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Optimality of Distributed Control for $n \times n$ Hyperbolic Systems with an Infinite Number of Variables

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ABSTRACT

In this paper, we study the existence of solutions for $2l$ order $(n \times n)$ cooperative systems governed by Dirichlet and Neumann problems involving hyperbolic operators with an infinite number of variables and with variable coefficients. The necessary and sufficient conditions for optimality of the distributed control with constraints are obtained and the set of inequalities that defining the optimal control of these systems are also obtained.

Keywords: Hyperbolic Operators; Cooperative Systems; Spaces With an Infinite Number of Variables; Existence and Uniqueness of Solutions; Distributed Control Problems; Control Constraints; Variable Coefficients

1. Introduction

The optimality conditions for systems consisting of only one equation and for $n \times n$ systems governed by different types of partial differential equations defined on spaces of functions of infinitely many variables have been discussed for example in [1-11].

In addition, optimal control problems for systems involving operators with an infinite number of variables for non-standard functional and time delay have been introduced in [12,13].

Furthermore, time-optimal control of systems with an infinite number of variables has been studied in [14-16].

Some applications of optimal control problem for systems involving Schrodinger operators are introduced for example in [17-21].

Making use of the theory of Lions [22] and Berezanskiĭ [23], we consider the optimal control problem of distributed type for $2l$ order $(n \times n)$ cooperative systems governed by Dirichlet and Neumann problems involving hyperbolic operators with an infinite number of variables and with variable coefficients. We first prove the existence and uniqueness of the state for these systems, then we find the set of equations and inequalities that characterize the optimal control of these systems. Finally, we impose some constraints on the control. Necessary and sufficient conditions for optimality with control constraints are derived.

This paper is organized as follows. In Section 1, we introduce spaces of functions of an infinite number of variables. In Section 2, we discuss the distributed control problem for these systems with Dirichlet conditions. In Section 3, we consider the problem with Neumann conditions.

2. Sobolev Spaces with an Infinite Number of Variables

This section covers the basic notations, definitions, and properties, which are necessary to present this work [24]. Let $\left(p_k(t)\right)_{k=1}^{\infty}$ be a sequence of continuous positive probability weights such that

$$0 \prec p_k(t) \in C^{\infty}\left(R^1\right), \int_{R^1} p_k(t)\,\mathrm{d}t = 1,$$

with respect to it we introduce on the region $R^{\infty} = R^1 \times R^1 \times \cdots$, the measure $\mathrm{d}\rho(x)$ by:

$$\mathrm{d}\rho(x) = p_1(x_1)\,\mathrm{d}x_1 \otimes p_2(x_2)\,\mathrm{d}x_2 \otimes \cdots,$$
$$\left(x = \left(x_k\right)_{k=1}^{\infty} \in R^{\infty}, x_k \in R^1\right).$$

On R^{∞} we construct the space $L^2\left(R^{\infty}, \mathrm{d}\rho(x)\right)$ with respect to this measure such that $L^2\left(R^{\infty}, \mathrm{d}\rho(x)\right)$ is the space of all square integrable functions on R^{∞} *i.e.*

$$\|u\|_{L^2\left(R^{\infty}, \mathrm{d}\rho(x)\right)} = \left(\int_{R^{\infty}} |u|^2\,\mathrm{d}\rho(x)\right)^{1/2} \prec \infty.$$

We shall set $L^2\left(R^{\infty}, \mathrm{d}\rho(x)\right) = L^2\left(R^{\infty}\right)$.

$L^2\left(R^\infty\right)$ is a Hilbert space for the scalar product

$$\left(u,v\right)_{L^2\left(R^\infty\right)} = \int_{R^\infty} u\left(x\right)v\left(x\right)\mathrm{d}\rho\left(x\right),$$

associated to the above norm.

We consider a Sobolev space in the case of an unbounded region. For functions which are continuously differentiable l times up to the boundary Γ of R^∞ and which vanish in a neighborhood of ∞, we introduce the scalar product

$$\left(u,v\right)_{W^l\left(R^\infty\right)} = \sum_{|\alpha|\le l}\left(D^\alpha u, D^\alpha v\right)_{L^2\left(R^\infty\right)},$$

where D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}\cdots}, |\alpha| = \sum_{i=1}^{\infty}\alpha_i,$$

and the differentiation is taken in the sense of generalized function on R^∞, and after the completion, we obtain the Sobolev space $W^l\left(R^\infty\right)$, which is a Hilbert space and dense in $L^2\left(R^\infty\right)$. The space $W^l\left(R^\infty\right)$ forms a positive space. We can construct the negative space $W^{-l}\left(R^\infty\right)$ with respect to the zero space $L^2\left(R^\infty\right)$ and then we have the following imbedding

$$W^l\left(R^\infty\right)\subseteq L^2\left(R^\infty\right)\subseteq W^{-l}\left(R^\infty\right),$$

$$\|y\|_{W^l\left(R^\infty\right)} \ge \|y\|_{L^2\left(R^\infty\right)} \ge \|y\|_{W^{-l}\left(R^\infty\right)}.$$

Analogous to the above chain we have a chain of the form

$$W_0^l\left(R^\infty\right)\subseteq L^2\left(R^\infty\right)\subseteq W_0^{-l}\left(R^\infty\right),$$

$$\|y\|_{W_0^l\left(R^\infty\right)} \ge \|y\|_{L^2\left(R^\infty\right)} \ge \|y\|_{W_0^{-l}\left(R^\infty\right)},$$

where

$$W_0^l\left(R^\infty\right) = \left\{u \,/\, u\in W^l\left(R^\infty\right),\ D^\beta u\big|_\Gamma = 0,\ |\beta|\le l-1\right\}$$

with the scalar product

$$\left(u,v\right)_{W_0^l\left(R^\infty\right)} = \sum_{|\alpha|=l}\left(D^\alpha u, D^\alpha v\right)_{L^2\left(R^\infty\right)}$$

and $W_0^{-l}\left(R^\infty\right)$ is its dual.

$L^2\left(0,T;W_0^l\left(R^\infty\right)\right)$ denotes the space of measurable function $t\to f(t)$ on open interval $(0,T)$ for the Lebesgue measure $\mathrm{d}t$ and such that

$$\|f\left(t\right)\|_{L^2\left(0,T;W^l\left(R^\infty\right)\right)} = \left(\int_0^T\|f\left(t\right)\|^2_{W^l\left(R^\infty\right)}\mathrm{d}t\right)^{1/2} \prec \infty,$$

endowed with the scalar product

$$\left(f\left(t\right),g\left(t\right)\right)_{L^2\left(0,T;W^l\left(R^\infty\right)\right)} = \int_0^T\left(f\left(t\right),g\left(t\right)\right)_{W^l\left(R^\infty\right)}\mathrm{d}t,$$

which is a Hilbert space.

Analogously, we can define the spaces

$$L^2\left(0,T;L^2\left(R^\infty\right)\right) = L^2\left(Q\right) \quad\text{and}\quad L^2\left(0,T;W^{-l}\left(R^\infty\right)\right),$$

then we have a chain in the form

$$L^2\left(0,T;W^l\left(R^\infty\right)\right)\subseteq L^2\left(0,T;L^2\left(R^\infty\right)\right)\subseteq L^2\left(0,T;W^{-l}\left(R^\infty\right)\right),$$

where $Q = R^\infty\times\left(0,T\right)$ with boundary $\Sigma = \Gamma\times\left(0,T\right)$.

By the Cartesian product, it is easy to construct the following Sobolev spaces $\left(W^l\left(R^\infty\right)\right)^n$ with the norm defined by

$$\|u\|_{\left(W^l\left(R^\infty\right)\right)^n} = \sum_{i=1}^n\|u_i\|_{W^l\left(R^\infty\right)},$$

where $u = \left(u_1,u_2,\cdots,u_n\right)$ is a vector function and $u_i\in W^l\left(R^\infty\right)$, also we can construct the Cartesian product for the above Hilbert spaces. Finally we have the following chain:

$$\left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n\subseteq\left(L^2\left(0,T;L^2\left(R^\infty\right)\right)\right)^n\subseteq\left(L^2\left(0,T;W^{-l}\left(R^\infty\right)\right)\right)^n,$$

$$\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n\subseteq\left(L^2\left(0,T;L^2\left(R^\infty\right)\right)\right)^n\subseteq\left(L^2\left(0,T;W_0^{-l}\left(R^\infty\right)\right)\right)^n$$

where

$$\left(L^2\left(0,T;W^{-l}\left(R^\infty\right)\right)\right)^n \quad\text{and}\quad \left(L^2\left(0,T;W_0^{-l}\left(R^\infty\right)\right)\right)^n$$

are the dual spaces of $\left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n$ and $\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n$ resp.

3. Dirichlet Problem for 2*l* Order (*n* × *n*) Cooperative Hyperbolic System with an Infinite Number of Variables and with Variable Coefficients

In this section, we study the existence and uniqueness of solutions for $2l$ order $\left(n\times n\right)$ cooperative systems governed by Dirichlet problems involving hyperbolic operators with an infinite number of variables and with variable coefficients, then we find the necessary and sufficient conditions of the optimal control of distributed type.

For $1\le i\le n$, we have the following system:

$$\begin{cases}\dfrac{\partial^2 y_i\left(x\right)}{\partial t^2} + A\left(t\right)y_i = \displaystyle\sum_{j=1}^n a_{ij}\left(x\right)y_j + f_i\left(x,t\right) & \text{in } Q,\\ y_i\big|_\Sigma = 0, \qquad 1\le i\le n, & \\ y_i\left(x,0\right) = y_{0,i}\left(x\right),\quad \dfrac{\partial y_i\left(x,0\right)}{\partial t} = y_{1,i}\left(x\right) & \text{in } R^\infty,\end{cases} \tag{1}$$

where $y_i, \frac{\partial y_i}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right)$, $f = \left(f_1, f_2, \cdots, f_n\right)$ is a given function, and $a_{ij}\left(x\right)$ are bounded functions such that

$$a_{ij}\left(x\right) \succ 0 \quad \text{for all } i \neq j, \text{ for all } x, \tag{2}$$

$$a_{ij} = a_{ji} \quad \text{for all } 1 \leq i, j \leq n. \tag{3}$$

System (1) is called cooperative if (2) holds.

The operator $B\left(t\right) = \frac{\partial^2}{\partial t^2} + A\left(t\right)$ in system (1) is $2l$ order hyperbolic operator with an infinite number of variables with

$$A\left(t\right) \in \mathcal{L}\left(\left(W_0^l\left(R^\infty\right)\right)^n ; \left(W_0^{-l}\left(R^\infty\right)\right)^n\right),$$

[23] is given by:

$$\begin{gathered} A\left(t\right) y_i\left(x\right) = \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left(-1\right)^{|\alpha|} D_k^{2\alpha} y_i\left(x\right) + q\left(x,t\right) y_i\left(x\right) \\ D_k y_i\left(x\right) = \frac{1}{\sqrt{p_k\left(x_k,t\right)}} \frac{\partial}{\partial x_k} \sqrt{p_k\left(x_k,t\right)} y_i\left(x\right), \end{gathered} \tag{4}$$

since $q(x,t)$ is a real valued function in x which is bounded and measurable on R^∞, such that

$$q\left(x,t\right) \geq c,\ 0 < c \leq 1,\ c \text{ is a constant.} \tag{5}$$

Definition 1:

For each $t \in (0,T)$, we define a bilinear form $\pi\left(t; y, \psi\right) : \left(W_0^l\left(R^\infty\right)\right)^n \times \left(W_0^l\left(R^\infty\right)\right)^n \to R$ by

$$\pi\left(t; y, \psi\right) = \left(S\left(t\right) y, \psi\right)_{\left(L^2\left(R^\infty\right)\right)^n},$$

$$y = \left(y_1, y_2, \cdots, y_n\right),\ \psi = \left(\psi_1, \psi_2, \cdots, \psi_n\right) \in \left(W_0^l\left(R^\infty\right)\right)^n,$$

where

$$\begin{aligned} S\left(t\right) y_i\left(x\right) = &\sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left(-1\right)^{|\alpha|} D_k^{2\alpha} y_i\left(x\right) + q\left(x,t\right) y_i\left(x\right) \\ &- \sum_{j=1}^{n} a_{ij}\left(x\right) y_j\left(x\right), \quad i = 1, 2, \cdots, n. \end{aligned}$$

Then,

$$\pi\left(t; y, \psi\right) = \sum_{i=1}^{n} \left(S\left(t\right) y_i, \psi_i\right)_{L^2\left(R^\infty\right)},$$

$$\begin{aligned} \pi\left(t; y, \psi\right) = &\sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} D_k^\alpha y_i\left(x\right) D_k^\alpha \psi_i\left(x\right) \mathrm{d}\rho\left(x\right) \\ &+ \sum_{i=1}^{n} \int_{R^\infty} q\left(x,t\right) y_i\left(x\right) \psi_i\left(x\right) \mathrm{d}\rho\left(x\right) \\ &- \sum_{i,j=1}^{n} \int_{R^\infty} a_{ij}\left(x\right) y_j\left(x\right) \psi_i\left(x\right) \mathrm{d}\rho \end{aligned} \tag{6}$$

3.1. The Existence and Uniqueness of Solution

Lemma 1:

The bilinear form (6) is coercive on $\left(W_0^l\left(R^\infty\right)\right)^n$, that is, there exists c, $c_1 \in R$, such that:

$$\begin{aligned} &\pi\left(t; y, y\right) + c_1 \left\|y\left(x\right)\right\|^2_{\left(L^2\left(R^\infty\right)\right)^n} \\ &\geq c \left\|y\left(x\right)\right\|^2_{\left(W_0^l\left(R^\infty\right)\right)^n},\ c, c_1 > 0 \end{aligned} \tag{7}$$

Proof:

We have,

$$\begin{aligned} \pi\left(t; y, y\right) = &\sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left|D_k^\alpha y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &+ \sum_{i=1}^{n} \int_{R^\infty} q\left(x,t\right) \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- \sum_{i,j=1}^{n} \int_{R^\infty} a_{ij}\left(x\right) y_i\left(x\right) y_j\left(x\right) \mathrm{d}\rho\left(x\right), \end{aligned}$$

thus,

$$\begin{aligned} \pi\left(t; y, y\right) = &\sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left|D_k^\alpha y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &+ \sum_{i=1}^{n} \int_{R^\infty} q\left(x,t\right) \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- \sum_{i=1}^{n} \int_{R^\infty} a_{ii}\left(x\right) \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- \sum_{i \neq j}^{n} \int_{R^\infty} a_{ij}\left(x\right) y_i\left(x\right) y_j\left(x\right) \mathrm{d}\rho\left(x\right). \end{aligned}$$

From (2), (3), and (5), we deduce

$$\begin{aligned} &\pi\left(t; y, y\right) \geq \sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left|D_k^\alpha y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &+ c \sum_{i=1}^{n} \int_{R^\infty} \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) - c_1 \sum_{i=1}^{n} \int_{R^\infty} \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- 2c_1 \sum_{i \prec j}^{n} \int_{R^\infty} y_i\left(x\right) y_j\left(x\right) \mathrm{d}\rho\left(x\right), \end{aligned}$$

then,

$$\begin{aligned} &\pi\left(t; y, y\right) \geq \sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left|D_k^\alpha y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- c_1 \sum_{i=1}^{n} \int_{R^\infty} \left|y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) - 2c_1 \sum_{i \prec j}^{n} \int_{R^\infty} y_i\left(x\right) y_j\left(x\right) \mathrm{d}\rho\left(x\right) \\ &\geq \sum_{i=1}^{n} \int_{R^\infty} \sum_{|\alpha| \leq l} \sum_{k=1}^{\infty} \left|D_k^\alpha y_i\left(x\right)\right|^2 \mathrm{d}\rho\left(x\right) \\ &- c_1 \left(\sum_{i=1}^{n} \left\|y_i\left(x\right)\right\|_{L^2\left(R^\infty\right)}\right)^2, \end{aligned}$$

$$\pi(t;y,y)+c_1\left(\sum_{i=1}^{n}\|y_i(x)\|_{L^2(R^\infty)}\right)^2 \geq \sum_{i=1}^{n}\int_{R^\infty}\sum_{|\alpha|\leq l}\sum_{k=1}^{\infty}\left|D_k^\alpha y_i(x)\right|^2 \mathrm{d}\rho(x),$$

then,

$$\pi(t;y,y)+c_1\sum_{i=1}^{n}\|y_i(x)\|^2_{L^2(R^\infty)} \geq \sum_{i=1}^{n}\sum_{|\alpha|\leq l}\sum_{k=1}^{\infty}\left\|D_k^\alpha y_i(x)\right\|^2_{L^2(R^\infty)} \geq \sum_{i=1}^{n}\|y_i(x)\|^2_{W_0^l(R^\infty)},$$

since $0<c\leq 1$, we have,

$$\pi(t;y,y)+c_1\|y(x)\|^2_{(L^2(R^\infty))^n} \geq c\|y(x)\|^2_{(W_0^l(R^\infty))^n},$$

which proves the coerciveness condition on $\left(W_0^l\left(R^\infty\right)\right)^n$.

Under all the a bove consideration, theorems of Lions [22] and using the Lax-Milgram lemma we have proved the following theorem.

Theorem 1:

Under the hypotheses (2), (3) and (7), if $f=(f_1,f_2,\cdots,f_n)$, $y_{0,i}(x)$ and $y_{1,i}(x)$ are given in $\left(L^2\left(0,T;W_0^{-l}\left(R^\infty\right)\right)\right)^n$, $W_0^l\left(R^\infty\right)$ and $L^2\left(R^\infty\right)$ resp., then there exists a unique solution

$$y=(y_1,y_2,\cdots,y_n)\in\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n$$

for system (1).

Proof:

Let $\psi\to L(\psi)$ be a continuous linear form defined on $\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n$ by

$$L(\psi)=\sum_{i=1}^{n}\int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t + \sum_{i=1}^{n}\int_{R^\infty} y_{1,i}(x)\psi_i(x,0)\mathrm{d}\rho(x),$$

$$\forall\psi=(\psi_1,\psi_2,\cdots,\psi_n)\in\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n,$$

then by Lax-Milgram lemma, there exists a unique element $y=(y_1,y_2,\cdots,y_n)\in\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n$ such that

$$\left(\frac{\partial^2 y}{\partial t^2},\psi\right)+\pi(t;y,\psi)=L(\psi)$$
$$\forall\psi=(\psi_1,\psi_2,\cdots,\psi_n)\in\left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n. \quad (9)$$

Now, let us multiply both sides of first equation of system (1) by $\psi_i(x)$, then integration over Q, we have:

$$\int_Q\left[\frac{\partial^2 y_i(x)}{\partial t^2}+\sum_{|\alpha|\leq l}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}y_i(x)+q(x,t)y_i(x) -\sum_{j=1}^{n}a_{ij}(x)y_j(x)\right]\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t = \int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by applying Green's formula

$$\int_Q\left[\frac{\partial^2 y_i(x)}{\partial t^2}\psi_i(x)+\sum_{|\alpha|\leq l}\sum_{k=1}^{\infty}D_k^\alpha y_i(x)D_k^\alpha\psi_i(x) +q(x,t)y_i(x)\psi_i(x)-\sum_{j=1}^{n}a_{ij}(x)y_j(x)\psi_i(x)\right]\mathrm{d}\rho(x)\mathrm{d}t$$
$$-\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}x-\int_\Sigma\psi_i\frac{\partial y_i}{\partial\nu_A}\mathrm{d}\Sigma = \int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by entering the summation on the both sides, we have

$$\sum_{i=1}^{n}\int_Q\left[\frac{\partial^2 y_i(x)}{\partial t^2}\psi_i(x)+\sum_{|\alpha|\leq l}\sum_{k=1}^{\infty}D_k^\alpha y_i(x)D_k^\alpha\psi_i(x) +q(x,t)y_i(x)\psi_i(x)-\sum_{j=1}^{n}a_{ij}(x)y_j(x)\psi_i(x)\right]\mathrm{d}\rho(x)\mathrm{d}t$$
$$-\sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}\rho(x)-\sum_{i=1}^{n}\int_\Sigma\psi_i\frac{\partial y_i}{\partial\nu_A}d\Sigma = \sum_{i=1}^{n}\int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by comparing the summation with (6), (8) and (9) we obtain:

$$\sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}\rho(x)+\sum_{i=1}^{n}\int_\Sigma\psi_i\frac{\partial y_i}{\partial\nu_A}\mathrm{d}\Sigma = \sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)y_{1,i}(x)\mathrm{d}\rho(x),$$

then we deduce that:

$$y_i\big|_\Sigma=0,\quad 1\leq i\leq n,$$
$$\frac{\partial\, y_i(x,0)}{\partial t}=y_{1,i}(x)\quad \text{in } R^\infty,$$

which completes the proof.

3.2. Formulation of Dirichlet Problem

The space $\left(L^2(Q)\right)^n$ being the space of controls. For a control $u=(u_1,u_2,\cdots,u_n)\in\left(L^2(Q)\right)^n$, the state

$$y(u) = (y_1(u), y_2(u), \cdots, y_n(u)) \in \left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^n$$

of system (1) is given by the solution of

$$\begin{cases} \dfrac{\partial^2 y_i(u)}{\partial t^2} + \left(\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|} D_k^{2\alpha} I + q(x,t)\right) y_i(u) \\ \quad = \sum_{j=1}^{n} a_{ij}(x) y_j(u) + f_i + u_i, \quad \text{in } Q, \\ y_i(u)\big|_\Sigma = 0,\ y_i(x,0,u) = y_{0,i}(x),\ \dfrac{\partial y_i(x,0,u)}{\partial t} = y_{1,i}(x), \\ \quad x \in R^\infty, \quad 1 \le i \le n, \end{cases} \tag{10}$$

$$y_i(u), \frac{\partial y_i(u)}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right).$$

The observation equation is given by

$$z(u) = (z_1(u), z_2(u), \cdots, z_n(u)) = y(u) = (y_1(u), y_2(u), \cdots, y_n(u)).$$

N is given as

$$N = (N_1, N_2, \cdots, N_n) \in \mathcal{L}\left(\left(L^2(Q)\right)^n, \left(L^2(Q)\right)^n\right)$$

such that,

$$(Nu,u)_{\left(L^2(Q)\right)^n} \ge \gamma \|u\|^2_{\left(L^2(Q)\right)^n}, \quad \gamma \succ 0.$$

For a given $z_d = (z_{d1}, z_{d2}, \cdots, z_{dn}) \in \left(L^2(Q)\right)^n$, the cost function is given by

$$J(v) = \sum_{i=1}^{n} \left\| y_i(v) - z_{di} \right\|^2_{L^2(Q)} + (Nv,v)_{\left(L^2(Q)\right)^n}. \tag{11}$$

Control Constraints

The set of admissible controls U_{ad} is a closed convex subset of $U = \left(L^2(Q)\right)^n$, Then the control problem is to find *inf* $J(v)$ over U_{ad}.

Then using the general theory of Lions [22], there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality.

Theorem 2:

Assume that (7) holds and the cost function is given by (11). The necessary and sufficient conditions for

$$u = (u_1, u_2, \cdots, u_n) \in \left(L^2(Q)\right)^n$$

to be an optimal control are the following equations and inequalities:

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} + \left(\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|} D_k^{2\alpha} I + q(x,t)\right) p_i(u) \\ \quad - \sum_{j=1}^{n} a_{ij}(x) p_j(u) = y_i(u) - z_{di} \quad \text{in } Q, \\ p_i(u)\big|_\Sigma = 0,\ \ p_i(x,T,u) = 0,\ \dfrac{\partial p_i(u)}{\partial t}(x,T,u) = 0, \\ \quad x \in R^\infty \qquad \text{for all } 1 \le i \le n, \end{cases} \tag{12}$$

with

$$y_i(u),\ p_i(u) \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right),$$

$$\frac{\partial y_i(u)}{\partial t}, \frac{\partial p_i(u)}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right),$$

$$\begin{aligned} &(p(u) + Nu, v-u)_{\left(L^2(Q)\right)^n} \ge 0 \\ &\forall v = (v_1, v_2, \cdots, v_n) \in U_{ad}, \end{aligned} \tag{13}$$

together with (10), where

$$p(u) = (p_1(u), p_2(u), \cdots, p_n(u))$$

is the adjoint state.

Proof:

The optimal control $u = (u_1, u_2, \cdots, u_n) \in \left(L^2(Q)\right)^n$ is characterized by [23]:

$$\sum_{i=1}^{n} J'(u)(v_i - u_i) \ge 0\ \forall v = (v_1, v_2, \cdots, v_n) \in U_{ad},$$

that is

$$\sum_{i=1}^{n}\left[\left(y_i(u) - z_{di}, y_i(v) - y_i(u)\right)_{L^2(Q)} + \left(N_i u_i, v_i - u_i\right)_{L^2(Q)}\right] \ge 0$$

this inequality can be written as

$$\sum_{i=1}^{n}\left[\int_0^T \left(y_i(u) - z_{di}, y_i(v) - y_i(u)\right)_{L^2\left(R^\infty\right)} \mathrm{d}t + \left(N_i u_i, v_i - u_i\right)_{L^2(Q)}\right] \ge 0. \tag{14}$$

Now, since

$$(p, By)_{\left(L^2(Q)\right)^n} = \sum_{i=1}^{n}\int_0^T \left(p_i, \frac{\partial^2 y_i(u)}{\partial t^2} + \sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|} D_k^{2\alpha} y_i + q(x,t) y_i - \sum_{j=1}^{n} a_{ij} y_j\right)_{L^2\left(R^\infty\right)} \mathrm{d}t$$

by using Green's formula, (3) and (10), we have

$$\left(p,By\right)_{\left(L^2(Q)\right)^n}=\sum_{i=1}^{n}\int_0^T\left(\frac{\partial^2 p_i\left(u\right)}{\partial t^2}+\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}\left(-1\right)^{|\alpha|}D_k^{2\alpha}p_i+q\left(x,t\right)p_i-\sum_{j=1}^{n}a_{ij}p_j,y_i\right)_{L^2\left(R^\infty\right)}\mathrm{d}t=(B^*p,y)_{\left(L^2(Q)\right)^n}.$$

Then

$$B^*p_i=\frac{\partial^2 p_i\left(u\right)}{\partial t^2}-\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}\left(-1\right)^{|\alpha|}D_k^{2\alpha}p_i+q\left(x,t\right)p_i-\sum_{j=1}^{n}a_{ij}p_j,\quad i=1,2,\cdots,n,$$

and

$$S^*p_i=\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}\left(-1\right)^{|\alpha|}D_k^{2\alpha}p_i+q\left(x,t\right)p_i-\sum_{j=1}^{n}a_{ij}p_j,\qquad i=1,2,\cdots,n, \tag{15}$$

Since the adjoint equation for hyperbolic systems in Lions [22] takes the following form: $\frac{\partial^2 p\left(u\right)}{\partial^2 t}+S^*p\left(u\right)$ $=y\left(u\right)-z_d$, then, from (15) we obtain the first equation in (12), and from theorem1, system (12) admits a unique solution which satisfies $p_i\left(u\right),\frac{\partial p_i\left(u\right)}{\partial t}\in L^2\left(0,T;W_0^l\left(R^\infty\right)\right)$.

Now, we transform (14) by using (12) as follows:

$$\sum_{i=1}^{n}\int_0^T\left(\left[\frac{\partial^2}{\partial t^2}+\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}I+q\left(x,t\right)\right]p_i-\sum_{j=1}^{n}a_{ij}p_j,y_i\left(v\right)-y_i\left(u\right)\right)_{L^2\left(R^\infty\right)}\mathrm{d}t+\left(Nu,v-u\right)_{\left(L^2(Q)\right)^n}\ge 0,$$

using Green's formula, (10) and (12), we obtain

$$\sum_{i=1}^{n}\int_0^T\left(p_i\left(u\right),\left[\frac{\partial^2}{\partial t^2}+\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}\left(-1\right)^{|\alpha|}D_k^{2\alpha}I+q\left(x,t\right)\right]y_i\left(v\right)-y_i\left(u\right)-\sum_{j=1}^{n}a_{ij}\left(y_j\left(v\right)-y_j\left(u\right)\right)\right)_{L^2\left(R^\infty\right)}\mathrm{d}t+\left(Nu,v-u\right)_{\left(L^2(Q)\right)^n}\ge 0,$$

using (10), we have

$$\sum_{i=1}^{n}\left[\int_0^T\left(p_i\left(u\right),v_i-u_i\right)_{L^2(Q)}\mathrm{d}t+\left(N_iu_i,v_i-u_i\right)_{L^2(Q)}\right]\ge 0,$$

which is equivalent to $\left(p\left(u\right)+Nu,v-u\right)_{\left(L^2(Q)\right)^n}\ge 0$.

Thus the proof is complete.

4. Neumann Problem for 2*l* Order (*n* × *n*) Cooperative Hyperbolic System with an Infinite Number of Variables and with Variable Coefficients

In this section, we discuss the optimal control for $2l$ order $\left(n\times n\right)$ cooperative **non-homogenous** Neumann systems involving hyperbolic operators with an infinite number of variables and with variable coefficients.

$$\begin{cases}\frac{\partial^2 y_i\left(x\right)}{\partial t^2}+D\left(t\right)y_i=\sum_{j=1}^{n}a_{ij}\left(x\right)y_j+f_i\left(x,t\right) & \text{in } Q,\\ \left.\frac{\partial y_i\left(x\right)}{\partial \nu}\right|_{\Sigma}=g_i,\quad 1\le i\le n, & \\ y_i\left(x,0\right)=y_{0,i}\left(x\right),\quad \frac{\partial y_i(x,0)}{\partial t}=y_{1,i}\left(x\right) & \text{in } R^\infty,\end{cases} \tag{16}$$

where $y_i,\frac{\partial y_i}{\partial t}\in L^2\left(0,T;W^l\left(R^\infty\right)\right)$, for all $1\le i\le n$, $g=\left(g_1,g_2,\cdots,g_n\right)$ is a given function in $L^2\left(0,T;W^{-l/2}\left(\Gamma\right)\right)$ and the operator $\frac{\partial^2}{\partial t^2}+D\left(t\right)$ in system (16) is *2l* order hyperbolic operator with an infinite number of variables with

$D(t) \in \mathcal{L}\left(\left(W^l\left(R^\infty\right)\right)^n;\left(W^{-l}\left(R^\infty\right)\right)^n\right)$ is given by:

$$D(t)y_i(x) = \sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}y_i(x) + q(x,t)y_i(x),$$

since $q(x,t)$ is defined as in (5).

For each $t \in (0,T)$, we define a bilinear form $\pi(t;y,\psi):\left(W^l\left(R^\infty\right)\right)^n \times \left(W^l\left(R^\infty\right)\right)^n \to R$ as in (6).

4.1. The Existence and Uniqueness of Solution

Lemma 2:

The bilinear form $\pi(t;y,\psi)$ is also coercive on $\left(W^l\left(R^\infty\right)\right)^n$, that is, there exists $c, c_1 \in R$, such that:

$$\pi(t;y,y) + c_1\left\|y(x)\right\|^2_{\left(L^2\left(R^\infty\right)\right)^n} \ge c\left\|y(x)\right\|^2_{\left(W^l\left(R^\infty\right)\right)^n},\ c, c_1 > 0 \tag{17}$$

Proof:

Since $\left(W_0^l\left(R^\infty\right)\right)^n$ is everywhere dense in $\left(W^l\left(R^\infty\right)\right)^n$ with topological inclusion, then we have

$$\|y\|_{\left(W_0^l\left(R^\infty\right)\right)^n} \ge \|y\|_{\left(W^l\left(R^\infty\right)\right)^n}. \tag{18}$$

By using (18) in (7), we obtain

$$\pi(t;y,y) + c_1\left\|y(x)\right\|^2_{\left(L^2\left(R^\infty\right)\right)^n} \ge c\left\|y(x)\right\|^2_{\left(W^l\left(R^\infty\right)\right)^n},$$

which proves the coerciveness condition on $\left(W^l\left(R^\infty\right)\right)^n$.

By the Lax-Milgram lemma, we shall introduce the following theorem which gives the existence and uniqueness of the state for system (16).

Theorem 3:

Under the hypotheses (2), (3) and (17), if $f = (f_1, f_2, \cdots, f_n)$, $y_{0,i}(x)$ and $y_{1,i}(x)$ are given in $\left(L^2\left(0,T;W^{-l}\left(R^\infty\right)\right)\right)^n$, $W^l\left(R^\infty\right)$ and $L^2\left(R^\infty\right)$ resp., then there exists a unique solution $y = (y_1, y_2, \cdots, y_n) \in \left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n$ for system (16).

Proof:

Let $\psi \to L(\psi)$ be a continuous linear form defined on $\left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n$ by

$$L(\psi) = \sum_{i=1}^{n}\int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t + \sum_{i=1}^{n}\int_\Sigma g_i(x)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t + \sum_{i=1}^{n}\int_{R^\infty} y_{1,i}(x)\psi_i(x,0)\mathrm{d}\rho(x), \tag{19}$$

$$\forall \psi = (\psi_1, \psi_2, \cdots, \psi_n) \in \left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n,$$

then by Lax-Milgram lemma, there exists a unique element $y = (y_1, y_2, \cdots, y_n) \in \left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n$ such that (9) is satisfied.

$$\int_Q\left\{\frac{\partial^2 y_i(x)}{\partial t^2} + \sum_{|\alpha|\le l}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}y_i(x) + q(x,t)y_i(x) - \sum_{j=1}^{n}a_{ij}(x)y_j(x)\right\}\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t = \int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by applying Green's formula

$$\int_Q\left\{\frac{\partial^2 y_i(x)}{\partial t^2}\psi_i(x) + \sum_{|\alpha|\le l}\sum_{k=1}^{\infty}D_k^{\alpha}y_i(x)D_k^{\alpha}\psi_i(x) + q(x,t)y_i(x)\psi_i(x) - \sum_{j=1}^{n}a_{ij}(x)y_j(x)\psi_i(x)\right\}\mathrm{d}\rho(x)\mathrm{d}t$$

$$-\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}x - \int_\Sigma \psi_i\frac{\partial y_i}{\partial \nu_A}\mathrm{d}\Sigma = \int_Q f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by entering the summation on the both sides, we have

$$\sum_{i=1}^{n}\int_{Q}\left\{\frac{\partial^2 y_i(x)}{\partial t^2}\psi_i(x)+\sum_{|\alpha|\le l}\sum_{k=1}^{\infty}D_k^{\alpha}y_i(x)D_k^{\alpha}\psi_i(x)+q(x,t)y_i(x)\psi_i(x)-\sum_{j=1}^{n}a_{ij}(x)y_j(x)\psi_i(x)\right\}\mathrm{d}\rho(x)\mathrm{d}t$$

$$-\sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}\rho(x)-\sum_{i=1}^{n}\int_{\Sigma}\psi_i\frac{\partial y_i}{\partial \nu_A}\mathrm{d}\Sigma=\sum_{i=1}^{n}\int_{Q}f_i(x,t)\psi_i(x)\mathrm{d}\rho(x)\mathrm{d}t,$$

by comparing the summation with (6), (8) and (9) we obtain:

$$\sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)\frac{\partial y_i(x,0)}{\partial t}\mathrm{d}\rho(x)+\sum_{i=1}^{n}\int_{\Sigma}\psi_i\frac{\partial y_i}{\partial \nu_A}\mathrm{d}\Sigma=\sum_{i=1}^{n}\int_{R^\infty}\psi_i(x,0)y_{1,i}(x)\mathrm{d}\rho(x),$$

then we deduce that:

$$\left.\frac{\partial y_i(x)}{\partial \nu}\right|_{\Sigma}=g_i,\quad 1\le i\le n,\quad \frac{\partial y_i(x,0)}{\partial t}=y_{1,i}(x)\quad \text{in } R^\infty,$$

which completes the proof.

4.2. Formulation of Neumann Problem

The space $\left(L^2(Q)\right)^n$ is the space of controls. The state $y(u)=\left(y_1(u),y_2(u),\cdots,y_n(u)\right)\in\left(L^2\left(0,T;W^l\left(R^\infty\right)\right)\right)^n$ of system (16) is given by the solution of

$$\begin{cases}\dfrac{\partial^2 y_i(u)}{\partial t^2}+D(t)y_i(u)=\displaystyle\sum_{j=1}^{n}a_{ij}(x)y_j(u)+f_i+u_i \quad \text{in } Q,\\ \left.\dfrac{\partial y_i(u)}{\partial \nu}\right|_{\Sigma}=g_i,\qquad 1\le i\le n,\\ y_i(x,0;u)=y_{0,i}(x),\ \dfrac{\partial y_i(x,0;u)}{\partial t}=y_{1,i}(x)\qquad \text{in } R^\infty,\end{cases}\tag{20}$$

$$y_i,\frac{\partial y_i}{\partial t}\in L^2\left(0,T;W^l\left(R^\infty\right)\right).$$

The observation equation is given by

$$z(u)=\left(z_1(u),z_2(u),\cdots,z_n(u)\right)=y(u)=\left(y_1(u),y_2(u),\cdots,y_n(u)\right).$$

For a given $z_d=\left(z_{d1},z_{d2},\cdots,z_{dn}\right)\in\left(L^2(Q)\right)^n$, the cost function is given by

$$J(v)=\sum_{i=1}^{n}\left\|y_i(v)-z_{di}\right\|_{L^2(Q)}^2+M\sum_{i=1}^{n}\left(v_i,v_i\right)_{L^2(Q)},\tag{21}$$

where M is a positive constant.

The control problem then is to find *inf* $J(v)$ over U_{ad} with the same control constraints in Section II.

Then as in Section II, there exists a unique optimal control $u\in U_{ad}$ such that

$$J(u)=\inf J(v)\text{ for all } v\in U_{ad}.\tag{22}$$

Under the given considerations, we may apply theorems of Lions [22] as in Section II to obtain the following theorem:

Theorem 4:

The necessary and sufficient conditions for optimality of the control problem (20), (21) and (22) are given by the following equations and inequalities:

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} + D(t)\, p_i(u) - \sum_{j=1}^{n} a_{ij}(x)\, p_j(u) = y_i(u) - z_{di} \text{ in } Q, \\ \left.\dfrac{\partial p_i(u)}{\partial \nu}\right|_{\Sigma} = 0,\ p_i(x,T,u) = 0,\ \dfrac{\partial p_i(u)}{\partial t}(x,T,u) = 0, x \in R \end{cases} \quad \text{for all} \quad 1 \le i \le n, \tag{23}$$

with $y_i(u), p_i(u) \in L^2\left(0,T;W^l\left(R^\infty\right)\right)$, $\dfrac{\partial y_i(u)}{\partial t}, \dfrac{\partial p_i(u)}{\partial t} \in L^2\left(0,T;W^l\left(R^\infty\right)\right)$, $\left(p(u) + Nu, v - u\right)_{\left(L^2(Q)\right)^n} \ge 0 \quad \forall v = \left(v_1, v_2, \cdots, v_n\right) \in U_{ad}$, together with (16).

The case of no constraints on the control:

In the case of no constraints on the control, *i.e.* $U_{ad} = \left(L^2(Q)\right)^n$, the condition (13) reduces to $p_i(u) + N_i u_i = 0$, $x \in Q$, hence $u_i = -N_i^{-1} p_i(u)$.

Example 1:

If we take $n = 2$ in Dirichlet problem (1) with the same conditions of coefficients (2) and (3), then the space of controls is $L^2(Q) \times L^2(Q)$. For a control $u = (u_1, u_2) \in \left(L_2(Q)\right)^2$, the state $y(u) = \left(y_1(u), y_2(u)\right) \in \left(L^2\left(0,T;W_0^l\left(R^\infty\right)\right)\right)^2$ of the system is given by the solution of

$$\begin{cases} \dfrac{\partial^2 y_1(u)}{\partial t^2} + A(t)\, y_1(u) = a_{11}(x)\, y_1(u) + a_{12}(x)\, y_2(u) + f_1 + u_1 \quad \text{in } Q, \\ \dfrac{\partial^2 y_2(u)}{\partial t^2} + A(t)\, y_2(u) = a_{21}(x)\, y_1(u) + a_{22}(x)\, y_2(u) + f_2 + u_2 \quad \text{in } Q, \\ \left. y_1(u)\right|_{\Sigma} = 0, \quad \left. y_2(u)\right|_{\Sigma} = 0, \\ y_1(x,0;u) = y_{0,1}(x), \quad \dfrac{\partial y_1(x,0;u)}{\partial t} = y_{1,1}(x) \quad \text{in } R^\infty, \\ y_2(x,0;u) = y_{0,2}(x), \quad \dfrac{\partial y_2(x,0;u)}{\partial t} = y_{1,2}(x) \quad \text{in } R^\infty. \end{cases} \tag{24}$$

$$y_1(u), y_2(u), \frac{\partial y_1(u)}{\partial t}, \frac{\partial y_2(u)}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right).$$

The necessary and sufficient conditions for the optimality are the following equations and inequalities:

$$\begin{cases} \dfrac{\partial^2 p_1(u)}{\partial t^2} + A(t)\, p_1(u) - a_{11}(x)\, p_1(u) - a_{12}(x)\, p_2(u) = y_1(u) - z_{d1} \quad \text{in } Q, \\ \dfrac{\partial^2 p_2(u)}{\partial t^2} + A(t)\, p_2(u) - a_{21}(x)\, p_1(u) - a_{22}(x)\, p_2(u) = y_2(u) - z_{d2} \quad \text{in } Q, \\ \left. p_1(u)\right|_{\Sigma} = 0, \ \left. p_2(u)\right|_{\Sigma} = 0, \\ p_1(x,T,u) = 0, \dfrac{\partial p_1(u)}{\partial t}(x,T,u) = 0, \quad x \in R^\infty, \\ p_2(x,T,u) = 0, \ \dfrac{\partial p_2(u)}{\partial t}(x,T,u) = 0, \quad x \in R^\infty, \end{cases} \tag{25}$$

$$y_1, y_2, \frac{\partial y_1(u)}{\partial t}, \frac{\partial y_2(u)}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right), p_1, p_2, \frac{\partial p_1(u)}{\partial t}, \frac{\partial p_2(u)}{\partial t} \in L^2\left(0,T;W_0^l\left(R^\infty\right)\right),$$

$$\left(p_1(u) + N_1 u_1, v_1 - u_1\right)_{L^2(Q)} + \left(p_2(u) + N_2 u_2, v_2 - u_2\right)_{L^2(Q)} \ge 0 \quad \forall v = \left(v_1, v_2\right) \in U_{ad}, \tag{26}$$

together with (24), where $p(u) = \left(p_1(u), p_2(u)\right)$ is the adjoint state.

Example 2:

If we take

$$U_{ad} = \left\{ u / u_1 \text{ arbitrary in } L^2\left(Q\right), u_2 \geq 0 \text{ a.e. in } Q \right\}. \tag{27}$$

Thus there are no constraints on u_1 then the inequality (26) is equivalent to

$$\begin{cases} p_1\left(u\right) + N_1 u_1 = 0, \\ p_2\left(u\right) + N_2 u_2 \geq 0, \quad u_2 \geq 0, \\ u_2\left(p_2\left(u\right) + N_2 u_2\right) = 0. \end{cases} \tag{28}$$

Thus the optimal control is given by the solution of the following set of equations and inequalities

$$\begin{cases} \dfrac{\partial^2 y_1\left(u\right)}{\partial t^2} + A\left(t\right) y_1\left(u\right) - a_{11}\left(x\right) y_1\left(u\right) - a_{12}\left(x\right) y_2\left(u\right) + N_1^{-1} p_1\left(u\right) = f_1 \quad \text{in } Q, \\ \dfrac{\partial^2 y_2\left(u\right)}{\partial t^2} + A\left(t\right) y_2\left(u\right) - a_{21}\left(x\right) y_1\left(u\right) - a_{22}\left(x\right) y_2\left(u\right) - f_2 \geq 0 \quad \text{in } Q, \\ \dfrac{\partial^2 p_1\left(u\right)}{\partial t^2} + A\left(t\right) p_1\left(u\right) - a_{11}\left(x\right) p_1\left(u\right) - a_{12}\left(x\right) p_2\left(u\right) - y_1\left(u\right) = -z_{d1} \quad \text{in } Q, \\ \dfrac{\partial^2 p_2\left(u\right)}{\partial t^2} + A\left(t\right) p_2\left(u\right) - a_{21}\left(x\right) p_1\left(u\right) - a_{22}\left(x\right) p_2\left(u\right) - y_2\left(u\right) = -z_{d2} \quad \text{in } Q, \\ p_2 + N_2 \left\{ \dfrac{\partial^2 y_2\left(u\right)}{\partial t^2} + A\left(t\right) y_2\left(u\right) - a_{21}\left(x\right) y_1\left(u\right) - a_{22}\left(x\right) y_2\left(u\right) - f_2 \right\} \geq 0, \\ \left(\dfrac{\partial^2 y_2\left(u\right)}{\partial t^2} + A\left(t\right) y_2\left(u\right) - a_{21}\left(x\right) y_1\left(u\right) - a_{22}\left(x\right) y_2\left(u\right) - f_2 \right) \\ \times \left(p_2 + N_2 \left(\dfrac{\partial^2 y_2(u)}{\partial t^2} + A\left(t\right) y_2\left(u\right) - a_{21}\left(x\right) y_1\left(u\right) - a_{22}\left(x\right) y_2\left(u\right) - f_2 \right) \right) = 0, \\ y_1\left(u\right)\big|_{\Sigma} = 0, \quad y_2\left(u\right)\big|_{\Sigma} = 0, \quad p_1\left(u\right)\big|_{\Sigma} = 0, \quad p_2\left(u\right)\big|_{\Sigma} = 0, \\ y_1\left(x,0;u\right) = y_{0,1}\left(x\right), \quad \dfrac{\partial y_1\left(x,0;u\right)}{\partial t} = y_{1,1}\left(x\right) \quad \text{in } R^{\infty}, \\ y_2\left(x,0;u\right) = y_{0,2}\left(x\right), \quad \dfrac{\partial y_2\left(x,0;u\right)}{\partial t} = y_{1,2}\left(x\right) \quad \text{in } R^{\infty}, \end{cases} \tag{29}$$

$$p_1\left(x,T,u\right) = p_2\left(x,T,u\right) = 0, \quad x \in R^{\infty},$$

$$\frac{\partial p_1\left(u\right)}{\partial t}\left(x,T,u\right) = \frac{\partial p_2\left(u\right)}{\partial t}\left(x,T,u\right) = 0, \quad x \in R^{\infty},$$

Further

$$u_1 = N_1^{-1} p_1\left(u\right), \quad u_2 = \frac{\partial^2 y_2\left(u\right)}{\partial t^2} + A\left(t\right) y_2\left(u\right) - a_{21}\left(x\right) y_1\left(u\right) - a_{22}\left(x\right) y_2\left(u\right) - f_2 \tag{30}$$

5. Conclusions

The main result of this paper finds the necessary and sufficient conditions of optimality of distributed control for $2l$ order ($n \times n$) cooperative systems governed by Dirichlet and Neumann problems involving hyperbolic operators with an infinite number of variables and with variable coefficients that give the characterization of optimal control (Theorem 2, 4).

Also it is evident that by modifying:

- the boundary conditions (Dirichlet, Neumann, mixed)
- the nature of the control (distributed, boundary),
- the nature of the observation (distributed, boundary),
- the initial differential system,
- the number of variables,
- the type of equation (elliptic, parabolic and hyperbolic),
- the type of coefficients (constant, variable),

- the type of system (non-cooperative, cooperative),
- the order of equation, many of variations on the above problems are possible to study with the help of Lions formalism.

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