it applies to Hardy-type inequalities.

Lemma. *local minimum of a function f is a global minimum if and only if f is strictly convex.*

Proof

The necessary part follows from the fact that if a point x is a local optimum of a convex function f. Then $f(z)\geq f(x)$ for any z in some neighborhood U of x. For any y, $z=\lambda x+(1-\lambda)y$ belongs to U and $\lambda<1$ sufficiently close to 1 implies that x is a global optimum. For the sufficient part, we let f be a strictly convex function with convex domain. Suppose f has a local minimum at a and b such that $a\neq b$ and assuming $f(a)\leq f(b)$. By strict convexity and for any $\lambda\in(0,1)$, we have,

$$f(\lambda a+(1-\lambda)b)<\lambda f(a)+(1-\lambda)f(b)\leq \lambda f(b)+(1-\lambda)f(b)=f(b).$$

Since any neighborhood of b contains points of the form $\lambda a+(1-\lambda)b$ with $\lambda\in[0,1]$, thus the neighborhood of b contains points x for which $f(x)<f(b)$. Hence, f does not have a local minimum at b, a contradiction. It must be that $a=b$, this shows that f has at most one local minimum.

Lemma. *Let* $0<b<\infty$ *and* $-\infty\leq a<c\leq\infty$. *If* φ *is a positive convex function on* (*a*,*c*), *then*

$$\int_0^b\varphi\left[\frac{1}{x^q}\int_0^x h(t)\,\mathrm{d}t\right]\mathrm{d}x \leq \frac{1}{1-q}\int_0^b\varphi(h(t))\left(b^{1-q}-t^{1-q}\right)\mathrm{d}t \tag{6}$$

Proof

$$\int_0^b\varphi\left[\frac{1}{x^q}\int_0^x h(t)\,\mathrm{d}t\right]\mathrm{d}x\leq\int_0^b\frac{1}{x^q}\left(\int_0^x\varphi(h(t))\,\mathrm{d}t\right)\mathrm{d}x$$
$$=\int_0^b\varphi(h(t))\left(\int_t^b\frac{1}{x^q}\,\mathrm{d}x\right)\mathrm{d}t=\int_0^b\varphi(h(t))\left(\frac{b^{1-q}-t^{1-q}}{1-q}\right)\mathrm{d}t$$
$$=\frac{1}{1-q}\int_0^b\varphi(h(t))\left(b^{1-q}-t^{1-q}\right)\mathrm{d}t$$

Hence the proof.

Lemma. *Let* $h(x,t)$ *be non-negative for* $x,t\geq 0$, λ *non decreasing and* $-\infty\leq a\leq b\leq\infty$. then

$$\int_a^x h(x,t)^{1/pq}\,\mathrm{d}\lambda(t) \leq\left[\int_a^x\mathrm{d}\lambda(t)\right]^{1-\frac{1}{p}}\left[\int_a^x h(x,t)^{1/q}\,\mathrm{d}\lambda(t)\right]^{\frac{1}{p}} \tag{7}$$

Proof

Let Φ be continuous and convex, If Φ has a continuous inverse which is neccessarily concave, then by Jensen's inequality we have

$$\phi^{-1}\left[\frac{\int_a^x h(x,t)\,\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}\right]\geq\frac{\int_a^x\phi^{-1}\left[h(x,t)\right]\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}$$

Taking $\phi(u)=u^p$, $p\geq 1$, we obtain

$$\left[\frac{\int_a^x h(x,t)\,\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}\right]^{\frac{1}{p}}\geq\frac{\int_a^x h(x,t)^{\frac{1}{p}}\,\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}$$

for $1\leq p\leq q$, we have

$$\left[\frac{\int_a^x h(x,t)^{\frac{1}{q}}\,\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}\right]^{\frac{1}{p}}\geq\frac{\int_a^x h(x,t)^{\frac{1}{pq}}\,\mathrm{d}\lambda(t)}{\int_a^x\mathrm{d}\lambda(t)}$$

which we write as

$$\int_a^x h(x,t)^{\frac{1}{pq}}\,\mathrm{d}\lambda(t)\leq\left[\int_a^x\mathrm{d}\lambda(t)\right]^{1-\frac{1}{p}}\left[\int_a^x h(x,t)^{\frac{1}{q}}\,\mathrm{d}\lambda(t)\right]^{\frac{1}{p}}$$

This complete the proof.

Theorem 5 *If* $0<b\leq\infty$ *and* $-\infty\leq a<c\leq\infty$, *let f, g be defined on* (0,*b*) *such that* $a<f(x),g(x)<c$, *then*

$$\int_0^b\exp\left[\frac{1}{x^q}\int_0^x\ln(fg)\,\mathrm{d}t\right]\mathrm{d}x \leq\frac{e}{1-2q}\int_0^b t(fg)\left(b^{1-2q}-t^{1-2q}\right)\mathrm{d}t \tag{8}$$

Proof

$$\int_0^b\exp\left[\frac{1}{x^q}\int_0^x\ln(fg)\,\mathrm{d}t\right]\mathrm{d}x$$
$$=\int_0^b\exp\left(\frac{1}{x^q}\int_0^x\left(\ln t(fg)-\ln t\right)\mathrm{d}t\right)\mathrm{d}x$$
$$=\int_0^b\left[\exp\left(\frac{1}{x^q}\int_0^x\ln t(fg)\mathrm{d}t\right)\times\exp\left(\frac{-1}{x^q}\int_0^x\ln t\mathrm{d}t\right)\right]\mathrm{d}x$$

Since $f(x)=\mathrm{e}^x$ is a convex function, applying Jensen's inequality to the above gives

$$\int_0^b\exp\left[\frac{1}{x^q}\int_0^x\ln(fg)\,\mathrm{d}t\right]\mathrm{d}x$$
$$\leq\int_0^b\frac{1}{x^q}\left[\int_0^x t(fg)\,\mathrm{d}t\times\frac{1}{x^{q-1}}\exp(-\ln x+1)\right]\mathrm{d}x$$
$$=\mathrm{e}\int_0^b\frac{1}{x^{2q}}\left(\int_0^x t(fg)\,\mathrm{d}t\right)\mathrm{d}x=\mathrm{e}\int_0^b t(fg)\left(\int_t^b\frac{1}{x^{2q}}\,\mathrm{d}x\right)\mathrm{d}t$$
$$=\frac{\mathrm{e}}{1-2q}\int_0^b t(fg)\left(b^{1-2q}-t^{1-2q}\right)\mathrm{d}t$$

The result follows.

Theorem 6 *Let g be a continuous and nondecreasing on* $[a,b]$, $0 \le a \le b \le \infty$, *with* $g(x) > 0$ *for* $x > 0$ *and* $a \le t < b$. *Let* $1 \le p \le q$ *and* $f(x)$ *be nonnegative and Lebesgue-Stieltjes integrable with respect to* $g(x)$ *on* $[a,b]$. *Suppose r is a real number such that* $0 > r > -\infty$ *then,*

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_0^x f(t)\mathrm{d}g(t)\right)^q \mathrm{d}g(x)\right]^{\frac{1}{q}} \le C(a,b,p,q,r)\left[\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p \mathrm{d}g(x)\right]^{\frac{1}{p}} \tag{9}$$

where

$$C(a,b,p,q,r) = \left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left(\frac{p}{p+rq}\right)^{\frac{1}{q}}\left(g(b)^{\frac{r-1}{r}} - g(a)^{\frac{r-1}{r}}\right)^{\frac{p-1}{p}}\left(g(b)^{\frac{p+rq}{p}} - g(a)^{\frac{p+rq}{p}}\right)^{\frac{1}{q}}$$

Proof

In the inequality (2.5), we let

$$h(x,t) = g(x)^{rq} g(t)^{\frac{pq}{r}} f(t)^{pq}$$

and

$$\mathrm{d}\lambda(t) = g(t)^{\frac{-1}{r}} \mathrm{d}g(t)$$

Then, the left hand side of (2.5) becomes

$$\int_a^x g(x)^{\frac{r}{p}} g(t)^{\frac{1}{r}} f(t) g(t)^{\frac{-1}{r}} \mathrm{d}g(t) = \int_a^x g(x)^{\frac{r}{p}} f(t)\mathrm{d}g(t) = g(x)^{\frac{r}{p}} \int_a^x f(t)\mathrm{d}g(t)$$

and the right hand side reduces to

$$\left[\int_a^x g(t)^{\frac{-1}{r}} \mathrm{d}g(t)\right]^{\frac{p-1}{p}}\left[\int_a^x g(x)^r g(t)^{\frac{p}{r}} f(t)^p g(t)^{\frac{-1}{r}} \mathrm{d}g(t)\right]^{\frac{1}{p}} = \left[\int_a^x g(t)^{\frac{-1}{r}} \mathrm{d}g(t)\right]^{\frac{p-1}{p}}\left[\int_a^x g(x)^r g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right]^{\frac{1}{p}}$$

$$= \left[\frac{r}{r-1} g(t)^{\frac{r-1}{r}}\Big|_a^x\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right]^{\frac{1}{p}}$$

$$= \left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left[g(x)^{\frac{r-1}{r}} - g(a)^{\frac{r-1}{r}}\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right]^{\frac{1}{p}}$$

Hence, inequality (2.5) becomes

$$g(x)^{\frac{r}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right) \le \left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left[g(x)^{\frac{r-1}{r}} - g(a)^{\frac{r-1}{r}}\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right]^{\frac{1}{p}}$$

for $q \ge p$, we have

$$g(x)^{\frac{rq}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right)^q \le \left(\frac{r}{r-1}\right)^{\frac{q(p-1)}{p}}\left[g(x)^{\frac{r-1}{r}} - g(a)^{\frac{r-1}{r}}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{rq}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right]^{\frac{q}{p}}$$

Integrating both sides with respect to $g(x)$ and then raising both sides to power $\frac{p}{q}$ yields

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right)^q \mathrm{d}g(x)\right]^{\frac{p}{q}}$$

$$\le \left[\left(\frac{r}{r-1}\right)^{\frac{q(p-1)}{p}} \int_a^b g(x)^{\frac{rq}{p}}\left(g(x)^{\frac{r-1}{r}} - g(a)^{\frac{r-1}{r}}\right)^{\frac{q(p-1)}{p}}\left(\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p \mathrm{d}g(t)\right)^{\frac{q}{p}} \mathrm{d}g(x)\right]^{\frac{p}{q}}$$

Applying Minkowski integral inequality to the right hand side implies

$$\le\left(\frac{r}{r-1}\right)^{p-1}\int_a^b g(t)^{\frac{p-1}{r}} f(t)^p\left[\int_t^b\left(g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{\frac{q(p-1)}{p}} g(x)^{\frac{rq}{p}}\,\mathrm{d}g(x)\right]^{\frac{p}{q}}\mathrm{d}g(t)$$

$$\le\left(\frac{r}{r-1}\right)^{p-1}\left(g(b)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{p-1}\int_a^b g(t)^{\frac{p-1}{r}} f(t)^p\left[\int_t^b g(x)^{\frac{rq}{p}}\,\mathrm{d}g(x)\right]^{\frac{p}{q}}\mathrm{d}g(t)$$

Since $r<0$

$$=\left(\frac{r}{r-1}\right)^{p-1}\left(\frac{p}{p+rq}\right)^{\frac{p}{q}}\left(g(b)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{p-1}\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p\left(g(b)^{\frac{p+rq}{p}}-g(t)^{\frac{p+rq}{p}}\right)^{\frac{p}{q}}\mathrm{d}g(x)$$

$$\le C(a,b,p,q,r)\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p\,\mathrm{d}g(x)$$

Hence, we have

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_0^x f(t)\,\mathrm{d}g(t)\right)^q \mathrm{d}g(x)\right]^{\frac{1}{q}}\le C(a,b,p,q,r)\left[\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p\,\mathrm{d}g(x)\right]^{\frac{1}{p}}$$

Which complete the proof of the Theorem.

3. Conclusion

This work obtained considerable improvement on Adeagbo-Sheikh and Imoru results and applications for measurable and convex functions are also given.

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