Advances in Pure Mathematics, 2013, 3, 586-589
http://dx.doi.org/10.4236/apm.2013.36075 Published Online September 2013 (http://www.scirp.org/journal/apm)

Estimates for Holomorphic Functions with Values in $\mathbb{C}\setminus\{0,1\}$

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Received June 29, 2013; revised July 28, 2013, accepted August 24, 2013

Abstract

Extension of classical Mandelbrojt's criterion for normality to several complex variables is given. Some inequalities for holomorphic functions which omit values 0 and 1 are obtained.

Keywords: Complex Space; Holomorphic Functions

1. Introduction

In 1929, Mandelbrojt [1] has asserted his criterion for normality of a family of holomorphic zero-free functions of one complex variables.

In [2], the author has proved a generalization of Mandelbrojt's criterion to several complex variables. In order to state this criterion precisely, we introduce some notations.

Let $\mathcal{F}$ be a family of zero-free holomorphic functions in a domain $\Omega\subset\mathbb{C}^n$ and D be a subdomain in Ω such that $\bar{D}\subset\Omega$. So that the quantities

$$m(f,D) = \begin{cases} \sup\limits_{z,w\in D}\dfrac{\ln|f(z)|}{\ln|f(w)|}, & \text{if } |f(w)|\neq 1 \text{ for all } w\in D;\\ \infty, & \text{if } |f(w)|=1 \text{ for some } w\in D,\end{cases}$$

$$m'(f,D)=\sup_{z,w\in D}\frac{|f(z)|}{|f(w)|},$$

$$L(f,D)=\min\left[m(f,D),m'(f,D)\right],$$

are well defined for each function $f\in\mathcal{F}$.

Theorem 1. (*See* [2].) Let $\mathcal{F}$ be a family of holomorphic functions in a domain Ω with values in $\mathbb{C}\setminus\{0,1\}$ Then $\mathcal{F}$ is normal in Ω if and only if for each point $z_0\in\Omega$ there exists a ball $\overline{B(z_0,r_0)}\subset\Omega$ such that the the set of quantities $L(f,B(z_0,r_0))$, $f\in\mathcal{F}$, is bounded.

It is well known that a family $\mathcal{F}$ of functions holomorphic on a domain Ω all of which omits the values 0 and 1 is normal, so by the Theorem $L(f,B(z_0,r_0))$ for some r_0 and all $f\in\mathcal{F}$. But for this case we may obtain a more plain inequalities:

Proposition 2. Let K_X be the Kobayashi distance on a connected complex space X. Let $\mathcal{F}$ be the family of all holomorphic functions on X with values in $\mathbb{C}\setminus\{0,1\}$. Then, for all $x,y\in X$ and all $f\in\mathcal{F}$,

$$\exp\left(-K_X(x,y)\right)\leq\frac{c+\left|\log|f(x)|\right|}{c+\left|\log|f(y)|\right|}\leq\exp\left(K_X(x,y)\right), \tag{1}$$

where

$$c=\frac{\Gamma\left(\frac{1}{4}\right)}{4\pi^2}=4.3768796\cdots.$$

Furthermore, if there exists continuous $\log f$ on X then

$$\exp\left(-K_X(x,y)\right)\leq\frac{c+|\log f(x)|}{c+|\log f(y)|}\leq\exp\left(K_X(x,y)\right). \tag{2}$$

In the proof of this proposition, we combine the result of Lai [3] with the definition of the Kobayashi metric and obtain a very elementary proof of Proposition 3 in [4].

2. The Proof of Mandelbrojt's Criterion

Proof of Theorem 1. $\Rightarrow$ Fix a point z_0 in Ω and consider a ball $B(z_0,r)\subset\Omega$. Suppose that $\mathcal{F}$ is normal in Ω but the set $L(f,B(z_0,r_0))$, $f\in\mathcal{F}$, for some $r_0<r$, is unbounded. Then there exists a sequence $\{f_j\}\subseteq\mathcal{F}$ suchthat

$$L(f,B(z_0,r_0))>j \text{ for all } j\in\mathbb{N}. \quad (3)$$

By hypothesis $\mathcal{F}$ is normal, and therefore, the following two cases exhaust all the possibilities for sequence $\{f_j\}$:

1) The sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ which converges uniformly on $\overline{B(z_0,r_0)}$ to a holomorphic function f;

2) The sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ which converges uniformly on $B(z_0,r_0)$ to Since $\mathcal{F}$ is a family of zero-free holomorphic functions in a domain Ω by Hurwit's theorem f is either nowhere zero or identically equal to zero.

Therefore the following three cases exhaust all the possibilities for sequence $\{f_j\}$:

a) The sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ which converges uniformly on $B(z_0,r_0)$ to the holomorphic function $f\equiv 0$;

b) The sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ which converges uniformly on $B(z_0,r_0)$ to a holomorphic function f which is zero-free on $\overline{B(z_0,r_0)}$;

c) The sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ which converges uniformly on $B(z_0,r_0)$ to ∞.

Since it follows readily from (3) that

$$L(f,B(z_0,r_0))>j \text{ for all } j\in\mathbb{N} \quad (4)$$

In case a) (respectively in case c)) we have $|f_{j_k}(z)|<1/2$ (respectively $|f_{j_k}(z)|>2$ for all $z\in\overline{B(z_0,r_0)}$ and all $k\in\mathbb{N}$ sufficiently large. Hence $\ln|f_{j_k}(z)|$ is a negative (respectively positive) pluriharmonic function in $B(z_0,r)$ Pluriharmonic functions form a subclass of the class of harmonic functions in $B(z_0,r)$ (obviously proper for $n>1$). So by Harnack's inequality there exists some constant

$$C=C\left(\overline{B(z_0,r_0)},B(z_0,r)\right),\quad C\in(1,\infty),$$

that

$$\frac{\ln|f_{j_k}(z)|}{\ln|f_{j_k}(w)|}\le C \text{ for all } z \text{ and } w\in\overline{B(z_0,r_0)},$$

and hence $m(f_{j_k},B(z_0,r_0))\le C$ for all $k\in\mathbb{N}$ sufficiently large.

In case b), we have $\lim_{k\to\infty}|f_{j_k}(z)|=f(z)$ for all $z\in\overline{B(z_0,r)}$. It follows

$$\lim_{k\to\infty}\frac{|f_{j_k}(z)|}{|f_{j_k}(w)|}=\frac{|f(z)|}{|f(w)|} \text{ uniformly for } z \text{ and } w\in\overline{B(z_0,r_0)}$$

The function $f(z)/f(w)$ is holomorphic on $\overline{B(z_0,r_0)}\times\overline{B(z_0,r_0)}$, it follows that $m'(f,B(z_0,r_0))$ is bounded.

Since $L(f_{j_k},B(z_0,r_0))$ is the minimum of

$$m(f_{j_k},B(z_0,r_0)) \text{ and } m'(f_{j_k},B(z_0,r_0))$$

the set of quantities $L(f_{j_k},B(z_0,r_0)),k\in\mathbb{N}$, is bounded, which is a contradiction to (4).

$\Leftarrow$Fix a point z_0 in Ω and define the families $\mathcal{J}$ and $\mathcal{H}$ by

$$\mathcal{J}=\{f\in\mathcal{F},|f(z_0)|\le 1\},$$

$$\mathcal{H}=\{f\in\mathcal{F},|f(z_0)|>1\}.$$

It will be shown that $\mathcal{J}$ is normal in $\mathcal{O}(\Omega)$ and that $\mathcal{H}$ is normal in $C(\Omega,\boldsymbol{C})$.

To prove that the family $\mathcal{J}=\{f\in\mathcal{F},|f(z_0)|\le 1\}$ is normal, it is sufficient to show that each sequence $\{f_j\}\subset\mathcal{J}$ contains a subsequence converging locally uniformly in $B(z_0,r_0)$ to a holomorphic function or to ∞. The following two cases exhaust all the possibilities:

a) There exists a subsequence $\{f_{j_k}\}$ such that for any $k\in\mathbb{N}$ the function $\ln|f_{j_k}|$ does not vanish in $B(z_0,r_0)$;

b) For each $j\in\mathbb{N}$ there exists $z_j\in B(z_0,r_0)$ such that $\ln|f_j(z_j)|=0$.

In case a), we have that $|f_{j_k}|<1$ in $B(z_0,r_0)$ for all elements of the sequence. Such a subsequence is normal in $B(z_0,r_0)$ by Montel's theorem and hence we are done in casea).

In case b), we have $m(f_j,B(z_0,r_0))=+\infty$ for all $j\in\mathbb{N}$. Therefore, according to the hypothesis, $m'(f_j,B(z_0,r_0))<C$ for all $j\in\mathbb{N}$ and some constant $C>0$. It follows that $|f_j|<C$ in $B(z_0,r_0)$ for all $j\in\mathbb{N}$, which means that $\{f_j\}$ is a normal family in $B(z_0,r_0)$ and hence finishes the proof in caseb).

If $f\in\mathcal{H}$, then $1/f$ is holomorphic on Ω because f never vanishes. Also $1/f$ never vanishes and $1/|f(z_0)|<1$. Hence reasoning similar to that in the above proof shows that $\tilde{\mathcal{H}}=\{1/f:f\in\mathcal{H}\}$ is also normal in $\mathcal{O}(B(z_0,r_0))$. So if $\{f_j\}$ is a sequence in $\mathcal{H}$ there is a subsequence $\{f_{j_k}\}$ and an analytic function h on $B(z_0,r_0)$ such that $\{1/f_{j_k}\}$ converges in $\mathcal{O}(B(z_0,r_0))$ to h. By the generalized Hurwitz's Theorem, either $h\equiv 0$ or h never vanishes. If $h\equiv 0$ it is easy to see that $f_{j_k}\to\infty$ uniformly on compact subsets of $B(z_0,r_0)$. If h never vanishes then $1/h$ is analytic and it follows that $f_{j_k}(z) \to 1/h(z)$ uniformly on compact subsets of $B(z_0, r_0)$.

It follows that $\mathcal{J}$ and $\mathcal{H}$ are normal at z_0 so that the union $\mathcal{F}$ is normal in $B(z_0, r_0)$ Since normality is a local property, $\mathcal{F}$ is a normal family in Ω This completes the proof of the theorem. □

Remark 1. It should be pointed out that the above theorem is not true if the condition "for each point $z_0 \in \Omega$ there exists a ball $B(z_0, r_0) \subset \Omega$. such that the the set of quantities $L(f, B(z_0, r_0))$, $f \in F$, is bounded" is replaced by the condition "the corresponding family of functions given by $\tilde{f}(z,w) = |f(z)|/|f(w)|$ is locally bounded on $\Omega \times \Omega$ (cf. [5, Theorem 2.2.8]).

To see this, consider the family $F := \{z^j\}_{j=1}^{\infty}$ of holomorphic functions. If we take $A := \{z \in \mathbb{C} : 1/2 < |z| < 1\}$, then $F|_A$ is a set of bounded (by 1) zero-free holomorphic functions in A so Montel's theorem guarantees that F is normal. It is plain by inspection that the family $\{|z|^j / |w|^j\}_{j=1}^{\infty}$ is not locally bounded on $A \times A$, while $\{\ln|z|^j / \ln|w|^j\}_{j=1}^{\infty}$ is a locally bounded family on $A \times A$. Hence Theorem 2.2.8 in [5] is not true.

3. Estimates for Holomorphic Functions Which Omit the Values 0 and 1

Proof of Proposition 2. The classical theorem of Landau may be stated in the form that if the function $f(z)$ is holomorphic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and does not take the values 0 and 1, then $|f'(0)|$ has a bound depending only on $|f(0)|$. In fact $|f'(0)|$ has a bound depending only on $|f(0)|$

$$|f'(0)| \le 2|f(0)|\left(c + \left|\log|f(0)|\right|\right) \tag{5}$$

where

$$c = \frac{\Gamma\left(\frac{1}{4}\right)}{4\pi^2} = 4.3768796\cdots$$

(see, for example, [4]).

Let ρ denote the Poincaré distance on Δ, *i.e.*, the distance function defined by the Poincaré metric

$$ds^2 = \frac{4dzd\bar{z}}{\left(1-|z|^2\right)^2}.$$

For $z \in \Delta$ define $\psi_z(w) = (w+z)/(1+\bar{z}w)$. Since $f \circ \psi_z$ does not take the values 0 and 1, from (5) we derive the following inequality

$$\begin{aligned}\left(1-|z|^2\right)\left|f'(\psi_z(0))\right| &= \left|(f \circ \psi_z)'(0)\right| \\ &\le 2\left|f \circ \psi_z(0)\right|\left(c + \left|\log\left|f \circ \psi_z(0)\right|\right|\right).\end{aligned} \tag{6}$$

Let x, y be a pair of points in X. Since K_X is an inner pseudometric (see [6]), for each $\varepsilon > 0$ there exist an integer $l > 1$, $\varphi_1, \cdots, \varphi_l \in Hol(\Delta, X)$, and $a_1, \cdots, a_l \subset (0,1)$ satisfying $\varphi_1(0) = y$, $\varphi_j(a_j) = \varphi_{j+1}(0)$, for $j = 1, \cdots, l-1$ and $\varphi_l(a_l) = x$, and

$$\sum_{j=1}^{l} \rho(0, a_j) < K_X(x,y) + \varepsilon/2.$$

Set $g_j = f \circ \varphi_j$. From inequality (6), we obtain

$$\frac{|g_j'(t)|}{|g_j(t)|\left(c + \left|\log|g_j(t)|\right|\right)} \le \frac{2}{1-t^2} \quad \text{for } t \in [0, a_j]. \tag{7}$$

Since for $t \in [0, a_j]$

$$\left|\frac{g_j'(t)}{g_j(t)}\right| = \frac{\partial}{\partial t}\left|\log g_j(t)\right| \ge \left|\frac{\partial}{\partial t}\log|g_j(t)|\right| \ge \frac{\partial}{\partial t}\left|\log|g_j(t)|\right|$$

from (7), we obtain

$$\frac{\partial}{\partial t}\left|\log\left(c + \log|g_j(t)|\right)\right| \le \frac{2}{1-t^2}. \tag{8}$$

If we integrate both sides from $t = 0$ to a_j, the result becomes

$$\log\left[\frac{c + \left|\log|g_j(a_j)|\right|}{c + \left|\log|g_j(0)|\right|}\right] \le \rho(0, a_j).$$

Then

$$\log\frac{c + \left|\log|f(x)|\right|}{c + \left|\log|f(y)|\right|} \le K_X(x,y) + \varepsilon.$$

Letting $\varepsilon \to 0$, we finally get

$$\frac{c + \left|\log|f(x)|\right|}{c + \left|\log|f(y)|\right|} \le \exp\left(K_X(x,y)\right)$$

so the second inequality in (1) is proved. Since x and y play symmetric roles, it is evident that the first inequality in (1) also holds.

For obtaining inequalities (2), let us notice that there exists continuous $\log(\log g_j)$ on $t \in [0, a_j]$. Since

$$\left|\frac{\partial}{\partial t}\log\log g_j(t)\right| \ge \frac{\partial}{\partial t}\log\left|\log g_j(t)\right|$$

we have

$$\left|\frac{g_j'(t)}{g'(t)}\right| = \left|\frac{\partial}{\partial t}\log g_j(t)\right| \ge \frac{\partial}{\partial t}\left|\log g_j(t)\right|$$
$$= \frac{\partial}{\partial t}\left(c + \left|\log g_j(t)\right|\right).$$

From this inequality and inequality (8), we obtain

$$\frac{\partial}{\partial t}\log\left(c + \left|\log g_j(t)\right|\right) \le \frac{2}{1-t^2} \text{ for } t \in \left[0, a_j\right].$$

Integrating both sides of this inequality as above we obtain the inequality (2).

The proof of the theorem is now complete. □

Remark 2. Proposition 2 holds also for holomorphic functions defined on an infinite dimensional complex Banach manifold with values in $\mathbb{C}\setminus\{0,1\}$, the same proof works. So we give here more simple proof of Proposition 3 in [4].

4. Acknowledgements

I would like to thank Professor Robert B. Burckel for all his help during my work on this paper.

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