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Nemytskii Operator in the Space of Set-Valued Functions of Bounded φ-Variation

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ABSTRACT

In this paper we consider the Nemytskii operator, *i.e.*, the composition operator defined by $(Nf)(t)=H(t,f(t))$, where H is a given set-valued function. It is shown that if the operator N maps the space of functions bounded φ_1-variation in the sense of Riesz with respect to the weight function α into the space of set-valued functions of bounded φ_2-variation in the sense of Riesz with respect to the weight, if it is globally Lipschitzian, then it has to be of the form $(Nf)(t)=A(t)f(t)+B(t)$, where $A(t)$ is a linear continuous set-valued function and B is a set-valued function of bounded φ_2-variation in the sense of Riesz with respect to the weight.

Keywords: Bounded Variation; Function of Bounded Variation in the Sense of Riesz; Variation Space; Weight Function; Banach Space; Algebra Space

1. Introduction

In [1], it was proved that every globally Lipschitz Nemytskii operator

$$(Nu)(t)=H(t,u(t))$$

mapping the space $\mathrm{Lip}([a,b];cc(Y))$ into itself admits the following representation:

$$(Nu)(t)=A(t)u(t)+B(t),$$
$$u\in \mathrm{Lip}([a,b];cc(Y)),\ t\in[a,b],$$

where $A(t)$ is a linear continuous set-valued function and B is a set-valued function belonging to the space $\mathrm{Lip}([a,b];cc(Y))$. The first such theorem for single-valued functions was proved in [2] on the space of Lipschitz functions. A similar characterization of the Nemytskii operator has also been obtained in [3] on the space of set-valued functions of bounded variation in the classical Jordan sense. For single-valued functions it was proved in [4]. In [5,6], an analogous theorem in the space of set-valued functions of bounded p-variation in the sense of Riesz was obtained. Also, they proved a similar result in the case in which that the Nemytskii operator N maps the space of functions of bounded p-variation in the sense of Riesz into the space of set-valued functions of bounded q-variation in the sense of Riesz, where $1\le q\le p<\infty$, and N is globally Lipschitz. In [7], they showed a similar result in the case where the Nemytskii operator N maps the space $RV_{\varphi_1}([a,b];K)$ of set-valued functions of bounded φ_1-variation in the sense of Riesz into the space $RW_{\varphi_2}([a,b];cc(Y))$ of set-valued functions of bounded φ_2-variation in the sense of Riesz and N is globally Lipschitz.

While in [8], we generalize article [6] by introducing a weight function. Now, we intend to generalize [7] in a similar form we did in [8], *i.e.*, the propose of this paper is proving an analogous result in which the Nemytskii operator N maps the space $RV_{\varphi_1,\alpha}([a,b];K)$ of set-valued functions of bounded φ_1-variation in the sense of Riesz with a weight α into the space $RW_{\varphi_2,\alpha}([a,b];cc(Y))$ of set-valued functions of bounded φ_2-variation in the sense of Riesz with a weight α and N is globally Lipschitz.

2. Preliminary Results

In this section, we introduce some definitions and recall known results concerning the Riesz φ-variation.

Definition 2.1 *By a φ-function we mean any non-decreasing continuous function* $\varphi:[0,+\infty)\to[0,+\infty)$

such that $\varphi(x)=0$ *if and only if* $x=0$, *and* $\varphi(x)\to\infty$ *as* $x\to\infty$.

Let $\mathcal{N}$ be the set of all convex continuous functions that satisfy Definition 2.1.

Definition 2.2 *Let* $(X,\|\cdot\|)$ *be a normed space and* φ *be a* φ*-function. Given* $I\subset\mathbb{R}$ *be an arbitrary* (*i.e.*, *closed*, *half-closed*, *open*, *bounded or unbounded*) *fixed interval and* $\alpha: I\to\mathbb{R}$ *a fixed continuous strictly increasing function called a it is weight. If* $\varphi\in\mathcal{N}$, *we define the* (*total*) *generalized* φ*-variation* $V_\varphi(f)\equiv V_\varphi(f,I,\alpha)$ *of the function* $f: I\to X$ *with respect to the weight function* α *in two steps as follows* (*cf.* [9]). *If* $I=[a,b]$ *is a closed interval and* π *is a partition* $\pi: a=t_0<t_1<\cdots<t_n=b$ *of the interval I* (*i.e.*, $n\in\mathbb{N}$), *we set*

$$V_\varphi(f,\pi,\alpha):=\sum_{i=1}^{n}\varphi\left(\frac{\|f(t_i)-f(t_{i-1})\|}{|\alpha(t_i)-\alpha(t_{i-1})|}\right)|\alpha(t_i)-\alpha(t_{i-1})|.$$

Denote by Π the set of all partitions of $[a,b]$, we set

$$V_\varphi(f)\equiv V_\varphi(f,[a,b],\alpha):=\sup\{V_\phi(f,\pi,\alpha):\pi\in\Pi\}.$$

If I is any interval in $\mathbb{R}$, we put

$$V_\varphi(f)\equiv V_\varphi(f,I,\alpha):=\sup\{V_\varphi(f,[a,b],\alpha):[a,b]\in I\}.$$

The set of all functions *of bounded generalized* φ*-variation with weight* α *will be denoted by*

$$RV_\varphi(I)\equiv RV_\varphi(I,\alpha)=\{f:[a,b]\to X \mid V_\varphi(f,I,\alpha)<\infty\}.$$

If $\alpha(t)=id(t)=t, t\in I=[a,b]$, and $\varphi(\rho)=\rho^q$, $\rho\ge 0$, $q>1$, the φ-variation $V_\varphi(f,I,\alpha)$, also written as $V_q(f)$, is the classical q*-variation* of f in the sense of Riesz [10], showing that $V_q(f)<\infty$ if and only if $f\in AC(I)$ (*i.e.*, $f: I\to\mathbb{R}$ is absolutely continuous) and its almost everywhere derivative f' is *Lebesgue* q-summable on I . Recall that, as it is well known, the space $RV_\varphi(I)$ with *I*, φ and α as above and endowed with the norm $|f|_q=|f(a)|+(V_q(f))^{1/q}$ is a Banach algebra for all $q\ge 1$.

Riesz's criterion was extended by *Medvedev* [11]: if $\varphi\in\mathcal{N}$, then $f\in RV_\varphi(I)$ if and only if $f\in AC(I)$ and $\int_I\varphi(|f'(t)|)\mathrm{d}t<\infty$. Functions of bounded generalized φ-variation with $\varphi\in\mathcal{N}$ and $\alpha=id$ (also called functions of bounded Riesz-Orlicz φ-variation) were studied by Cybertowicz and Matuszewska [12]. They showed that if $f\in RV_\varphi(I)$, then

$$V_\varphi(f)=\int_I\varphi(|f'(t)|)\mathrm{d}t,$$

and that the space

$$GV_\varphi(I)=\{f\in\mathbb{R}^I \text{ such that } \lim_{\lambda\to+0}V_\varphi(\lambda f)=0\}$$

is a semi-normed linear space with the *Luxemburg-Nakano* (cf. [13,14]) seminorm given by

$$p_\varphi(f)=\inf\{r>0 \mid V_\varphi(f/r)\le 1\}.$$

Later, Maligranda and Orlicz [15] proved that the space $GV_\varphi(I)$ equipped with the norm

$$\|f\|_\varphi=\sup_{t\in I}|f(t)|+p_\varphi(f)$$

is a Banach algebra.

3. Generalization of Medvedev Lemma

We need the following definition:

Definition 3.1 *Let* φ *be a* φ*-function. We say* φ *satisfies condition* ∞_1 *if*

$$\limsup_{t\to\infty}\frac{\varphi(t)}{t}=\infty. \quad (1)$$

For φ convex, (1) is just $\lim_{t\to\infty}\varphi(t)/t=\infty$. Clearly, for $id=1$ the space $RV_\alpha(f,[a,b],id)$ coincides with the classical space $BV(f,[a,b])$ of functions of bounded variation. In the particular case when $X=\mathbb{R}$ and $1<p<\infty$, we have the space $RV_{p,\alpha}(f,[a,b];X)$ of functions of bounded Riesz p-variation. Let $([a,b],\sum,\mu_\alpha)$ be a measure space with the Lebesgue-Stieltjes measure defined in σ-algebra $\sum$ and

$$L_{p,\alpha}[a,b]:=$$
$$\left\{f:[a,b]\to\mathbb{R}/f \text{ is } \mu_\alpha \text{ integrable and } \int_a^b|f|^p\,\mathrm{d}\alpha<+\infty\right\}.$$

Moreover, let α be a function strictly increasing and continuous in $[a,b]$. We say that $E\subset[a,b]$ has μ_α-measure 0, if given $\varepsilon>0$ there is a countable cover $\{(a_n,b_n)/n\in\mathbb{N}\}$ by open intervals of E , such that

$$\sum_{n=1}^{\infty}[\alpha(b_n)-\alpha(a_n)]<\varepsilon.$$

Since α is strictly increasing, the concept of "μ_α measure 0" coincides with the concept of "measure 0" of Lebesgue. [cf. [16], § 25].

Definition 3.2 (Jef) *A function* $f:[a,b]\to\mathbb{R}$ *is said to be absolutely continuous with respect to* α , *if for every* $\varepsilon>0$, *there exists* $\delta>0$ *such that*

$$\sum_{j=1}^{n}\varphi(|f(b_j)-f(a_j)|)\le\varepsilon,$$

for every finite number of nonoverlapping intervals (a_j,b_j), $j=1,\cdots,n$ *with* $[a_j,b_j]\subset[a,b]$ *and*

$$\sum_{j=1}^{n}|\alpha(b_j)-\alpha(a_j)|\le\delta.$$

The space of all absolutely continuous functions $f:[a,b]\to\mathbb{R}$, with respect to a function α strictly increasing, is denoted by $\alpha-AC$. Also the following characterization of [17,18] is well-known:

Lemma 3.3 *Let* $f \in \alpha - AC[a,b]$. *Then* f'_α *exists and is finite in* $[a,b]$, *except on a set of* μ_α*-measure* 0.

Lemma 3.4 *Let* $f \in \alpha - AC[a,b]$. *Then* f'_α *is integrable in the sense Lebesgue-Stieltjes and*

$$f(x) = f(a) + (L-S)\int_a^x f'_\alpha(t)\,\mathrm{d}\alpha(t), \quad x \in [a,b].$$

Lemma 3.5 *Let* $\varphi \in \mathcal{N}$ *such that satisfies the* ∞_1 *condition. If* $f \in RV_\varphi(f,[a,b],\alpha)$, *then* f *is* α*-absolutely* continuous *in* $[a,b]$, *i.e.*,

$$RV_\varphi(f,[a,b],\alpha) \subset \alpha - AC[a,b].$$

Also the following is a generalization of Medvedev Lemma [11]:

Theorem 3.6 (Generalization a Medvedev Lemma) *Let* $\varphi \in \mathcal{N}$ *such that satisfies the* ∞_1 *condition*, $f:[a,b] \to X$. *Then*

1) If f is α-absolutely continuous on $[a,b]$ and

$$\int_a^b \varphi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x) < +\infty,$$

then

$$f \in RV_\varphi(f,[a,b],\alpha)$$

and

$$RV_\varphi(f,[a,b],\alpha) \le \int_a^b \varphi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x).$$

2) If $f \in RV_\varphi(f,[a,b],\alpha)$ (*i.e.*, $RV_\varphi(f) < +\infty$), then f is α-absolutely continuous on $[a,b]$ and

$$\int_a^b \varphi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x) \le RV_\varphi(f,[a,b],\alpha).$$

Proof. 1) Since f is α absolutely continuous, there exists $f_{\alpha'}$ a.e. in $[a,b]$ by Lemma 3.3. Let $t_1, t_2 \in [a,b]$, $t_1 < t_2$

$$\varphi\left(\frac{|f(t_2)-f(t_1)|}{|\alpha(t_2)-\alpha(t_1)|}\right)|\alpha(t_2)-\alpha(t_1)|$$

$$= \varphi\left(\frac{\left|\int_{t_1}^{t_2} f_{\alpha'}(t)\,\mathrm{d}\alpha(t)\right|}{|\alpha(t_2)-\alpha(t_1)|}\right)|\alpha(t_2)-\alpha(t_1)|$$

by Lemma 3.4 and φ is strictly increasing

$$\le \varphi\left(\frac{\int_{t_1}^{t_2} |f_{\alpha'}(t)|\,\mathrm{d}\alpha(t)}{|\alpha(t_2)-\alpha(t_1)|}\right)|\alpha(t_2)-\alpha(t_1)|$$

$$= \varphi\left(\frac{\int_{t_1}^{t_2} |f_{\alpha'}(t)|\,\mathrm{d}\alpha(t)}{\int_{t_1}^{t_2} |\mathrm{d}\alpha(t)|}\right)|\alpha(t_2)-\alpha(t_1)|$$

using the generalized *Jenssen*'s inequality

$$\le \frac{\int_{t_1}^{t_2} \varphi(|f_{\alpha'}(t)|)\,\mathrm{d}\alpha(t)}{\int_{t_1}^{t_2} \mathrm{d}\alpha(t)}|\alpha(t_2)-\alpha(t_1)|$$

$$= \int_{t_1}^{t_2} \varphi(|f_{\alpha'}(t)|)\,\mathrm{d}\alpha(t).$$

Let $\pi: a = t_0 < \cdots < t_n = b$ be any partition of interval $[a,b]$; then

$$\sum_{i=1}^n \varphi\left(\frac{|f(t_2)-f(t_1)|}{|\alpha(t_2)-\alpha(t_1)|}\right)|\alpha(t_2)-\alpha(t_1)|$$

$$\le \sum_{i=1}^n \int_{t_{i-1}}^{t_j} \varphi(|f_{\alpha'}(t)|)\,\mathrm{d}\alpha(t) = \int_a^b \varphi(|f_{\alpha'}(t)|)\,\mathrm{d}\alpha(t) < \infty,$$

and we have

$$V_\varphi(f,[a,b],\alpha) \le \int_a^b \varphi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x).$$

Thus $f \in RV_\varphi(f,[a,b],\alpha)$.

2) Let $f \in RV_\varphi[a,b]$. Then f is α-absolutely continuous on $[a,b]$ by Lemma 3.5 and f'_α exist a.e. on $[a,b]$.

For every $n \in \mathbb{N}$, we consider

$$\pi_n : a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b$$

a partition of the interval $[a,b]$ define by

$$t_{i,n} = a + i\frac{b-a}{n}, \quad i = 0,1,\cdots,n.$$

Let $\{f_n\}_n$ be a sequence of step functions, defined by $f_n:[a,b] \to \mathbb{R}$

$$t \mapsto f_n(t) = \begin{cases} \dfrac{f(t_{i+1,n})-f(t_{i,n})}{\alpha(t_{i+1,n})-\alpha(t_{i,n})}, & t_{i,n} \le t < t_{i+1,n} \\ 0, & t = b \end{cases}$$

$\{f_n\}_{n\in\mathbb{N}}$ converge to f'_α a.e. on $[a,b]$. It is sufficient to prove $\{f_n\} \to f'_\alpha$ in those points where f is α-differentiable and different from $t_{i,n}$, $i = 0,\cdots,n$ for $n \in \mathbb{N}$, *i.e.*, in

$$\mathcal{A} = \{t \in [a,b] / f'_\alpha(t) \text{ exists}\} - \{t_{i,n} / n \in \mathbb{N}, i = 0,1,\cdots,n\}$$

For $t \in \mathcal{A}$, and each $n \in \mathbb{N}$, there exists $k \in \{0,\cdots,n\}$ such that $t_{k,n} \le t < t_{k+1,n}$, so

$$f_n(t) = \frac{f(t_{k+1,n})-f(t_{k,n})}{\alpha(t_{k+1,n})-\alpha(t_{k,n})}$$

$$= \frac{\alpha(t_{k+1,n})-\alpha(t)}{\alpha(t_{k+1,n})-\alpha(t_{k,n})}\frac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)} + \frac{\alpha(t)-\alpha(t_{k,n})}{\alpha(t_{k+1,n})-\alpha(t_{k,n})}\frac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})}.$$

Therefore, $f_n(t)$ is a convex combination of points

$$\frac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)} \text{ and } \frac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})}.$$

Now if $n\to\infty$, then $t_{k,n}\to t$ and $t_{k+1,n}\to t$ and since f is α-differentiable for t, the expressions

$$\frac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)} \text{ and } \frac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})}.$$

tend $f'_\alpha(t)$ to which is α-differentiable from f in t. So results

$$\lim_{n\to\infty} f_n(t)=f'_\alpha(t) \ \left(t\in\mathcal{A} \text{ a.e. in } [a,b]\right).$$

Since φ is continuous, we have

$$\lim_{n\to\infty}\varphi(|f_n(t)|)=\varphi\left(\lim_{n\to\infty}|f_n(t)|\right)=\varphi(|f'_\alpha(t)|) \ \ t\in\mathcal{A}.$$

Using the *Fatou*'s Lemma and definition of f'_n sequence, results that

$$\begin{aligned}\int_a^b\varphi(|f'_\alpha(t)|)\mathrm{d}\alpha(t)&=\int_a^b\lim_{n\to\infty}\varphi(|f_n(t)|)\mathrm{d}\alpha(t)\\&\le\liminf_{n\to\infty}\int_a^b\varphi(|f_n(t)|)\mathrm{d}\alpha(t)\\&=\liminf_{n\to\infty}\sum_{i=0}^{n-1}\int_{t_{i,n}}^{t_{i+1,n}}\varphi(|f_n(t)|)\mathrm{d}\alpha(t).\end{aligned}$$

By definition from f_n

$$\begin{aligned}&=\liminf_{n\to\infty}\sum_{i=0}^{n-1}\int_{t_{i,n}}^{t_{i+1,n}}\varphi\left(\left|\frac{f(t_{i+1,n})-f(t_{i,n})}{\alpha(t_{i+1,n})-\alpha(t_{i,n})}\right|\right)\left|\alpha(t_{i+1,n})-\alpha(t_{i,n})\right|\\&=\liminf_{n\to\infty}\sum_{i=0}^{n-1}\varphi\left(\frac{|f(t_{i+1,n})-f(t_{i,n})|}{|\alpha(t_{i+1,n})-\alpha(t_{i,n})|}\right)\int_{t_{i,n}}^{t_{i+1,n}}\mathrm{d}\alpha(t)\\&=\liminf_{n\to\infty}\sum_{i=0}^{n-1}\varphi\left(\frac{|f(t_{i+1,n})-f(t_{i,n})|}{|\alpha(t_{i+1,n})-\alpha(t_{i,n})|}\right)\left|\alpha(t_{i+1,n})-\alpha(t_{i,n})\right|\\&\le V_\alpha(f,[a,b])<+\infty.\end{aligned}$$

which is what we wished to demonstrate.

Corollary 3.7 *Let $\varphi\in\mathcal{N}$ such that satisfies the ∞_1 condition, then $f\in RV_\varphi(I)$ if and only if f is α-absolutely continuous on $[a,b]$ and*

$$\int_a^b\varphi(|f'_\alpha(x)|)\mathrm{d}\alpha(x)<+\infty.$$

Also

$$\int_a^b\varphi(|f'_\alpha(x)|)\mathrm{d}\alpha(x)=RV_\varphi(f,[a,b],\alpha).$$

Corollary 3.8 *Let $\varphi\in\mathcal{N}$ such that satisfies the ∞_1 condition. If $f\in RV_\varphi(I)$, then f is α-absolutely continuous on $[a,b]$ and*

$$\int_a^b\varphi(|f'_\alpha(x)|)\mathrm{d}\alpha(x)=RV_\varphi(f,[a,b],\alpha).$$

4. Set-Valued Function

Let $cc(X)$ be the family of all non-empty convex compact subsets of X and D be the *Hausdorff metric* in $cc(X)$, *i.e.*,

$$D(A,B):=\inf\{t>0:A\subseteq B+tS,\ B\subseteq A+tS\},$$

where $S=\{y\in X:\|y\|\le1\}$, or equivalently,

$$D(A,B)=\max\{e(A,B),e(A,B):A,B\in cc(X)\},$$

where

$$\begin{cases}e(A,B)=\sup\{d(x,B):x\in A\},\\ d(x,B)=\inf\{d(x,y):y\in B\}.\end{cases} \tag{2}$$

Definition 4.1 *Let $\varphi\in\mathcal{N}$, α a fixed continuous strictly increasing function and $F:[a,b]\to cc(X)$. We say that F has bounded φ-variation in the sense of Riesz if*

$$W_\varphi(F,[a,b],\alpha):=\sup_\pi\sum_{i=1}^n\varphi\left(\frac{D(F(t_i),F(t_{i-1}))}{|\alpha(t_i)-\alpha(t_{i-1})|}\right)|\alpha(t_i)-\alpha(t_{i-1})|<\infty, \tag{3}$$

where the supremum is taken over all partitions π of $[a,b]$.

Definition 4.2 *Denote by*

$$RW_\varphi^*(F,[a,b],\alpha):=\{F:[a,b]\to cc(X):W_\varphi(F,[a,b],\alpha)<\infty\} \tag{4}$$

and

$$RW_\varphi(F,[a,b],\alpha):=\{F:[a,b]\to cc(X):RW_\varphi^*(\lambda F)<\infty \text{ for some } \lambda>0\}, \tag{5}$$

both equipped with the metric

$$D_\varphi(F_1,F_2):=D(F_1(a),F_2(a))+\inf\{\varepsilon>0:W_\varphi(F_1/\varepsilon,F_2/\varepsilon)\le1\}, \tag{6}$$

where

$$W_\varphi(F_1,F_2)=\sup_\pi\sum_{i=1}^n\varphi\left(\frac{D(F_1(t_i)+F_2(t_{i-1}),F_1(t_{i-1})+F_2(t_i))}{|\alpha(t_i)-\alpha(t_{i-1})|}\right)\cdot|\alpha(t_i)-\alpha(t_{i-1})|.$$

Now, let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be two normed spaces and K be a convex cone in X. Given a set-valued function $H:[a,b]\times K\to cc(Y)$ we consider the *Nemytskii operator* $N:K^{[a,b]}\to Y^{[a,b]}$ generated by H, that is the composition operator defined by:

$$(Nf)(t) := H(t, f(t)),\ f:[a,b]\to K; t\in[a,b].$$

We denote by $L(K;cc(Y))$ the space of all set-valued function $A:K\to cc(Y)$, *i.e.*, additive and positively homogeneous, we say that A is linear if $A\in L(K;cc(Y))$.

In the proof of the main results of this paper, we will use some facts which we list here as lemmas.

Lemma 4.3 ([19]) *Let $(X,\|\cdot\|)$ be a normed space and let A,B,C be subsets of X. If A,B are convex compact and C is non-empty and bounded, then*

$$D(A+C,B+C)=D(A,B). \quad (7)$$

Lemma 4.4 ([20]) *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be normed spaces and K be a convex cone in X. A set-valued function $F:K\to cc(Y)$ satisfies the Jensen equation*

$$F\left(\frac{x+y}{2}\right)=\frac{1}{2}\left(F(x)+F(y)\right),\ x,y\in K, \quad (8)$$

if and only if there exists an additive set-valued function $A:K\to cc(Y)$ and a set $B\in cc(Y)$ such that

$$F(x)=A(t)+B,\quad x\in K.$$

We will extend the results of *Aziz*, *Guerrero*, *Merentes and Sánchez* given in [8] and [21] to set-valued functions of φ-bounded variation with respect to the weight function α.

5. Main Results

Lemma 5.1 *If $\varphi\in\mathcal{N}$ such that satisfies the ∞_1 condition and*

$$F\in RW_\varphi([a,b];cc(Y),\alpha),$$

then $F:[a,b]\to cc(X)$ is continuous.

Proof. Since $F\in RW_\varphi([a,b],\alpha)$, exists $M>0$ such that

$$\sum_{i=1}^{n}\varphi\left(\frac{D(F(t_i),F(t_{i-1}))}{|\alpha(t_i)-\alpha(t_{i-1})|}\right)|\alpha(t_i)-\alpha(t_{i-1})|\le M, \quad (9)$$

for all partitions of $[a,b]$, in particular given $t,t_0\in[a,b]$, we have

$$\varphi\left(\frac{D(F(t),F(t_0))}{|\alpha(t)-\alpha(t_0)|}\right)|\alpha(t)-\alpha(t_0)|\le M. \quad (10)$$

Since φ is convex φ-function, from the last inequality, we get

$$D(F(t),F(t_0))\le\frac{\varphi^{-1}\left(\dfrac{M}{|\alpha(t)-\alpha(t_0)|}\right)}{\dfrac{1}{|\alpha(t)-\alpha(t_0)|}}. \quad (11)$$

By (1),

$$\begin{aligned}&\lim_{t\to t_0}D(F(t),F(t_0))\\ &\le\lim_{t\to t_0}\frac{\varphi^{-1}\left(\dfrac{M}{|\alpha(t)-\alpha(t_0)|}\right)}{\dfrac{1}{|\alpha(t)-\alpha(t_0)|}}=\lim_{\rho\to\infty}\frac{M\rho}{\varphi(\rho)}=0.\end{aligned} \quad (12)$$

This proves the continuity of F at t_0. Thus F is continuous on $[a,b]$.

Now, we are ready to formulate the main result of this work.

Main Theorem 5.2 *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be normed spaces, K be a convex cone in X and φ_1,φ_2 be two convex φ-functions in X, strictly increasing, that satisfy ∞_1 condition and such that there exists constants c and T_0 with $\varphi_2(t)\le\varphi_1(ct)$ for all $t\ge T_0$. If the Nemitskii operator N generated by a set-valued function $H:[a,b]\times K\to cc(Y)$ maps the space $RV_{\varphi_1}(f,[a,b],\alpha;K)$ into the space $RW_{\varphi_2}(f,[a,b],\alpha;cc(Y))$ and if it is globally Lipschitz, then the set-valued function H satisfies the following conditions*:

1) For every $t\in[a,b]$ there exists $M(t)\in[0,+\infty)$, such that

$$D(H(t,x),H(t,y))\le M(t)\|x-y\|\ (x,y\in X). \quad (13)$$

2) There are functions $A:[a,b]\to L(K,cc(Y))$ and $B\in RW_{\varphi_2}(f,[a,b],\alpha;cc(Y))$ such that

$$H(t,x)=A(t)x+B(t)\ (t\in[a,b],\ x\in K) \quad (14)$$

Proof. 1) Since N is globally *Lipschitz*, there exists a constant $M\in[0,+\infty)$ such that

$$\begin{aligned}&D_{\varphi_2}(Nf_1,Nf_2)\\ &\le M\|f_1-f_2\|_{\varphi_1}\ (f_1,f_2\in RV_{\varphi_1}([a,b],\alpha;K)).\end{aligned} \quad (15)$$

Using the definitions of the operator N and metric D_{φ_2} we have

$$\begin{aligned}&D(Nf_1(a),Nf_2(a))+\inf\left\{\varepsilon>0:\sup_{\pi}\sum_{i=1}^{n}\varphi_2\left(\frac{D(h_{t_i,t_{i-1}}N_{f_1,f_2},h_{t_{i-1},t_i}N_{f_1,f_2})}{\varepsilon|\alpha(t_i)-\alpha(t_{i-1})|}\right)|\alpha(t_i)-\alpha(t_{i-1})|\le1\right\}\\ &\le M\|f_1-f_2\|_{\varphi_1}\quad (f_1,f_2\in RV_{\varphi_1}([a,b],\alpha;K)),\end{aligned}$$

where $h_{s,t}N_{f_1,f_2} := (Nf_1)(s) + (NF_2)(t)$. In particular,

$$\inf\left\{\varepsilon > 0 : \varphi_2\left(\frac{D\left(d_{f_1,f_2}(H,t,\bar{t}), d_{f_1,f_2}(H,\bar{t},t)\right)}{\varepsilon\left|\alpha(\bar{t}) - \alpha(t)\right|}\right)\left|\alpha(\bar{t}) - \alpha(t)\right| \le 1\right\} \le M\|f_1 - f_2\|_{\varphi_1},$$

for all $f_1, f_2 \in RV_{\varphi_1}([a,b],\alpha;K)$ and $t,\bar{t} \in [a,b]$, $t \ne \bar{t}$, where

$$d_{f_1,f_2}(H,s,t) = H(s, f_1(s)) + H(t, f_2(t)).$$

Since φ_1 and φ_2 satisfy

$$\varphi_i\left(\varphi_i\left(\frac{1}{\left|\alpha(\bar{t}) - \alpha(t)\right|}\right)\right)\left|\alpha(\bar{t}) - \alpha(t)\right| = 1, \quad i = 1,2, \quad (16)$$

we obtain

$$\inf\left\{\varepsilon > 0 : \varphi_2\left(\frac{D\left(d_{f_1,f_2}(H,t,\bar{t}), d_{f_1,f_2}(H,\bar{t},t)\right)}{\varepsilon\left|\alpha(\bar{t}) - \alpha(t)\right|}\right)\left|\alpha(\bar{t}) - \alpha(t)\right| \le 1\right\} = D\left(d_{f_1,f_2}(H,t,\bar{t}), d_{f_1,f_2}(H,\bar{t},t)\right).$$

Therefore

$$D\left(d_{f_1,f_2}(H,t,\bar{t}), d_{f_1,f_2}(H,\bar{t},t)\right) \le M\|f_1 - f_2\|_{\varphi_1}\left|\alpha(\bar{t}) - \alpha(t)\right|\varphi_2\left(\frac{1}{\left|\alpha(\bar{t}) - \alpha(t)\right|}\right). \quad (17)$$

Define the auxiliary function $\eta : [a,b] \to [0,1]$ by:

$$\eta(\tau) := \begin{cases} \dfrac{\alpha(\tau) - \alpha(a)}{\alpha(t) - \alpha(a)}, & a \le \tau \le t \\ 1, & t \le \tau \le b. \end{cases} \quad (18)$$

Then $\eta \in RV_{\varphi_1}([a,b],\alpha)$ and

$$V_{\varphi_1}(\eta,[a,b],\alpha) = \varphi_1\left(\frac{1}{\left|\alpha(t) - \alpha(a)\right|}\right)\left|\alpha(t) - \alpha(a)\right|.$$

Let us fix $x, y \in K$ and define the functions $f_i : [a,b] \to K$ $(i = 1,2)$ by:

$$\begin{aligned} &f_1(\tau) := x, \\ &f_2(\tau) := \eta(\tau)(y - x) + x, \\ &\tau \in [a,b]. \end{aligned} \quad (19)$$

Then the functions $f_i \in RV_{\varphi_1}([a,b],\alpha;K)$ $(i = 1,2)$ and

$$\begin{aligned} \|f_1 - f_2\|_{\varphi_1} &= \|f_1(a) - f_2(a)\| \\ &+ \inf\left\{\varepsilon > 0 : \sup_{\pi}\sum_{i=1}^{n}\varphi_1\left(\frac{\left\|(f_1 - f_2)(t_i) - (f_1 - f_2)(t_{i-1})\right\|}{\varepsilon\left|\alpha(t_i) - \alpha(t_{i-1})\right|}\right)\left|\alpha(t_i) - \alpha(t_{i-1})\right| \le 1\right\} \end{aligned} \quad (20)$$

From the definition of f_1 and f_2, we have

$$\|f_1 - f_2\|_{\varphi_1} = \inf\left\{\varepsilon > 0 : \varphi_1\left(\frac{\|x - y\|}{\varepsilon\left|\alpha(t) - \alpha(a)\right|}\right)\left|\alpha(t) - \alpha(a)\right| \le 1\right\}. \quad (21)$$

From (16), we get

$$\inf\left\{\varepsilon > 0 : \varphi_1\left(\frac{\|x - y\|}{\varepsilon\left|\alpha(t) - \alpha(a)\right|}\right)\left|\alpha(t) - \alpha(a)\right|\right\} = \frac{\|x - y\|}{\left|\alpha(t) - \alpha(a)\right|\varphi_1^{-1}\left(\dfrac{1}{\left|\alpha(t) - \alpha(a)\right|}\right)} \quad (22)$$

Hence,

$$D\left(d_{f_1,f_2}\left(H,t,\bar{t}\right),d_{f_1,f_2}\left(H,\bar{t},t\right)\right)\le\frac{M\left|\alpha\left(\bar{t}\right)-\alpha\left(t\right)\right|\varphi_2^{-1}\left(\frac{1}{\left|\alpha\left(\bar{t}\right)-\alpha\left(t\right)\right|}\right)\left\|x-y\right\|}{\left|\alpha\left(t\right)-\alpha\left(a\right)\right|\varphi_1^{-1}\left(\frac{1}{\left|\alpha\left(t\right)-\alpha\left(a\right)\right|}\right)} \tag{23}$$

Hence, substituting in inequality (5) the particular functions f_i $(i=1,2)$ defined by (19) and taking $\alpha(\bar{t})=\alpha(a)$ in (23), we obtain

$$D\left(H(t,x)+H(a,x),H(a,x)+H(t,y)\right)\le M\frac{\varphi_2^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(a)\right|}\right)}{\varphi_1^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(a)\right|}\right)}\left\|x-y\right\|, \tag{24}$$

for all $t\in[a,b],\ x,y\in K$.

By Lemma 4.3 and the inequality (24), we have

$$D\left(H(t,x),H(t,y)\right)\le M\frac{\varphi_2^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(a)\right|}\right)}{\varphi_1^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(a)\right|}\right)}\left\|x-y\right\|,$$

for all $t\in[a,b],\ x,y\in K$.

Now, we have to consider the case $\alpha(t)=\alpha(b)$. Define the function $\eta_1:[a,b]\to[0,1]$ by

$$\eta_1(\tau):=\frac{\alpha(\tau)-\alpha(a)}{\alpha(b)-\alpha(a)},\quad \tau\in[a,b]. \tag{25}$$

Then the function $\eta_1\in RV_{\varphi_1}([a,b],\alpha)$ and

$$V_{\varphi_1}(\eta_1)=\varphi_1\left(\frac{1}{\left|\alpha(b)-\alpha(a)\right|}\right)\left|\alpha(b)-\alpha(a)\right|.$$

Let us fix $x,y\in K$ and define the functions $f_i:[a,b]\to K$ $(i=1,2)$ by

$$f_1(\tau):=x,\ f_2(\tau):=\eta_1(\tau)(x-y)+y;\ \tau\in[a,b]. \tag{26}$$

Then the functions $f_i\in RV_{\varphi_1}([a,b],\alpha;K)$ $(i=1,2)$ and

$$\begin{aligned}\left\|f_1-f_2\right\|_{\varphi_1}&=\left\|x-y\right\|+\inf\left\{\varepsilon>0:\varphi_1\left(\frac{\left\|x-y\right\|}{\varepsilon\left|\alpha(b)-\alpha(a)\right|}\right)\left|\alpha(b)-\alpha(a)\right|\le 1\right\}\\&=\left\|x-y\right\|+\frac{\left\|x-y\right\|}{\left|\alpha(b)-\alpha(a)\right|\varphi_1^{-1}\left(\frac{1}{\left|\alpha(b)-\alpha(a)\right|}\right)}\\&=\left\|x-y\right\|\left(1+\frac{1}{\left|\alpha(b)-\alpha(a)\right|\varphi_1^{-1}\left(\frac{1}{\left|\alpha(b)-\alpha(a)\right|}\right)}\right).\end{aligned}$$

Substituting $\alpha(\bar{t})=\alpha(a)$ and $\alpha(t)=\alpha(b)$, and consider $\alpha=\alpha(b)-\alpha(a)$, we obtain

$$\begin{aligned}&D\left(H(b,x)+H(a,y),H(a,x)+H(b,x)\right)\\&\le MK\left(a,b,x,y,\varphi_1^{-1},\varphi_2^{-1}\right)\end{aligned} \tag{27}$$

for all $x,y\in K$, where

$$\begin{aligned}&K\left(a,b,x,y,\varphi_1^{-1},\varphi_2^{-1}\right)\\&=\left|\alpha\right|\varphi_2^{-1}\left(\frac{1}{\left|\alpha\right|}\right)\left\|x-y\right\|\left(1+\frac{1}{\left|\alpha\right|\varphi_1^{-1}\left(\frac{1}{\left|\alpha\right|}\right)}\right)\end{aligned}$$

By Lemma 4.3 and the above inequality, we get

$$D\big(H(a,y),H(a,x)\big) \le M|\alpha|\varphi_2^{-1}\left(\frac{1}{|\alpha|}\right)\|x-y\|\left(1+\frac{1}{|\alpha|\varphi_1^{-1}\left(\frac{1}{|\alpha|}\right)}\right)$$

for all $x,y\in K$. Define the function $M:[a,b]\to\mathbb{R}$ by

$$M(t)=\begin{cases} M\dfrac{\varphi_2^{-1}\left(\dfrac{1}{|\alpha(t)-\alpha(a)|}\right)}{\varphi_1^{-1}\left(\dfrac{1}{|\alpha(t)-\alpha(a)|}\right)}, a<t\le b \\ M|\alpha|\varphi_2^{-1}\left(\dfrac{1}{|\alpha|}\right)\|x-y\|\left(1+\dfrac{1}{|\alpha|\varphi_1^{-1}\left(\dfrac{1}{|\alpha|}\right)}\right), t=a. \end{cases}$$

Hence

$$D\big(H(t,x),H(t,y)\big) \le M(t)\|x-y\|\big(x,y\in X,\ t\in[a,b]\big),$$

and, consequently, for every $t\in[a,b]$ the function $H:[a,b]\times K\to cc(Y)$ is continuous.

This completes the proof of part 1).

Now we shall prove that H satisfies equality 2).

Let us fix $t,t_0\in[a,b]$ such that $t_0<t$. Since the Nemytskii operator N is globally Lipschitzian, there exists a constant M, such that

$$D\big(d_{u,v}(H,t,t_0),d_{u,v}(H,t_0,t)\big) \le M\|u-v\|_{\varphi_1}|\alpha(t_0)-\alpha(t)|\varphi_2\left(\frac{1}{|\alpha(t_0)-\alpha(t)|}\right), \tag{28}$$

where $d_{u,v}(H,s,t)=H(s,u(s))+H(t,v(t))$. Define the function $\eta_2:[a,b]\to[0,1]$ by

$$\eta_2(\tau)=\begin{cases}\dfrac{\alpha(\tau)-\alpha(a)}{\alpha(t_0)-\alpha(a)}, & a\le\tau\le t_0, \\ \dfrac{\alpha(t)-\alpha(\tau)}{\alpha(t)-\alpha(t_0)}, & t_0\le\tau\le t, \\ 0, & t\le\tau\le b.\end{cases}$$

The function $\eta_2\in RV_{\varphi_1}([a,b],\alpha)$.

Let us fix $x,y\in K$ and define the functions $f_i:[a,b]\to K$ by

$$\begin{cases} f_1(\tau):=\dfrac{1}{2}\eta_2(\tau)x+\left(1-\dfrac{1}{2}\eta_2(\tau)\right)y, & \tau\in[a,b]; \\ f_2(\tau):=\dfrac{1}{2}(1+\eta_2(\tau))x+\dfrac{1}{2}(1-\eta_2(\tau))y, & \tau\in[a,b]. \end{cases} \tag{29}$$

The functions $f_i\in RV_{\varphi_1}([a,b],\alpha;K)$ $(i=1,2)$ and

$$\|f_1-f_2\|_{\varphi_1}=\frac{\|x-y\|}{2}.$$

Hence, substituting in the inequality (28) the particular functions f_i $(i=1,2)$ defined by (29), we obtain

$$D\left(H(t_0,x)+H(t,y),H\left(t_0,\frac{x+y}{2}\right)+H\left(t,\frac{x+y}{2}\right)\right) \le\frac{1}{2}M|\alpha(t)-\alpha(t_0)|\varphi_2^{-1}\left(\frac{1}{|\alpha(t)-\alpha(t_0)|}\right)\|x-y\|. \tag{30}$$

Since N maps $RV_{\varphi_1}([a,b],\alpha;K)$ into $RW_{\varphi_2}([a,b],\alpha;cc(Y))$, then $H(\cdot,z)$ is continuous for all $z\in K$. Hence letting $t_0\uparrow t$ in the inequality (30), we get

$$D\left(H(t,x)+H(t,y),H\left(t,\frac{x+y}{2}\right)+H\left(t,\frac{x+y}{2}\right)\right) =0, \tag{31}$$

for all $t\in[a,b]$ and $x,y\in K$.

Thus for all $t\in[a,b]$, $x,y\in K$, we have

$$H\left(t,\frac{x+y}{2}\right)+H\left(t,\frac{x+y}{2}\right)=H(t,x)+H(t,y). \tag{32}$$

Since H is convex, we have

$$H\left(t,\frac{x+y}{2}\right)=\frac{1}{2}\big[H(t,x)+H(t,y)\big], \tag{33}$$

for all $t\in[a,b]$, $x,y\in K$. Thus for all $t\in[a,b]$, the set-valued function $H(t,\cdot):K\to cc(Y)$ satisfies the *Jensen* Equation (33). Now by Lemma 4.4, there exists an additive set-valued function $A(t):K\to cc(Y)$ and a set $B(t)\in cc(Y)$, such that

$$H(t,x)=A(t)x+B(t),\ \big(x\in K,t\in[a,b]\big). \tag{34}$$

Substituting $H(t,x)=A(t)x+B(t)$ into inequality (13), we deduce that for all $t\in[a,b]$ there exists $M(t)\in[0,+\infty)$, such that

$$D\big(A(t)x,A(t)y\big)\le M(t)\|x-y\|\ \ (x,y\in K),$$

consequently, for every $t\in[a,b]$ the set-valued function $A(t):K\to cc(Y)$ is continuous, and

$$A(t)(\cdot)\in L\big(K,cc(Y)\big).$$

Since $A(t)(\cdot)$ is additive and $0\in K$, then $A(t)=\{0\}$ for all $t\in[a,b]$, thus $H(\cdot,0)=B(\cdot)$.

The Nemytskii operator N maps the space $RV_{\varphi_1}([a,b],\alpha;K)$ into the space $RW_{\varphi_2}([a,b];cc(Y))$, then

$$H(\cdot,0)=B(\cdot)\in RW_{\varphi_2}([a,b],\alpha;K).$$

Consequently the set-valued function H has to be of the form

$$H(t,x)=A(t)x+B(t),\ t\in[a,b],\ x\in K,$$

where $A(t)\in L(K,cc(Y))$ and

$$B\in RW_{\varphi_2}([a,b],\alpha;cc(Y)).$$

Theorem 5.3 *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be normed spaces, K a convex cone in X and φ_1,φ_2 be two convex φ-functions in X, strictly increassing, satisfying ∞_1 condition and $\lim_{t\to\infty}\varphi_2^{-1}(\varphi_1(t))/t=\infty$. If the Nemytskii operator N generated by a set-valued function $H:[a,b]\times K\to cc(Y)$ maps the space $RV_{\varphi_2}([a,b]\alpha;K)$ into the space $RW_{\varphi_1}([a,b],\alpha;cc(Y))$ and if it is globally Lipschizian, then the set-valued function H satisfies the following condition*

$$H(t,x)=H(t,0)\ \ (t\in[a,b],\ x\in K);$$

i.e., the Nemytskii operator is constant.

Proof. Since the Nemytskii operator N is globally Lipschizian between $RV_{\varphi_1}([a,b],\alpha;K)$ and the space $RW_{\varphi_1}([a,b],\alpha;cc(Y))$, then there exists a constant M, such that

$$D_{\varphi_1}(Nf_1,Nf_2)\le M\|f_1-f\|_{2\varphi_2}\ (f_1,f_2\in RV_{\varphi_2}([a,b],\alpha;K)). \tag{35}$$

Let us fix $t,t_0\in[a,b]$ such that $t_0<t$. Using the definitions of the operator N and of the metric D_{φ_1}, we have

$$\begin{aligned}&D\big(H(t,f_1(t))+H(t_0,f_2(t_0)),\\&H(t_0,f_1(t_0))+H(t,f_2(t))\big)\\&\le M|\alpha(t)-\alpha(t_0)|\|f_1-f_2\|_{\varphi_2}\varphi_1^{-1}\left(\frac{1}{|\alpha(t)-\alpha(t_0)|}\right)\\&(f_1,f_2\in RV_{\varphi_2}([a,b],\alpha;K))\end{aligned} \tag{36}$$

Define the auxiliary function $\eta_3:[a,b]\to[0,1]$. by

$$\eta_3(\tau):=\begin{cases}1, & a\le\tau\le t_0,\\ -\dfrac{\alpha(\tau)-\alpha(t)}{\alpha(t)-\alpha(t_0)}, & t_0\le\tau\le t,\\ 0, & t\le\tau\le b.\end{cases}$$

The function $\eta_3\in RV_{\varphi_2}([a,b],\alpha)$ and

$$V_{\varphi_2}(\eta_3;[a,b])=|\alpha(t)-\alpha(t_0)|\varphi_2^{-1}\left(\frac{1}{|\alpha(t)-\alpha(t_0)|}\right).$$

Let us fix $x\in K$ and define the functions $f_i:[a,b]\to K$ $(i=1,2)$ by

$$f_1(\tau):=x,\quad f_2(\tau):=\eta_3(\tau)x,\quad \tau\in[a,b]. \tag{37}$$

The functions $f_i\in RV_{\varphi_2}([a,b],\alpha;K)$ $(i=1,2)$ and

$$\begin{aligned}\|f_1-f_2\|_{\varphi_2}&=\|f_1(a)-f_2(a)\|+\inf\left\{\varepsilon>0:\sup_{\pi}\sum_{i=1}^{n}\varphi_2\left(\frac{\|(f_1-f_2)(t_i)-(f_1-f_2)(t_{i-1})\|}{\varepsilon|\alpha(t_i)-\alpha(t_{i-1})|}\right)|\alpha(t_i)-\alpha(t_{i-1})|\le1\right\}\\&=\inf\left\{\varepsilon>0:\varphi_2\left(\frac{\|x\|}{\varepsilon|\alpha(t)-\alpha(t_0)|}\right)|\alpha(t)-\alpha(t_0)|\le1\right\}\\&=\frac{\|x\|}{|\alpha(t)-\alpha(t_0)|\varphi_2^{-1}\left(\dfrac{1}{|\alpha(t)-\alpha(t_0)|}\right)}.\end{aligned}$$

Hence, substituting in the inequality (36) the auxiliary functions f_i $(i=1,2)$ defined by (37), we obtain

$$D\big(H(t,x)+H(t_0,x),H(t_0,x)+H(t,0)\big)\le M|\alpha(t)-\alpha(t_0)|\frac{\varphi_1^{-1}\left(\dfrac{1}{|\alpha(t)-\alpha(t_0)|}\right)}{|\alpha(t)-\alpha(t_0)|\varphi_2^{-1}\left(\dfrac{1}{|\alpha(t)-\alpha(t_0)|}\right)}\|x\|.$$

By Lemma 4.3 and the above inequality, we get

$$D\left(H(t,x),H(t,0)\right)\le M\left|\alpha(t)-\alpha(t_0)\right|\frac{\varphi_1^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(t_0)\right|}\right)}{\left|\alpha(t)-\alpha(t_0)\right|\varphi_2^{-1}\left(\frac{1}{\left|\alpha(t)-\alpha(t_0)\right|}\right)}\|x\|.$$

Since $\lim_{t\to\infty}\varphi_2^{-1}\left(\varphi_1(t)\right)/t=\infty$, letting $t_0\uparrow t$ in the above inequality, we have

$$D\left(H(t,x),H(t,0)\right)=0.$$

Thus for all $t\in[a,b]$ and for all $x\in K$, we get

$$H(t,x)=H(t,0).$$

Theorem 5.4 *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be normed spaces, K a convex cone in X and φ be a convex φ-function in X satisfying the ∞_1 condition. If the Nemytskii operator N generated by a set-valued function $H:[a,b]\times K\to cc(Y)$ maps the space $RV_\varphi([a,b],\alpha;K)$ into the space $BW([a,b];cc(Y))$ and if it is globally Lipschizian, then the left regularization $H^*:[a,b]\times K\to cc(Y)$ of the function H defined by*

$$H^*(t,x):=\begin{cases}H(t,x), & t\in(a,b],\ x\in K;\\ \lim_{s\downarrow a}H(s,x), & t=a,\ x\in K,\end{cases}$$

satisfies the following conditions:

- for all $t\in[a,b]$ there exists $M(t)$, such that

$$D\left(H^*(t,x),H^*(t,y)\right)\le M(t)\|x-y\|\quad (x,y\in K).$$

- $H^*(t,x)=A(t)x+B(t)\quad \left(t\in[a,b]x\in K\right)$, where $A(t)$ is a linear continuous set-valued function, and $B\in BW([a,b];cc(Y))$.

Proof. We take $t\in[a,b]$, and define the auxiliary function $\eta_4:[a,b]\to[0,1]$ by:

$$\eta_4(\tau):=\begin{cases}1, & a\le\tau\le t,\\ \frac{\alpha(\tau)-\alpha(b)}{\alpha(t)-\alpha(b)}, & t\le\tau\le b.\end{cases}$$

The function $\eta_4\in RV_\varphi([a,b],\alpha;K)$ and

$$V_\varphi(\eta_4,[a,b])=\varphi\left(\frac{1}{\left|\alpha(b)-\alpha(t)\right|}\right)\left|\alpha(b)-\alpha(t)\right|.$$

Let us fix $x,y\in K$ and define the functions $f_i:[a,b]\to K\quad (i=1,2)$ by

$$f_1(\tau):=x,\ f_2(\tau):=\eta_4(\tau)(y-x)+x,\quad \tau\in[a,b]. \tag{38}$$

The functions $f_i\in RV_\varphi([a,b],\alpha;K)\quad (i=1,2)$ and

$$\|f_1-f_2\|_\varphi=\|f_1(a)-f_2(a)\|+\inf\left\{\varepsilon>0:\sup_\pi\sum_{i=1}^n\varphi\left(\frac{\left\|(f_1-f_2)(t_i)-(f_1-f_2)(t_{i-1})\right\|}{\varepsilon\left|\alpha(t_i)-\alpha(t_{i-1})\right|}\right)\left|\alpha(t_i)-\alpha(t_{i-1})\right|\le1\right\} \tag{39}$$

From the definition of f_1 and f_2, we obtain

$$\|f_1-f_2\|_\varphi=\|x-y\|\left(1+\frac{1}{\left|\alpha(b)-\alpha(t)\right|\varphi^{-1}\left(\frac{1}{\left|\alpha(b)-\alpha(t)\right|}\right)}\right). \tag{40}$$

Since the Nemytskii operator N is globally Lipschitzian between $RV_\varphi([a,b],\alpha;K)$ and $BW([a,b];cc(Y))$, then there exists a constant M, such that

$$D\left(H(b,f_1(b))+H(t,f_2(t)),H(t,f_1(t))+H(b,f_2(b))\right)\le M\|f_1-f_2\|_\varphi.$$

for $f_1,f_2\in RV_\varphi([a,b],\alpha;K)$. By Lemma 4.3, substituting the particular functions $f_i\quad (i=1,2)$ defined by (38) in the above inequality, we obtain

$$D\left(H(b,f_1(b))+H(t,f_2(t)),H(t,f_1(t))+H(b,f_2(b))\right)\le M(t)\|x-y\|_\varphi, \tag{41}$$

for all $x, y \in K, t \in [a,b]$. By Lemma 4.3, we get

$$D(H(t,x), H(t,y)) \le M(t)\|x-y\|_\varphi \quad (42)$$

for all $t \in [a,b)$ and $x, y \in K$.

In the case where $t = b$, by a similar reasoning as above, we obtain that there exists a constant $M(b)$, such that

$$D(H(b,x), H(b,y)) \le M(b)\|x-y\|_\varphi \quad (x, y \in K). \quad (43)$$

Define the function $M : [a,b] \to \mathbb{R}$ by

$$M(t) = \begin{cases} M\left(1 + \dfrac{1}{|\alpha(b)-\alpha(t)|\varphi^{-1}\left(\dfrac{1}{|\alpha(b)-\alpha(t)|}\right)}\right), & a \le t < b, \\ M(b), & t = b. \end{cases} \quad (44)$$

Hence,

$$D(H(t,x), H(t,y)) \le M(t)\|x-y\|_\varphi,$$
$$t \in [a,b], x, y \in K.$$

By passing to the limit in the inequality (41) by the inequality (43) and the definition of H^* we have for all $t \in [a,b]$ that there exists $M(t)$, such that

$$D(H^*(t,x), H^*(t,y)) \le M(t)\|x-y\|_\varphi$$
$$(t \in [a,b] x, y \in K)$$

Now we shall prove that H^* satisfies the following equality

$$H^*(t,x) = A(t)x + B(t) \quad (t \in [a,b], x \in K),$$

where $A(t)$ is a linear continuous set-valued functions, and

$$B \in BW([a,b]; cc(Y)).$$

Let us fix $t, t_0 \in [a,b], n \in \mathbb{N}$ such that $t_0 < t$. Define the partition π_n of the interval $[t_0, t]$ by

$$\pi_n : a < t_0 < t_1 < \cdots < t_{2n-1} < t_{2n} = t,$$
$$\text{where } t_i - t_{i-1} = \frac{t-t_0}{2n}, \quad i = 1, 2, \cdots, 2n.$$

The Nemytskii operator N is globally Lipschitzian between $RV_\varphi([a,b], \alpha; K)$ and $BW([a,b]; cc(Y))$, then there exists a constant $M > 0$, such that

$$\sum_{i=1}^{n} D\left(d_{f_1,f_2}(H, t_{2i}, t_{2i-1}), d_{f_1,f_2}(H, t_{2i-1}, t_{2i})\right) \le M\|f_1 - f_2\|_\varphi, \quad (45)$$

where

$$f_1, f_2 \in RV_\varphi([a,b], \alpha; K)$$

and

$$d_{f_1,f_2}(H, s, t) = H(s, f_1(s)) + H(t, f_2(t)).$$

We define the function $\eta_5 : [a,b] \to [0,1]$ in the following way:

$$\eta_5(\tau) := \begin{cases} 0, & a \le \tau \le t_0; \\ \dfrac{\alpha(\tau)-\alpha(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})}, & t_{i-1} \le \tau \le t_i, i = 1, 3, \cdots, 2n-1; \\ -\dfrac{\alpha(\tau)-\alpha(t_i)}{\alpha(t_i)-\alpha(t_{i-1})}, & t_{i-1} \le \tau \le t_i, i = 2, 4, \cdots, 2n; \\ 0, & t \le \tau \le b. \end{cases}$$

The function $\eta_5 \in RV_\varphi([a,b], \alpha; K)$ and

$$V_\varphi(\eta_5, \alpha; [a,b]) = |\alpha(t) - \alpha(t_0)|\varphi\left(\frac{2n}{|\alpha(t) - \alpha((t_0)|}\right).$$

Let us fix $x, y \in K$ and define the functions $f_i : [a,b] \to K$ by:

$$\begin{cases} f_1(\tau) := \frac{1}{2}\eta_5(\tau)x + \left[1 - \frac{1}{2}\eta_5(\tau)\right]y, & \tau \in [a,b]; \\ f_2(\tau) := \frac{1}{2}\left[1 + \eta_5(\tau)\right]x + \frac{1}{2}\left[1 - \eta_5(\tau)\right]y, & \tau \in [a,b]. \end{cases} \tag{46}$$

The functions $f_i \in RV_\varphi([a,b],\alpha;K)\ (i = 1,2)$ and

$$\|f_1 - f_2\|_\varphi = \frac{\|x-y\|}{2}.$$

Substituting in the inequality (45) the particular functions $f_i \quad (i = 1,2)$ defined in (46), we obtain

$$\sum_{i=1}^{n} D\left(H(t_{2i-1},x) + H(t_{2i},y), H\left(t_{2i-1}, \frac{x+y}{2}\right) + H\left(t_{2i}, \frac{x+y}{2}\right)\right) \le \frac{1}{2} M \|x-y\|_\varphi \quad x,y \in K. \tag{47}$$

Since the *Nemytskii* operator N maps the spaces $RV_\varphi([a,b],\alpha;K)$ into $BW([a,b];cc(Y))$, then for all $z \in K$, the function $H(\cdot,z) \in BW([a,b];cc(Y))$. Letting $t_0 \uparrow t$ in the inequality (47), we get

$$D\left(H^*(t,x) + H^*(t,y), H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right)\right)$$
$$\le \frac{M}{2n}\|x-y\|_\varphi.$$

for all $x,y \in K$ and $n \in \mathbb{N}$. By passing to the limit when $n \to \infty$, we get

$$H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right) = H^*(t,x) + H^*(t,y),$$
$$t \in [a,b], x,y \in K.$$

Since $H^*(t,x)$ is a convex function, then

$$H^*\left(t, \frac{x+y}{2}\right) = \frac{1}{2}\left[H^*(t,x) + H^*(t,y)\right]$$
$$(t \in [a,b], x,y \in K).$$

Thus for every $t \in [a,b]$, the set-valued function $H^*(t,\cdot): K \to cc(Y)$ satisfies the *Jensen* equation. By Lemma 4.4 and by the property (a) previously established, we get that for all $t \in [a,b]$ there exist an additive set-valued function $A(\cdot): K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$H^*(t,x) = A(t)x + B(t) \quad (t \in [a,b], x \in K).$$

By the same reasoning as in the proof of Theorem 5.2, we obtain that

$$A(t)(\cdot) \in L(K, cc(Y)) \text{ and } B \in BW([a,b]; cc(Y)).$$

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