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Weak Integrals and Bounded Operators in Topological Vector Spaces

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ABSTRACT

Let X be a topological vector space and let S be a locally compact space. Let us consider the function space $C_0(S,X)$ of all continuous functions $f:S\to X$, vanishing outside a compact set of S, equipped with an appropriate topology. In this work we will be concerned with the relationship between bounded operators $T:C_0(S,X)\to X$, and X-valued integrals on $C_0(S,X)$. When X is a Banach space, such relation has been completely achieved via Bochner integral in [1]. In this paper we investigate the context of locally convex spaces and we will focus attention on weak integrals, namely the Pettis integrals. Some results in this direction have been obtained, under some special conditions on the structure of X and its topological dual X^*. In this work we consider the case of a semi reflexive locally convex space and prove that each Pettis integral with respect to a signed measure μ, on S gives rise to a unique bounded operator $T:C_0(S,X)\to X$, which has the given Pettis integral form.

Keywords: Bounded Operators; Integral Representation; Pettis Integral

1. Topological Preliminaries

Suppose that S is a locally compact space and let X be a locally convex TVS. We denote by $C_0(S,X)$ the set of all continuous functions $f:S\to X$ vanishing outside a compact set of S, put $C_0(S,X)=C_0(S)$ if $X = R$. We are interested in representing linear bounded operators $T:C_0(S,X)\to X$, by means of weak integrals against scalar measures on the Borel σ-field B_S of S. Before handling more closely this problem, we need some topological facts about the space $C_0(S,X)$.

If K is a compact set in S, let $C(S,K,X)$ be the set of all continuous functions $f:S\to X$, vanishing outside K. It is clear that $C(S,K,X)$ is a linear subspace of $C_0(S,X)$. We equip $C(S,K,X)$ with the topology τ_K generated by the family of seminorms:

$$(*)\, f\in C(S,K,X), \tilde{p}_{\alpha,K}=\operatorname{Sup}_{t\in K}p_\alpha(f(t))$$

where $\{p_\alpha\}$ is the family of seminorms generating the locally convex topology of X. The topology τ_K is the topology of uniform convergence on K.

Next let us observe that $C_0(S,X)=_K C(S,K,X)$, the union being performed over all the compact subsets K of S. On the other hand if K_1 is a subset of K_2, then the natural embedding $i_{K_1K_2}:C(S,K_1,X)\to C(S,K_2,X)$ is continuous. This allows one to provide the space $C_0(S,X)$ with the inductive topology τ induced by the subspaces $C(S,K,X)$, τ_K. The facts we need about the space, $C_0(S,X)$ is well known:

1.1. Proposition

1) The space $C_0(S,X)$, τ is locally convex Hausdorff and for each compact K, the relative topology of τ on $C(S,K,X)$ is τ_K, this means that the canonical embedding $i_k:C(S,K,X)\to C_0(S,X)$ is continuous.

2) Let $T:C_0(S,X)\to V$ be a linear operator of $C_0(S,X)$ into the locally convex Hausdorff space V, then T is continuous if and only if the restriction Toi_K of T to the subspace $C(S,K,X)$ is continuous for each compact K.

1.2. Definition

For each θ in the topological dual X^* of X and for each function $f \in C_0(S,X)$, define the function $U_\theta f$ on S by $U_\theta f(s) = \theta(f(s)) = \langle \theta, f(s)\rangle$. Then U_θ sends $C_0(S,X)$ into $C_0(S)$. Recall that $C_0(S)$ is equipped with the uniform norm.

1.3. Lemma

The operator U_θ is linear and bounded. Moreover for each $\theta \neq 0$, U_θ is onto.

Proof: First it is clear that $U_\theta f = C_0(S)$. Now by Proposition 1.1(b), we have to show that for each compact set K of S the operator $U_\theta oi_K : C(S,K,X) \to C_0(S)$ is bounded. Since θ is bounded, there is a seminorm p_α on X and a constant M such that $|\theta(x)| \le Mp_\alpha(x)$ for all $x \in X$. So we have $|\theta(f(s))| \le Mp_\alpha(f(s))$ if $f \in C(S,K,X)$, and $U_\theta oi_K(f)(s) = \theta(f(s))$, $s \in S$; it follows that

$$\|U_\theta f\| = \mathrm{Sup}_{t\in K} |\theta(f(s))| \le \mathrm{MSup}_{t\in K}\, p_\alpha(f(s)).$$

Since by Formula (*), the right side of this inequality is $M\tilde{p}_{\alpha,K}(f)$, we deduce that U_θ is continuous. Now suppose $\theta \neq 0$. Then there exists $x \in X$ such that $x \neq 0$ and $\theta(x) \neq 0$. It is clear that we can assume $\theta(x) = 1$. Now let $h \in C_0(S)$ and define $f : S \to X$ by $f(t) = h(t).x$, then $f \in C_0(S,X)$ and we have $U_\theta(f)(s) = U_\theta(h(s)x) = h(s)$, because $\theta(x) = 1$. It follows that U_θ is onto. ■

Now we consider the relationship between bounded operators $T : C_0(S,X) \to X$, and weak integrals in the sense of the following definition. Such relationship is reminiscent to the classical Riesz theorem [2].

1.4. Definition

We say that a bounded operator $T : C_0(S,X) \to X$ has a Pettis integral form if there exists a scalar measure of bounded variation μ on B_S such that, for every continuous functional θ in X^*, we have:

$$f \in C_0(S,X), \langle \theta, Tf\rangle = \int \langle \theta, f(s)\rangle \mathrm{d}\mu(s)$$

See Reference [3] for details on Pettis integral.

2. Integral Representation by Pettis Integral

In what follows, we introduce a class of bounded operators $T : C_0(S,X) \to X$, which is, in this context, similar to the class C_{XX} used in [1].

2.1. Definition

Let P be the class of all bounded operators $T : C_0(S,X) \to X$ satisfying the following condition:

(*I*) For $\theta, \sigma \in X^*$ and $f, g \in C_0(S,X)$, if $U_\theta f = U_\sigma g$ then $\theta(Tf) = \sigma(Tg)$.

It is easy to check that P is a subspace of the space $L(C_0(S,X),X)$ of all bounded operators from $C_0(S,X)$ to X. Also one can prove that P is closed in the weak operator topology of $L(C_0(S,X),X)$. Note also that for a given bounded $T : C_0(S,X) \to X$, Definition 1.4 implies condition (*I*) *i.e.* $T \in P$. The crucial point is that condition (*I*) implies the Pettis integral form of Definition 1.4, for some bounded scalar measure μ on B_S. This is the content of the following theorem proved in [4].

2.2. Theorem

Let $T : C_0(S,X) \to X$ be in the class P. Then there is a unique bounded signed measure μ on B_S such that $\langle \theta, Tf\rangle = \int \langle \theta, f(s)\rangle \mathrm{d}\mu(s)$ holds for all θ in X^* and $f \in C_0(S,X)$. Moreover for each seminorm p_α on X we have $|T|_{p_\alpha} = |\mu|$, where $|\mu|$ is the total variation of μ and $|T|_{p_\alpha}$ is the p_α-norm of T defined by

$$|T|_{p_\alpha} = \mathrm{Sup}\left\{p_\alpha(Tf) : f \in \tilde{B}_{p_\alpha}\right\}$$

with

$$\tilde{B}_{p_\alpha} = \left\{f \in C_0(S,X) : \mathrm{Sup}_S\, p_\alpha(f(s)) \le 1\right\}.$$

By this theorem we may denote each operator T in the class P by the conventional symbol

$$(W)\, f \in C_0(S,X), Tf = P - \int f(s)\mathrm{d}\mu(s)$$

where the letter P stands for Pettis integral.

3. Operators Associated to Scalar Measures via Pettis Integrals

In this section we start with a bounded scalar measure μ on B_S and we seek for a linear bounded $T : C_0(S,X) \to X$ such that the correspondence between μ and T would be given by formula (W). First let us make some observations.

3.1. Operators via Pettis Integrals

A little inspection of (W) suggests the following quite plausible observations: First the integral $\int \langle \theta, f(s)\rangle \mathrm{d}\mu(s)$, as a linear functional of θ on X^*, should beat least continuous for some convenient topology on X^* Also the existence of the corresponding Tf in (W) will require that such topology on X should be compatible for the dual pair $\langle X^*, X\rangle$. Finally, to get the continuity of the functional $\theta \to_S \langle \theta, f(s)\rangle \mathrm{d}\mu(s)$, one can seek conditions such that if $\theta \to 0$ in an appropriate manner, then $\langle \theta, f(s)\rangle$ goes to 0 uniformly for $s \in S$. Since μ is bounded this will give $\int \langle \theta, f(s)\rangle \mathrm{d}\mu(s) \to 0$.

Such a program has been realized in [4], for a locally convex space having the convex compactness property [5], according to the following theorems (see [4] for details).

3.2. Theorem

Let X be a locally convex space with the convex compactness property, and whose dual X^* is equipped with the Mackey topology $\tau\left(X^*,X\right)$. If μ is a bounded scalar measure on B_S, then there is a unique bounded operator $T:C_0\left(S,X\right)\to X$ in the class P satisfying (W), with $\left|T\right|_{p_\alpha}=\left|\mu\right|$ for each seminorm p_α on X.

3.3. Theorem

Let X be a locally convex Hausdorff space whose dual X_τ^* is a barrelled space. If μ is a bounded signed measure on B_S, then there is a unique bounded operator $T:C_0\left(S,X\right)\to X$ in the class P satisfying (W) with respect to μ and such that $\left|T\right|_{p_\alpha}=\left|\mu\right|$.

Most of these results have been obtained for a space whose dual is a Mackey space. It is natural to ask if similar representations can be established if the dual is endowed with another topology, e.g. the strong topology.

3.4. Definition

The strong topology $\beta\left(X^*,X\right)$ of X^* is the topology generated by the family of the seminorms:

$$(**)\theta\in X^*,P_B\left(\theta\right)=\mathrm{Sup}_\mathrm{B}\left|\theta\left(x\right)\right|$$

where B is running over all the bounded sets of X.

It is the topology of uniform convergence on the bounded sets of X. When we restrict $(**)$ to the finite sets B of X we get the so called weak * topology $\sigma\left(X^*,X\right)$, which is the topology of simple convergence on X. We shall denote by $X_\beta^*\left(X_\sigma^*\right)$ the space X^* equipped with the $\beta\left(X^*,X\right)$-topology (the $\sigma\left(X^*,X\right)$-topology). Then we have:

3.5. Proposition

1) For each $x^{**}\in\left(X_\sigma^*\right)^*$ there exists a unique $x\in X$ such that: $x^{**}\left(\theta\right)=\theta\left(x\right),\forall\theta\in X_\sigma^*$.

2) $\left(X_\sigma^*\right)^*\subset\left(X_\beta^*\right)^*$, that is, every weak * continuous functional on X^* is strongly continuous .

3.6. Definition

We say that the space X is semireflexive if $\left(X_\sigma^*\right)^*=\left(X_\beta^*\right)^*$.

Now we are in a position to state the main results of this paper.

3.7. Theorem

Let X be a locally convex Hausdorff semireflexive space. If μ is a bounded signed measure on B_S, then there is a unique bounded operator $T:C_0\left(S,X\right)\to X$ in the class P satisfying:

$$\forall f\in C_0\left(S,X\right),\left\langle\theta,Tf\right\rangle=\int\left\langle\theta,f\left(s\right)\right\rangle\mathrm{d}\mu\left(s\right),$$

$$\left|T\right|_{p_\alpha}=\left|\mu\right|.$$

where $\left|\mu\right|$ is the variation of μ.

Proof: Fix f in $C_0\left(S,X\right)$ and define the functional $\Delta_f:X^*\to R$, by $\Delta_f\left(\theta\right)=\int\left\langle\theta,f\left(s\right)\right\rangle\mathrm{d}\mu\left(s\right)$. It is clear that Δ_f is linear. Moreover $\Delta_f\in\left(X_\beta^*\right)^*$. Indeed it is enough to prove that $\lim_{\theta\to0}\Delta_f\left(\theta\right)=0$. If $\theta\to0$, in X_β^*, then for each bounded subset B of X, $\theta\left(x\right)\to0$ uniformly for $x\in B$. But since $f\in C_0\left(S,X\right)$, the set $\left\{f\left(s\right):s\in S\right\}$ is bounded, so $\left\langle\theta,f\left(s\right)\right\rangle\to0$ uniformly in $s\in S$. Therefore, $\int\left\langle\theta,f\left(s\right)\right\rangle\mathrm{d}\mu\left(s\right)\to0$, because the measure μ is of bounded variation. Hence $\Delta_f\in\left(X_\beta^*\right)^*$. Since X is semireflexive, $\Delta_f\in\left(X_\sigma^*\right)^*$; by Proposition 3.5(a), there is a unique $\gamma_f\in X$ such that $\Delta_f\left(\theta\right)=\left\langle\theta,\gamma_f\right\rangle$, $\forall\theta\in X^*$. Now let us define the operator $T:C_0\left(S,X\right)\to X$, by $Tf=\gamma_f$, $f\in C_0\left(S,X\right)$. It is easily checked that T is linear, and satisfies the condition of the theorem by construction. We have to show that T is bounded. Let p_α be a seminorm on X, and let K be a compact subset of X. For $f\in C\left(S,K,X\right)$, we have:

$$\begin{aligned}p_\alpha\left(\gamma_f\right)&=p_\alpha\left(Tf\right)=\mathrm{Sup}_{\theta\in B_{p_\alpha}^\alpha}\left|\theta o\ Tf\right|\\&=\mathrm{Sup}_{\theta\in B_{p_\alpha}^\alpha}\left|\int<\theta,f\left(s\right)>\mathrm{d}\mu\left(s\right)\right|\\&\le\mathrm{Sup}_{\theta\in B_{p_\alpha}^\alpha}\mathrm{Sup}_{s\in K}\left|\left\langle\theta,f\left(s\right)\right\rangle\right|.\left|\mu\right|\\&=\mathrm{Sup}_{s\in K}\mathrm{Sup}_{\theta\in B_{p_\alpha}^\alpha}\left|\left\langle\theta,f\left(s\right)\right\rangle\right|.\left|\mu\right|\\&=\tilde{p}_{\alpha,K}\left(f\right).\left|\mu\right|\end{aligned}$$

which proves the continuity of T.

Now to compute $\left|T\right|_{p_\alpha}$, observe from the integral form of θoTf that $\left|\theta oTf\right|\le\mathrm{Sup}\left\{\left|\left\langle\theta,f\left(s\right)\right\rangle\right|:s\in S\right\}.\left|\mu\right|$. Taking the supremum in both sides over $\theta\in B_{p_\alpha}^0$, the polar set of the unit ball $B_{p_\alpha}=\left\{x\in X,p_\alpha\left(x\right)\le1\right\}$ of X, we get:

$$\begin{aligned}&\mathrm{Sup}_{\theta\in B_{p_\alpha}^0}\left|\theta o\ Tf\right|=p_\alpha\left(Tf\right)\\&\le\mathrm{Sup}_{\theta\in B_{p_\alpha}^0}\mathrm{Sup}_{s\in S}\left|\left\langle\theta,f\left(s\right)\right\rangle\right|.\left|\mu\right|\\&=\mathrm{Sup}_{s\in S}\mathrm{Sup}_{\theta\in B_{p_\alpha}^0}\left|\left\langle\theta,f(s)\right\rangle\right|.\left|\mu\right|\\&=\mathrm{Sup}\,p_\alpha\left(f\left(s\right)\right)\cdot\left|\mu\right|\le\left|\mu\right|\ \text{ for } f\in\tilde{B}_{p_\alpha}\end{aligned}$$

So we deduce that $\left|T\right|_{p_\alpha}\le\left|\mu\right|$. To see the reverse inequality, let us consider a function $f\in C_0\left(S,X\right)$ of the form $f=g\cdot x$, with $g\in C_0\left(S\right)$ satisfying $\left\|g\right\|\le1$

and x fixed in X such that $p_\alpha(x)=1$. With this choice, the function f belongs to the unit ball $\tilde{B}_{p_\alpha}$. Then we have

$$\langle\theta, f(s)\rangle = g(s)\cdot\theta(x)$$

and

$$\langle\theta, Tf\rangle = \int\langle\theta, f(s)\rangle \mathrm{d}\mu(s) = \theta(s)\int g(s)\mathrm{d}\mu(s);$$

so that

$$\mathrm{Sup}_{\theta\in B^0_{p_\alpha}} |\theta o\ Tf| = p_\alpha(Tf) = p_\alpha(x)\left|\int g(s)\mathrm{d}\mu(s)\right|$$
$$= \left|\int g(s)\mathrm{d}\mu(s)\right|,$$

since $p_\alpha(x)=1$. So we get

$$p_\alpha(Tf) = \left|\int g(s)\mathrm{d}\mu(s)\right| \le |T|_{p_\alpha}$$

because $f\in\tilde{B}_{p_\alpha}$.

Therefore

$$\mathrm{Sup}\left\{\left|\int g(s)\mathrm{d}\mu(s)\right|, g\in C_0(S), g\le 1\right\} = |\mu| \le |T|_{p_\alpha} \qquad \blacksquare$$

By appealing to theorem 2.3, we get the following rather precise theorem:

3.8. Theorem

Let X be a locally convex Hausdorff semireflexive space. Then there is a one to one correspondence between the bounded operators $T: C_0(S,X)\to X$ of the class P and the X-valued Pettis integrals with respect to some bounded signed measure μ on B_S. This correspondence is given by the relation (W):

$$f\in C_0(S,X), Tf = P - \int f(s)\mathrm{d}\mu(s)$$

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REFERENCES

[1] L. Meziani, "Integral Representation for a Class of Vector Valued Operators," *Proceedings of the American Mathematical Society*, Vol. 130, No. 7, 2002, pp. 2067-2077. doi:10.1090/S0002-9939-02-06336-0

[2] S. Kakutani, "Concrete Representation of Abstract (M). Spaces," *Annals of Mathematics*, Vol. 42, No. 4, 1941, pp. 994-1024. doi:10.2307/1968778

[3] J. Diestel and J. J. Uhl, "Vector Measures," *Mathematical Surveys*, No. 15, 1977.

[4] L. Meziani, "A Theorem of Riesz Type with Pettis Integrals in TVS," *Journal of Mathematical Analysis and Applications*, Vol. 340, No. 2, 2008, pp. 817-824. doi:10.1016/j.jmaa.2007.09.005

[5] A. Wilanski, "Modern Methodsin Topological Vector Spaces," McGraw-Hill, New York, 1978.

