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# The Equivalence of Certain Norms on the Heisenberg Group

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## ABSTRACT

Let $IH_n$ be the $(2n+1)$-dimensional Heisenberg group. In this paper, we shall give among other things, the properties of some homogeneous norms relative to dilations on the $IH_n$ and prove the equivalence of these norms.

**Keywords:** Heisenberg Group; Heisenberg Norms; Equivalent Norms; Homogeneous Group

## 1. Introduction

The Heisenberg group (of order $n$), $IH_n$ is a non-commutative nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times IR$ with coordinates $(z,t) = (z_1, z_2, \cdots, z_n, t)$ and group law given by

$$(z,t)(z',t') = (z+z', t+t'+2\Im mz\cdot z')$$

where $z\cdot z' = \sum_{j=1}^{n} z_j \bar{z}_j \quad z\in\mathbb{C}^n, t\in IR.$

Setting $z_j = x_j + y_j$, then $(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, t)$ forms a real coordinate system for $IH_n$. In this coordinate system, we define the following vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

The set $\{X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n, T\}$ forms basis for the left invariant vector fields on $IH_n$ [1]. These vector fields span the Lie algebra $\mathfrak{h}_n$ of $IH_n$ and the following commutation relations hold:

$$[Y_j, X_k] = 4\delta_{jk}T,\ [Y_j, Y_j] = [X_j, T] = [Y_j, T] = 0.$$

Similarly, we obtain the complex vector fields by setting

$$\left.\begin{aligned} Z_j &= \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}\frac{\partial}{\partial t} \\ \bar{Z}_j &= \frac{1}{2}(X_j + Y_j) = \frac{\partial}{\partial \bar{z}_j} - iz\frac{\partial}{\partial t} \end{aligned}\right\}$$

In the complex coordinate, we also have the commutation relations

$$[Z_j, \bar{Z}_k] = -2\delta_{jk}T,$$
$$[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_j, T] = 0.$$

If we identify $IH_n$ with $IR^{2n+1}$, then each element of $IH_n$ is given by $u = (x,y,t) \in IR^n \times IR^n \times IR$ and the group law becomes

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t'+2\langle x, y'\rangle)$$

where $(x,y) \mapsto \langle x, y'\rangle = \sum_{j=1}^{n} x_j y_j$ denotes the scalar product of $IR^n$. The neutral element $1_{IH_n}$ of $IH_n$ is of the form $(0,0,0)$ and the inverse element

$$(x,y,t)^{-1} = (-x, -y, -t + \langle x, y\rangle).$$

The centre of $IH_n$ is given by

$$\mathcal{Z} = \{(0,0,t) : t \in IR\}$$

and therefore isomorphic to the additive locally compact topological group $IR$. The Haar measure on $IH_n$ is the Lebesgue measure $dxdydt$ on $IR^{2n}\times IR$ [1].

On the group, we introduce the group $\{\delta_r : 0 < r < \infty\}$ of dilations defined for each element $u = (z,t)$ of $IH_n$ by $\delta_r(z,t) = (rz, r^2 t)$ on the complex coordinates and by $\delta_r(x,y,t) = (rx, ry, r^2 t)$ on the real coordinates. The family of dilations $\{\delta_\lambda\}_\lambda > 0$ forms a one-parameter group of automorphisms of $IH_n$ Indeed, we have the following properties of this family of dilations.

(i) $\delta_{rs}(u)=\delta_r(\delta_s(u)),\ \forall u\in IH_n,\ r,s>0,$

(ii) $\delta_r(u\cdot u')=\delta_r(u)\delta_r(u')$. Moreover,

(iii) $(\delta_r)^{-1}(u)=\delta_{r^{-1}}(u)$. Properties (i) and (iii) can be easily seen [2,3]. To see (ii), we notice that: For $(x,y,t),(x',y',t')\in IH_n$ and $\delta_r: IH_n\to IH_n$, we have

$$\delta_r((x,y,t)(x',y',t'))$$
$$=\delta_r(x+x',y+y',t+t'+2(xy'-x'y))$$
$$=(rx+rx',ry+ry',r^2t+r^2t'+2r^2(xy'-x'y))$$
$$=\delta_r(x,y,t)\delta_r(x',y',t').\Box$$

With these dilations as automorphisms of $(IR^{2n}\times IR,\circ)$, $IH_n:=(IR^{2n}\times IR,\circ,\delta_r)$ becomes a stratified Lie group whose generators are the defined vector fields [4]. Similarly, $IH_n$ and its Lie structure equipped with this family of dilations is a homogeneous group of dimension $(2n+1)$ [5].

## 2. Homogeneous Norms on $IH_n$

**Definition 2.1:** A norm on the Heisenberg group, is a function

$$|\cdot|_{IH_n}: IH_n\to[0,\infty) \tag{2.1}$$

satisfying the following properties:

(i) $|\delta_r u|_{IH_n}=r|u|_{IH_n}$,

(ii) $|u|=0\Leftrightarrow u=0$,

(iii) $|u^{-1}|=|u|$,

(iv) $|u_1u_2|\le|u_1|+|u_2|$ for all $u$ and $r>0$, where $u=(z,t)$.

The value $|(z,t)|_{IH_n}=\left(|z|^4+t^2\right)^{1/4}$ is called the Heisenberg distance of $(z,t)$ from the origin and $\left(|z|^4+t^2\right)^{1/4}<1$ is the Heisenberg unit ball [6]. We say the norm in (2.1) is homogeneous of degree $Q$ with respect to the dilations if for any $u\in IH_n$, we have $|\delta_r u|_{IH_n}=r^Q|u|_{IH_n}$. The value given by

$$|(z,t)|:=\left(|z|^4+16t^2\right)^{1/4}=\left||z|^2\pm 4it\right|^{1/2}$$

is the popular Koranyi norm on $IH_n$ which is always positive definite [7].

Property (i) is the homogeneity of the Heisenberg norm while property (iv) indicates the subadditivity of the Heisenberg norm. The proof of properties (i)-(iii) is trivial and that of (iv) can be found in [8].

Following [9], we shall further define the following norms on $IH_n$. For $u=(z,t)$, define

$$\left.\begin{aligned}&|u|_0=\max\left\{|z_1|,\cdots,|z_n|,|t|^{1/2}\right\}\\&|u|_1=1\Leftrightarrow|z|^2+t^2=1\text{ and extended by homogeneity.}\\&|u|_2=\left(|z_1|^4+\cdots+|z_n|^4+t^2\right)^{1/4}=\left(\sum_{j=1}^n|z_j|^4+|t|^2\right)^{1/4}\end{aligned}\right\} \tag{2.2}$$

We notice that $|u|_0$ gives a choice which is not smooth away from the origin. The norm $|u|_2=\left(|z|^4+t^2\right)^{1/4}$ and the properties above do not uniquely determine the norm. For if $\phi$ is positive, smooth away from 0, and homogeneous of degree 0 in the Heisenberg group dilation structure, then $|u|_h^*\equiv\phi(u)|u|_h$ gives another norm [10].

Since $IH_n=\mathbb{C}^n\times IR$, it can be equipped with the Euclidean norm in $IR^{2n+1}$ denoted by $|u|_e$ and defined by

$$|u|_e=\left(|z|^2+|t|^2\right)^{1/2},\quad u=(z,t)\in IH_n.$$

We have the following:

**Proposition 2.3 [10]:** For $|u|_e^2\le\frac{1}{2}$, we have

$$|u|_e\le|u|_{IH_n}\le|u|_e^{1/2}.$$

We notice however, that this norm is not homogeneous. In what follows, we show that homogeneous norms on the Heisenberg group are equivalent following [10].

**Lemma 2.4:** Let $|\cdot|_{IH_n}$ be a homogeneous norm on $IH_n$ Then, there is a constant $M>0$ such that

$$M^{-1}|u|_2\le|\cdot|_{IH_n}\le M|u|_2\quad\forall u\in IH_n$$

where $|u|_2$ is as defined in (2.2).

**Proof:** Now observe that $|u|_{IH_n}$ is homogeneous of degree $2n+2$ and by hypothesis, $|\cdot|_2$ is homogeneous. Let

$$R:=\sup\left\{|u|_{IH_n}:|u|=1\right\}<\infty$$
$$\text{and } r:=\inf\left\{|u|_{IH_n}:|u|=1\right\}>0$$

and set

$$M:=\max\left\{R,\frac{1}{r}\right\}.$$

Now, if we identify $IH_n$ as $IR^{2n+1}$, then sup is actually a maximum and inf is a minimum. Thus $M\ne 0$ exists and the inequality in the theorem holds. This is possible since $R<\infty$ and $r>0$ follows from the fact that $\{u:|u|=1\}$ is a compact subset of $IH_n$ not containing the origin and $|\cdot|_{IH_n}$ is a continuous function which is strictly positive in $IH_n \setminus \{0\}$.□

**Corollary 2.5:** For every fixed homogeneous norm $|\cdot|_{IH_n}$ on $IH_n$ there exists a constant $M > 0$ such that

$$M^{-1}|u|_{IH_n} \le |u^{-1}|_{IH_n} \le M|u|_{IH_n} \quad \forall x \in IH_n.$$

**Proof:** We notice that the norm function is continuous and therefore, $|x| = |x^{-1}|$. Now consider the the group of dilations $\{\delta_r : r > 0\}$ on $IH_n$ Then $\delta_r(x^{-1}) = (\delta_r(x))^{-1}$ is an automorphism of $G$. Therefore, by Lemma 2.4, the result follows.□

**Theory 2.6:** Any two homogeneous norms on $IH_n$ are equivalent.

**Proof:** We apply the previous method as follows: Let

$$W := \{u \in IH_n : |u|^{\delta_r} \le 1\}$$

and define $\varphi : W \to [0, \infty)$ by

$$\varphi(u) = |u|^{\delta_r} = r^q|u|, \ q \ge 1.$$

Then

$$p : (IH_n, |u|_1) \to (IH_n, |u|_2)$$

is obviously continuous by the homogeneity property with respect to $|u|_1$. Since $W$ is bounded with respect to $|u|_1, \varphi$ attains it bounds and therefore, $\sup \varphi$ exists. Thus, $\exists M > 0$ such that $\varphi(u) \le M$. If $0 \ne u \in IH_n$, then there exists $r, R \ge 0$ such that $\delta_R(u)|u|_2^{\delta_r} \in IH_n$ so that

$$\varphi\left(\frac{\delta_R(u)}{|u|_2^{\delta_r}}\right) \le \frac{R|u|}{r|u|} = R\frac{1}{r} \le KM = M'.$$

The theorem then follows by Lemma 2.4.□

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