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The Continuous Wavelet Transform Associated with a Dunkl Type Operator on the Real Line

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ABSTRACT

We consider a singular differential-difference operator Λ on $\mathbf{R}$ which includes as a particular case the one-dimensional Dunkl operator. By using harmonic analysis tools corresponding to Λ, we introduce and study a new continuous wavelet transform on $\mathbf{R}$ tied to Λ. Such a wavelet transform is exploited to invert an intertwining operator between Λ and the first derivative operator $\mathrm{d}/\mathrm{d}x$.

Keywords: Differential-Difference Operator; Generalized Wavelets; Generalized Continuous Wavelet Transform

1. Introduction

In this paper we consider the first-order singular differential-difference operator on $\mathbf{R}$

$$\Lambda f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \left(\gamma + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} + q(x) f(x)$$

where $\gamma > -1/2$ and q is a C^∞ real-valued odd function on $\mathbf{R}$. For $q = 0$, we regain the differential-difference operator

$$\Lambda f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \left(\gamma + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator with parameter $\gamma + 1/2$ associated with the reflection group $\mathbf{Z}_2$ on $\mathbf{R}$. Those operators were introduced and studied by Dunkl [1-3] in connection with a generalization of the classical theory of spherical harmonics. Besides its mathematical interest, the Dunkl operator has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner's commutation rules [4-6].

Put

$$a_\gamma = \frac{\Gamma(\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1/2)} \tag{1}$$

and

$$Q(x) = \exp\left(-\int_0^x q(t)\,\mathrm{d}t\right), \; x \in \mathbf{R}. \tag{2}$$

The authors [7] have proved that the integral transform

$$Xf(x) = a_\gamma Q(x)\int_{-1}^{1} f(tx)\left(1-t^2\right)^{\gamma-1/2}(1+t)\,\mathrm{d}t \tag{3}$$

is the only automorphism of the space $E(\mathbf{R})$ of C^∞ functions on $\mathbf{R}$, satisfying

$$X\frac{\mathrm{d}}{\mathrm{d}x} f = \Lambda X f \text{ and } Xf(0) = f(0),$$

for all $f \in E(\mathbf{R})$. The intertwining operator X has been exploited to initiate a quite new commutative harmonic analysis on the real line related to the differential-difference operator Λ in which several analytic structures on $\mathbf{R}$ were generalized. A summary of this harmonic analysis is provided in Section 2. Through this paper, the classical theory of wavelets on $\mathbf{R}$ is extended to the differential-difference operator Λ. More explicitly, we call generalized wavelet each function g in $L^2\left(\mathbf{R}, |x|^{2\gamma+1}\mathrm{d}x\right)$ satisfying almost all $\lambda \in \mathbf{R}$:

$$0 < C_g = \int_0^\infty \left|F_\Lambda(g)(a\lambda)\right|^2 \frac{\mathrm{d}a}{a} < \infty,$$

where F_Λ denotes the generalized Fourier transform related to Λ given by

$$F_\Lambda(g)(\lambda) = \int_{\mathbf{R}} g(x)\Psi_{-\lambda}(x)|x|^{2\gamma+1}\mathrm{d}x, \; \lambda \in \mathbf{R},$$

$\Psi_{-\lambda}$ being the solution of the differential-difference equation

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$$\Lambda f(x) = -i\lambda f(x),\ f(0) = 1.$$

Starting from a single generalized wavelet g we construct by dilation and translation a family of generalized wavelets by putting

$$g_{a,b}(x) = {}^tT^b g_a(x),\ a > 0,\ b \in \mathbf{R},$$

where ${}^tT^b$ stand for the generalized dual translation operators tied to the differential-difference operator Λ, and g_a is the dilated function of g given by the relation

$$F_\Lambda(g_a)(\lambda) = F_\Lambda(g)(a\lambda).$$

Accordingly, the generalized continuous wavelet transform associated with Λ is defined for regular functions f on $\mathbf{R}$ by

$$\Phi_g(f)(a,b) = \int_{\mathbf{R}} f(x)\, \overline{g_{a,b}(x)}\, |x|^{2\gamma+1}.$$

In Section 3, we exhibit a relationship between the generalized and Dunkl continuous wavelet transforms. Such a relationship allows us to establish for the generalized continuous wavelet transform a Plancherel formula, a point wise reconstruction formula and a Calderon reproducing formula. Finally, we exploit the intertwining operator X to express the generalized continuous wavelet transform in terms of the classical one. As a consequence, we derive new inversion formulas for dual operator ${}^tX$ of X.

In the classical setting, the notion of wavelets was first introduced by J. Morlet, a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by A. Grossmann and J. Morlet in [8]. The harmonic analyst Y. Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [9-11]). Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [12-14] and the references therein).

2. Preliminaries

Notation. We denote by

- $L^p_\gamma(\mathbf{R})$, $1 \le p \le \infty$, the class of measurable functions f on $\mathbf{R}$ for which $\|f\|_{p,\gamma} < \infty$, where

$$\|f\|_{p,\gamma} = \left(\int_{\mathbf{R}} |f(x)|^p |x|^{2\gamma+1}\,\mathrm{d}x\right)^{1/p}, \quad \text{if } p < \infty,$$

  and $\|f\|_{\infty,\gamma} = \operatorname{ess\,sup}_{x\in\mathbf{R}} |f(x)|$.
- $L^p_Q(\mathbf{R})$, $1 \le p \le \infty$, the class of measurable functions f on $\mathbf{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\gamma} < \infty$, where Q is given by (2).
- $L^p_{1/Q}(\mathbf{R})$, $1 \le p \le \infty$, the class of measurable functions f on $\mathbf{R}$ for which $\|f\|_{p,1/Q} = \|f/Q\|_{p,\gamma} < \infty$.

Remark 1. Clearly the map

$$Mf(x) = Q(x) f(x) \tag{4}$$

is an isometry
- from $L^p_Q(\mathbf{R})$ onto $L^p_\gamma(\mathbf{R})$;
- from $L^p_\gamma(\mathbf{R})$ onto $L^p_{1/Q}(\mathbf{R})$.

2.1. Generalized Fourier Transform

The following statement is proved in [7].

Lemma 1. 1) For each $\lambda \in \mathbf{C}$, the differential-difference equation

$$\Lambda u = i\lambda u,\ u(0) = 1,$$

admits a unique C^∞ solution on $\mathbf{R}$, denoted Ψ_λ, given by

$$\Psi_\lambda(x) = Q(x)\, e_\gamma(i\lambda x), \tag{5}$$

where e_γ denotes the one-dimensional Dunkl kernel defined by

$$e_\gamma(z) = j_\gamma(iz) + \frac{z}{2(\gamma+1)} j_{\gamma+1}(iz)\ \ (z \in \mathbf{C}),$$

j_γ being the normalized spherical Bessel function of index γ given by

$$j_\gamma(z) = \Gamma(\gamma+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\gamma+1)}\ \ (z \in \mathbf{C}).$$

2) For all $x \in \mathbf{R}$, $\lambda \in \mathbf{C}$ and $n = 0,1,\cdots$, we have

$$\left|\frac{\partial^n}{\partial \lambda^n} \Psi_\lambda(x)\right| \le Q(x) |x|^n \mathrm{e}^{|\mathrm{Im}\lambda||x|}. \tag{6}$$

3) For each $x \in \mathbf{R}$ and $\lambda \in \mathbf{C}$, we have the Laplace type integral representation

$$\Psi_\lambda(x) = a_\gamma\, Q(x) \int_{-1}^{1} \left(1-t^2\right)^{\gamma-1/2} (1+t)\, \mathrm{e}^{i\lambda xt}\,\mathrm{d}t, \tag{7}$$

where a_γ is given by (1).

The generalized Fourier transform of a function f in $L^1_Q(\mathbf{R})$ is defined by

$$F_\Lambda(f)(\lambda) = \int_{\mathbf{R}} f(x) \Psi_{-\lambda}(x) |x|^{2\gamma+1}\,\mathrm{d}x. \tag{8}$$

Remark 2. 1) By (6) and (7), it follows that the generalized Fourier transform F_Λ maps continuously and injectively $L^1_Q(\mathbf{R})$ into the space $C_0(\mathbf{R})$ of continuous functions on $\mathbf{R}$ vanishing at infinity.

2) Recall that the one-dimensional Dunkl transform is defined for a function $f \in L^1_\gamma(\mathbf{R})$ by

$$F_\gamma(f)(\lambda) = \int_{\mathbf{R}} f(x) e_\gamma(-i\lambda x) |x|^{2\gamma+1}\,\mathrm{d}x. \tag{9}$$

Notice by (5), (8) and (9) that

$$F_\Lambda = F_\gamma \circ M, \tag{10}$$

where M is given by (4).

Two standard results about the generalized Fourier transform F_Λ are as follows.

Theorem 1 (inversion formula). Let $f \in L^1_Q(\mathbf{R})$ such that $F_\Lambda(f) \in L^1_\gamma(\mathbf{R})$. Then for almost all $x \in \mathbf{R}$ we have

$$f(x)(Q(x))^2 = m_\gamma \int_{\mathbf{R}} F_\Lambda(f)(\lambda)\Psi_\lambda(x)|\lambda|^{2\gamma+1}\,\mathrm{d}\lambda,$$

where

$$m_\gamma = \frac{1}{2^{2\gamma+2}(\Gamma(\gamma+1))^2}. \tag{11}$$

Theorem 2 (Plancherel). 1) For every $f \in L^2_Q(\mathbf{R})$, we have the Plancherel formula

$$\int_{\mathbf{R}} |f(x)|^2 (Q(x))^2 |x|^{2\gamma+1}\,\mathrm{d}x = m_\gamma \int_{\mathbf{R}} |F_\Lambda(f)(\lambda)|^2 |\lambda|^{2\gamma+1}\,\mathrm{d}\lambda.$$

2) The generalized Fourier transform F_Λ extends uniquely to an isometric isomorphism from $L^2_Q(\mathbf{R})$ onto $L^2_\gamma(\mathbf{R})$.

2.2. Generalized Convolution

Recall that the Dunkl translation operators $\tau^x_\gamma, x \in R$, are defined by

$$\tau^x_\gamma f(y) = \int_{\mathbf{R}} f(z)\,\mathrm{d}\mu^\gamma_{x,y}(z), \tag{12}$$

where $\mu^\gamma_{x,y}$ is a finite signed measure on **R**, of total mass 1, with support

$$\left[-|x|-|y|, -\big||x|-|y|\big|\right] \cup \left[\big||x|-|y|\big|, |x|+|y|\right],$$

and such that $\|\mu^\gamma_{x,y}\| \le 2$. For the explicit expression of the measure $\mu^\gamma_{x,y}$, see [15].

Define the generalized translation operators T^x, $x \in \mathbf{R}$, associated with Λ by

$$T^x f(y) = Q(x)Q(y)\int_{\mathbf{R}} \frac{f(z)}{Q(z)}\,\mathrm{d}\mu^\gamma_{x,y}(z). \tag{13}$$

By (12) and (13) observe that

$$T^x f(y) = Q(x)Q(y)\tau^x_\gamma(f/Q)(y). \tag{14}$$

The generalized dual translation operators are given by

$${}^tT^x f(y) = \frac{Q(x)}{Q(y)}\,\tau^{-x}_\gamma(Qf)(y). \tag{15}$$

We claim the following statement.

Proposition 1. 1) Let f be in $L^p_{1/Q}(\mathbf{R})$, $1 \le p \le \infty$. Then for all $x \in \mathbf{R}$, $T^x f$ is a well defined element in $L^p_{1/Q}(\mathbf{R})$, and

$$\|T^x f\|_{p,1/Q} \le 2Q(x)\|f\|_{p,1/Q}$$

2) Let f be in $L^p_Q(\mathbf{R})$, $1 \le p \le \infty$. Then for all $x \in \mathbf{R}$, ${}^tT^x f$ is well defined as a function in $L^p_Q(\mathbf{R})$, and

$$\|{}^tT^x f\|_{p,Q} \le 2Q(x)\|f\|_{p,Q}$$

3) For $f \in L^p_Q(\mathbf{R})$, $p = 1$ or 2, we have

$$F_\Lambda({}^tT^x f)(\lambda) = \Psi_{-\lambda}(x)F_\Lambda(f)(\lambda).$$

4) Let $1 \le p_1$, $p_2 \le \infty$ such that $1/p_1 + 1/p_2 = 1$. If $h_1 \in L^{p_1}_{1/Q}(\mathbf{R})$ and $h_2 \in L^{p_2}_Q(\mathbf{R})$, then we have the duality relation

$$\int_{\mathbf{R}} T^x(h_1)(y)h_2(y)|y|^{2\gamma+1}\,\mathrm{d}y = \int_{\mathbf{R}} h_1(y)\,{}^tT^x(h_2)(y)|y|^{2\gamma+1}\,\mathrm{d}y.$$

Proof. 1) By (14) and [13, Equation (8)] we have

$$\|T^x f\|_{p,1/Q} = \|(T^x f)/Q\|_{p,\gamma} = Q(x)\|\tau^x_\gamma(f/Q)\|_{p,\gamma} \le 2Q(x)\|f/Q\|_{p,\gamma} = 2Q(x)\|f\|_{p,1/Q}.$$

2) By (15) and [13, Equation (8)] we have

$$\|{}^tT^x f\|_{p,Q} = \|Q\,{}^tT^x f\|_{p,\gamma} = Q(x)\|\tau^{-x}_\gamma(Q f)\|_{p,\gamma} \le 2Q(x)\|Qf\|_{p,\gamma} = 2Q(x)\|f\|_{p,Q}.$$

3) By (5), (10), (15) and [1, Theorem 11] we have

$$\begin{aligned} F_\Lambda({}^tT^x f)(\lambda) &= F_\gamma(Q\,{}^tT^x f)(\lambda) \\ &= Q(x)F_\gamma(\tau^{-x}_\gamma(Q f))(\lambda) \\ &= Q(x)e_\gamma(-\mathrm{i}\lambda x)F_\gamma(Q f)(\lambda) \\ &= \Psi_{-\lambda}(x)F_\Lambda(f)(\lambda). \end{aligned}$$

4) By (14), (15) and [1, Theorem 11] we have

$$\begin{aligned} &\int_{\mathbf{R}} T^x(h_1)(y)h_2(y)|y|^{2\gamma+1}\,\mathrm{d}y \\ &= Q(x)\int_{\mathbf{R}} \tau^x_\gamma(h_1/Q)(y)Q(y)h_2(y)|y|^{2\gamma+1}\,\mathrm{d}y \\ &= Q(x)\int_{\mathbf{R}} (h_1/Q)(y)\tau^{-x}_\gamma(Qh_2)(y)|y|^{2\gamma+1}\,\mathrm{d}y \\ &= \int_{\mathbf{R}} h_1(y)\,{}^tT^x(h_2)(y)|y|^{2\gamma+1}\,\mathrm{d}y. \end{aligned}$$

This concludes the proof. ■

The generalized convolution product of two functions f and g on **R** is defined by

$$f \# g(x) = \int_{\mathbf{R}} {}^tT^y(f)(x)g(y)|y|^{2\gamma+1}\,\mathrm{d}y. \tag{16}$$

Remark 3. Recall that the Dunkl convolution product of two functions f and g on **R** is defined by

$$f *_\gamma g(x) = \int_{\mathbf{R}} \tau^x_\gamma(f)(-y)g(y)|y|^{2\gamma+1}\,\mathrm{d}y \tag{17}$$

By virtue of (15), (16) and (17) it is easily seen that

$$f \# g(x) = \frac{(Qf) *_\gamma (Qg)(x)}{Q(x)}. \tag{18}$$

By use of (10), (18) and the properties of the Dunkl convolution product mentioned in [16], we obtain the next statement.

Proposition 2. 1) Let $p_1, p_2, p_3 \in [1,\infty]$ such that $\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{p_3}$ If $f \in L_Q^{p_1}(\mathbf{R})$ and $g \in L_Q^{p_2}(\mathbf{R})$, then $f \# g \in L_Q^{p_3}(\mathbf{R})$ and

$$\| f \# g\|_{p_3,Q} \le 2 \| f\|_{p_1,Q} \|g\|_{p_2,Q}.$$

2) For $f \in L_Q^1(\mathbf{R})$ and $g \in L_Q^p(\mathbf{R})$, $p = 1$ or 2, we have

$$F_\Lambda(f \# g) = F_\Lambda(f) F_\Lambda(g).$$

2.3. Intertwining Operators

According to [7], the dual of the intertwining operator X given by (3), takes the form

$$^tXf(y) = a_\gamma \int_{|x|\ge|y|} f(x)Q(x)\operatorname{sgn}(x)\left(x^2-y^2\right)^{\gamma-1/2} \times (x+y)\mathrm{d}x$$

It was shown that ^tX is an automorphism of the space $D(\mathrm{R})$ of C^∞ compactly supported functions on $\mathbf{R}$, satisfying the intertwining relation

$$\frac{\mathrm{d}}{\mathrm{d}x} {}^tX f = {}^tX \tilde{\Lambda} f, \quad f \in D(\mathbf{R}),$$

where $\tilde{\Lambda}$ is the dual operator of Λ defined by

$$\tilde{\Lambda}f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \left(\gamma+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x} - q(x)f(x).$$

Moreover, we have the factorizations

$$X = M \circ V_\gamma, \qquad {}^tX = {}^tV_\gamma \circ M, \tag{19}$$

where V_γ and $^tV_\gamma$ are respectively the Dunkl inter-twining operator and its dual given by

$$V_\gamma f(x) = a_\gamma \int_{-1}^1 f(tx)\left(1-t^2\right)^{\gamma-1/2}(1+t)\mathrm{d}t,$$

$$^tV_\gamma f(y) = a_\gamma \int_{|x|\ge|y|} f(x)\operatorname{sgn}(x)\left(x^2-y^2\right)^{\gamma-1/2}(x+y)\mathrm{d}x.$$

Using (19) and the properties of V_γ and $^tV_\gamma$ provided by [17], we easily derive the next statement.

Proposition 3. 1) If $f \in L^\infty(\mathbf{R})$ then $Xf \in L_{1/Q}^\infty(\mathbf{R})$ and $\|Xf\|_{\infty,1/Q} \le \|f\|_\infty$.

2) If $f \in L_Q^1(\mathbf{R})$ then $^tXf \in L^1(\mathbf{R})$ and $\|{}^tXf\|_1 \le \|f\|_{1,Q}$.

3) For every $f \in L^\infty(\mathbf{R})$ and $g \in L_Q^1(\mathbf{R})$, we have the duality relation

$$\int_{\mathbf{R}} Xf(x)g(x)|x|^{2\gamma+1}\mathrm{d}x = \int_{\mathbf{R}} f(y)\,{}^tXg(y)\mathrm{d}y.$$

4) For every $f \in L_Q^1(\mathbf{R})$ we have the identity

$$F_\Lambda(f) = F_u \circ {}^tX(f), \tag{20}$$

where F_u denotes the usual Fourier transform on $\mathbf{R}$ given by

$$F_u(h)(\lambda) = \int_{\mathbf{R}} h(x)\mathrm{e}^{-i\lambda x}\mathrm{d}x, \quad h \in L^1(\mathbf{R}).$$

5) Let $f, g \in L_Q^1(\mathbf{R})$. Then

$$^tX(f \# g) = {}^tXf * {}^tXg,$$

where $*$ denotes the usual convolution product on $\mathbf{R}$ given by

$$h_1 * h_2(x) = \int_{\mathbf{R}} h_1(x-y)h_2(y)\mathrm{d}y.$$

6) Let $f \in L_Q^1(\mathbf{R})$ and $g \in L^\infty(\mathbf{R})$. Then

$$X\left({}^tXf * g\right) = Q^2 f \# \left(\frac{Xg}{Q^2}\right). \tag{21}$$

3. Generalized Wavelets

Notation. For a function f on $\mathbf{R}$ put

$$f^\sim(x) = \overline{f(-x)}, \quad x \in \mathbf{R}.$$

3.1. Dunkl Wavelets

Definition 1. A Dunkl wavelet is a function $g \in L_\gamma^2(\mathbf{R})$ satisfying the admissibility condition

$$0 < C_g^\gamma = \int_0^\infty \left|F_\gamma(g)(a\lambda)\right|^2 \frac{\mathrm{d}a}{a} < \infty, \tag{22}$$

for almost all $\lambda \in \mathbf{R}$.

Notation. For a function g in $L_\gamma^2(\mathbf{R})$ and for $(a,b) \in (0,\infty)\times\mathbf{R}$ we write

$$g_{a,b}^\gamma(x) = \tau_\gamma^{-b} g_a^\gamma(x), \tag{23}$$

where τ_γ^{-b} are the Dunkl translation operators given by (12), and

$$g_a^\gamma(x) = \frac{1}{a^{2\gamma+2}} g\left(\frac{x}{a}\right), \quad x \in \mathbf{R}. \tag{24}$$

Definition 2. Let $g \in L_\gamma^2(\mathbf{R})$ be a Dunkl wavelet. The Dunkl continuous wavelet transform is defined for smooth functions f on $\mathbf{R}$ by

$$S_g^\gamma(f)(a,b) = \int_{\mathbf{R}} f(x)\overline{g_{a,b}^\gamma(x)}|x|^{2\gamma+1}\mathrm{d}x, \tag{25}$$

which can also be written in the form

$$S_g^\gamma(f)(a,b) = f *_\gamma \left(g_a^\gamma\right)^\sim(b), \tag{26}$$

where $*_\gamma$ is the Dunkl convolution product given by (17).

The Dunkl continuous wavelet transform has been investigated in depth in [17] from which we recall the following fundamental properties.

Theorem 3. Let $g \in L_\gamma^2(\mathbf{R})$ be a Dunkl wavelet. Then

1) For all $f \in L_\gamma^2(\mathbf{R})$ we have the Plancherel formula

$$\int_{\mathbf{R}} \left| f(x) \right|^2 |x|^{2\gamma+1} \mathrm{d}x = \frac{1}{C_g^\gamma} \int_0^\infty \int_{\mathbf{R}} \left| S_g^\gamma(f)(a,b) \right|^2 |b|^{2\gamma+1} \mathrm{d}b \frac{\mathrm{d}a}{a}.$$

2) For $f \in L_\gamma^1(\mathbf{R})$ such that $F_\gamma(f) \in L_\gamma^1(\mathbf{R})$, we have

$$f(x) = \frac{1}{C_g^\gamma} \int_0^\infty \left(\int_{\mathbf{R}} S_g^\gamma(f)(a,b) g_{a,b}^\gamma(x) |b|^{2\gamma+1} \mathrm{d}b \right) \frac{\mathrm{d}a}{a}$$

for almost all $x \in \mathbf{R}$.

3) Assume that $F_\gamma(g) \in L^\infty(\mathbf{R})$. For $f \in L_\gamma^2(\mathbf{R})$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_g^\gamma} \int_\varepsilon^\delta \int_{\mathbf{R}} S_g^\gamma(f)(a,b) g_{a,b}^\gamma(x) |b|^{2\gamma+1} \mathrm{d}b \frac{\mathrm{d}a}{a}$$

belongs to $L_\gamma^2(\mathbf{R})$ and satisfies

$$\lim_{\varepsilon \to 0, \delta \to \infty} \left\| f^{\varepsilon,\delta} - f \right\|_{2,\gamma} = 0.$$

3.2. Generalized Wavelets

Definition 3. We say that a function $g \in L_Q^2(\mathbf{R})$ is a generalized wavelet if it satisfies the admissibility condition

$$0 < C_g = \int_0^\infty \left| F_\Lambda(g)(a\lambda) \right|^2 \frac{\mathrm{d}a}{a} < \infty, \tag{27}$$

for almost all $\lambda \in \mathbf{R}$.

Remark 4. 1) The admissibility condition (27) can also be written as

$$0 < C_g = \int_0^\infty \left| F_\Lambda(g)(\lambda) \right|^2 \frac{\mathrm{d}\lambda}{\lambda} = \int_0^\infty \left| F_\Lambda(g)(-\lambda) \right|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty.$$

2) If g is real-valued we have $F_\Lambda(g)(-\lambda) = \overline{F_\Lambda(g)(\lambda)}$, so (27) reduces to

$$0 < C_g = \int_0^\infty \left| F_\Lambda(g)(\lambda) \right|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty.$$

3) If $0 \neq g \in L_Q^2(\mathbf{R})$ is real-valued and satisfies $\exists \eta > 0$ such that $F_\Lambda(g)(\lambda) - F_\Lambda(g)(0) = O(\lambda^\eta)$, as $\lambda \to 0^+$, then (27) is equivalent to $F_\Lambda(g)(0) = 0$.

4) According to (10), (22) and (27), $g \in L_Q^2(\mathbf{R})$ is a generalized wavelet if and only if, $Qg \in L_\gamma^2(\mathbf{R})$ is a Dunkl wavelet, and we have

$$C_g = C_{Qg}^\gamma. \tag{28}$$

Notation. For a function g on $\mathbf{R}$ and $a > 0$, put

$$g_a(x) = \frac{Q(x/a) g(x/a)}{a^{2\gamma+2} Q(x)}, \quad x \in \mathbf{R}. \tag{29}$$

Remark 5. Notice by (24) and (29) that

$$g_a(x) = \frac{(Qg)_a^\gamma(x)}{Q(x)}. \tag{30}$$

Proposition 4. 1) Let $a > 0$ and $g \in L_Q^p(\mathbf{R})$ for some $1 \le p \le \infty$. Then $g_a \in L_Q^p(\mathbf{R})$ and

$$\left\| g_a \right\|_{p,Q} = a^{-2(\gamma+1)/q} \left\| g \right\|_{p,Q}$$

where q is such that $1/p + 1/q = 1$.

2) For $a > 0$ and $g \in L_Q^p(\mathbf{R})$, $p = 1$ or 2, we have

$$F_\Lambda(g_a)(\lambda) = F_\Lambda(g)(a\lambda).$$

Proof. 1) By (30) and [13, Equation (13)], we have

$$\left\| g_a \right\|_{p,Q} = \left\| Qg_a \right\|_{p,\gamma} = \left\| (Qg)_a^\gamma \right\|_{p,\gamma} = a^{-2(\gamma+1)/q} \left\| Qg \right\|_{p,\gamma} = a^{-2(\gamma+1)/q} \left\| g \right\|_{p,Q}.$$

2) By (10), (30) and [13, Equation (11)], we have

$$F_\Lambda(g_a)(\lambda) = F_\gamma(Qg_a)(\lambda) = F_\gamma\left((Qg)_a^\gamma\right)(\lambda) = F_\gamma(Qg)(a\lambda) = F_\Lambda(g)(a\lambda),$$

which achieves the proof. ■

Definition 4. Let $g \in L_Q^2(\mathbf{R})$ be a generalized wavelet. We define for regular functions f on $\mathbf{R}$, the generalized continuous wavelet transform by

$$\Phi_g(f)(a,b) = \int_{\mathbf{R}} f(x) \overline{g_{a,b}(x)} \left(Q(x)\right)^2 |x|^{2\gamma+1} \mathrm{d}x, \tag{31}$$

where $a > 0$, $b \in \mathbf{R}$,

$$g_{a,b}(x) = {}^tT^b g_a(x), \tag{32}$$

and ${}^tT^b$ are the dual generalized translation operators given by (15).

Remark 6. A combination of (15), (23) and (32) yields

$$g_{a,b}(x) = \frac{Q(b)}{Q(x)} (Qg)_{a,b}^\gamma(x). \tag{33}$$

Proposition 5. Let $g \in L_Q^2(\mathbf{R})$ be a generalized wavelet. Then for all $f \in L_Q^p(\mathbf{R})$, p = 1 or 2, we have

$$\Phi_g(f)(a,b) = Q(b) S_{Qg}^\gamma(Qf)(a,b) = \left(Q(b)\right)^2 f \# (g_a)^\sim(b), \tag{34}$$

where # is the generalized convolution product given by (16).

Proof. By (18), (25), (26), (30), (31) and (33), we have

$$\begin{aligned}&\Phi_g(f)(a,b)\\&=Q(b)\int_{\mathbf{R}}Q(x)f(x)\overline{(Qg)^{\gamma}_{a,b}(x)}\,|x|^{2\gamma+1}\,\mathrm{d}x\\&=Q(b)S^{\gamma}_{Qg}(Qf)(a,b)\\&=Q(b)(Qf)*_{\gamma}\left[(Qg)^{\gamma}_{a}\right]^{\sim}(b)\\&=Q(b))(Qf)*_{\gamma}\left[Qg_a\right]^{\sim}(b)\\&=Q(b)(Qf)*_{\gamma}\left[Q(g_a)^{\sim}\right](b)\\&=(Q(b))^2 f\#(g_a)^{\sim}(b),\end{aligned}$$

which ends the proof. ■

A combination of Theorem 3 with identities (28), (33) and (34) yields the following basic results for the generalized continuous wavelet transform.

Theorem 4 (Plancherel formula). Let $g\in L^2_Q(\mathbf{R})$ be a generalized wavelet. Then for all $f\in L^2_Q(\mathbf{R})$ we have

$$\begin{aligned}&\int_{\mathbf{R}}|f(x)|^2(Q(x))^2|x|^{2\gamma+1}\,\mathrm{d}x\\&=\frac{1}{C_g}\int_0^{\infty}\int_{\mathbf{R}}|\Phi_g(f)(a,b)|^2\frac{|b|^{2\gamma+1}}{(Q(b))^2}\,\mathrm{d}b\frac{\mathrm{d}a}{a}.\end{aligned}$$

Theorem 5 (inversion formula). Let $g\in L^2_Q(\mathbf{R})$ be a generalized wavelet. If $f\in L^1_Q(\mathbf{R})$ and $F_{\Lambda}(f)\in L^1_{\gamma}(\mathbf{R})$ then we have

$$f(x)=\frac{1}{C_g}\int_0^{\infty}\left(\int_{\mathbf{R}}\Phi_g(f)(a,b)g_{a,b}(x)\frac{|b|^{2\gamma+1}}{(Q(b))^2}\,\mathrm{d}b\right)\frac{\mathrm{d}a}{a}$$

for almost all $x\in\mathbf{R}$.

Theorem 6 (Calderon's formula). Let $g\in L^2_Q(\mathbf{R})$ be a generalized wavelet such that $F_{\Lambda}(g)\in L^{\infty}(\mathbf{R})$. Then for $f\in L^2_Q(\mathbf{R})$ and $0<\varepsilon<\delta<\infty$, the function

$$f^{\varepsilon,\delta}(x)=\frac{1}{C_g}\int_{\varepsilon}^{\delta}\int_{\mathbf{R}}\Phi_g(f)(a,b)g_{a,b}(x)\frac{|b|^{2\gamma+1}}{(Q(b))^2}\,\mathrm{d}b\frac{\mathrm{d}a}{a}$$

belongs to $L^2_Q(\mathbf{R})$ and satisfies

$$\lim_{\varepsilon\to0,\delta\to\infty}\left\|f^{\varepsilon,\delta}-f\right\|_{2,Q}=0.$$

3.5. Inversion of the Intertwining Operator ${}^tX$ Using Generalized Wavelets

In order to invert ${}^tX$ we need the following two technical lemmas.

Lemma 2. Let $0\neq g\in L^1\cap L^2(\mathbf{R})$ such that $F_u(g)\in L^1(\mathbf{R})$ and satisfying

$$\exists\eta>\gamma \text{ such that } F_u(g)(\lambda)=O\left(|\lambda|^{\eta}\right), \quad (35)$$

as $\lambda\to0$. Let $G=Xg/Q^2$. Then $G\in L^2_Q(\mathbf{R})$ and

$$F_{\Lambda}(G)(\lambda)=\frac{F_u(g)(\lambda)}{2\pi m_{\gamma}|\lambda|^{2\gamma+1}},$$

where m_{γ} is given by (11).

Proof. We have

$$g(x)=\frac{1}{2\pi}\int_{\mathbf{R}}F_u(g)(\lambda)\mathrm{e}^{i\lambda x}\mathrm{d}\lambda, \text{ a.e.}$$

As by (3) and (7),

$$\Psi_{\lambda}(x)=X\left(e^{i\lambda\cdot}\right)(x),$$

we deduce that

$$Xg(x)=m_{\gamma}\int_{\mathbf{R}}h(\lambda)\Psi_{\lambda}(x)|\lambda|^{2\gamma+1}\,\mathrm{d}\lambda, \text{ a.e.} \quad (36)$$

with

$$h(\lambda)=\frac{F_u(g)(\lambda)}{2\pi m_{\gamma}|\lambda|^{2\gamma+1}}.$$

Clearly, $h\in L^1_{\gamma}(\mathbf{R})$. So it suffices, in view of (36) and Theorem 2, to prove that h belongs to $h\in L^2_{\gamma}(\mathbf{R})$. We have

$$\begin{aligned}&\int_{\mathbf{R}}|h(\lambda)|^2|\lambda|^{2\gamma+1}\,\mathrm{d}\lambda\\&=(2\pi m_{\gamma})^{-2}\int_{\mathbf{R}}|\lambda|^{-2\gamma-1}|F_u(g)(\lambda)|^2\mathrm{d}\lambda\\&=(2\pi m_{\gamma})^{-2}\left(\int_{|\lambda|\le1}+\int_{|\lambda|\ge1}\right)|\lambda|^{-2\gamma-1}|F_u(g)(\lambda)|^2\,\mathrm{d}\lambda\\&=(2\pi m_{\gamma})^{-2}(I_1+I_2).\end{aligned}$$

By (35) there is a positive constant k such that

$$I_1\le k\int_{|\lambda|\le1}|\lambda|^{2\eta-2\gamma-1}\mathrm{d}\lambda=\frac{k}{\eta-\gamma}<\infty.$$

From the Plancherel theorem for the usual Fourier transform, it follows that

$$\begin{aligned}I_2&=\int_{|\lambda|\ge1}|\lambda|^{-2\gamma-1}|F_u(g)(\lambda)|^2\,\mathrm{d}\lambda\le\int_{\mathbf{R}}|F_u(g)(\lambda)|^2\mathrm{d}\lambda\\&=2\pi\int_{\mathbf{R}}|g(x)|^2\mathrm{d}x<\infty,\end{aligned}$$

which ends the proof. ■

Lemma 3. Let $0\neq g\in L^1\cap L^2(\mathbf{R})$ be real-valued such that $F_u(g)\in L^1(\mathbf{R})$ and satisfying $\exists\eta>2\gamma+1$ such that

$$F_u(g)(\lambda)=O\left(\lambda^{\eta}\right), \quad (37)$$

as $\lambda\to0^+$. Let $G=Xg/Q^2$. Then $G\in L^2_Q(\mathbf{R})$ is a generalized wavelet and $F_{\Lambda}(G)\in L^{\infty}(\mathbf{R})$.

Proof. By using (37) and Lemma 2 we see that $G\in L^2_Q(\mathbf{R})$, $F_{\Lambda}(G)$ is bounded and

$$F_{\Lambda}(G)(\lambda)=O\left(\lambda^{\eta-2\gamma-1}\right)\text{as }\lambda\to0^+.$$

Thus, in view of Remark 4 3), the function Xg/Q^2

satisfies the admissibility condition (27). ■

Recall that the classical continuous wavelet transform is defined for suitable functions f on $\mathbf{R}$ by

$$W_g(f)(a,b)=\int_{\mathbf{R}} f(x)\frac{1}{a}\overline{g\left(\frac{x-b}{a}\right)}\mathrm{d}x, \tag{38}$$

where $a>0$, $b\in\mathbf{R}$, and $g\in L^2(\mathbf{R})$ is a classical wavelet on $\mathbf{R}$, *i.e.*, satisfying the admissibility condition

$$0<c(g)=\int_0^\infty |F_u(g)(a\lambda)|^2\frac{\mathrm{d}a}{a}<\infty, \tag{39}$$

for almost all $\lambda\in\mathbf{R}$. A more complete and detailed discussion of the properties of the classical continuous wavelet transform can be found in [10].

Remark 7. 1) According to [10], each function satisfying the conditions of Lemma 3 is a classical wavelet.

2) In view of (20), (27) and (39), $g\in D(\mathbf{R})$ is a generalized wavelet, if and only if, ${}^tXg$ is a classical wavelet and we have

$$c({}^tXg)=C_g.$$

In the next statement we exhibit a formula relating the generalized continuous wavelet transform to the classical one.

Proposition 6. Let g be as in Lemma 3. Let $G=Xg/Q^2$. Then for all $f\in L_Q^p(\mathbf{R})$, $p=1$ or 2, we have

$$\Phi_G(f)(a,b)=\frac{1}{a^{2\gamma+1}}X\left[W_g({}^tXf)(a,\cdot)\right](b).$$

Proof. By (34) we have

$$\Phi_G(f)(a,b)=(Q(b))^2 f\#(G_a)^{\sim}(b).$$

But

$$(G_a)^{\sim}=\frac{X\left[(g_a^\gamma)^{\sim}\right]}{Q^2}$$

by virtue of (3), (24) and (29). So using (21) and (38) we find that

$$\begin{aligned}\Phi_G(f)(a,b)&=(Q(b))^2 f\#\left(\frac{X\left[(g_a^\gamma)^{\sim}\right]}{Q^2}\right)(b)\\&=X\left[{}^tXf*(g_a^\gamma)^{\sim}\right](b)\\&=\frac{1}{a^{2\gamma+1}}X\left[W_g({}^tXf)(a,\cdot)\right](b),\end{aligned}$$

which gives the desired result.

Combining Theorems 5, 6 with Lemma 3 and Proposition 6 we get

Theorem 7. Let g be as in Lemma 3. Let $G=Xg/Q^2$. Then we have the following inversion formulas for the integral transform ${}^tX$:

1) If $f\in L_Q^1(\mathbf{R})$ and $F_\Lambda(f)\in L_\gamma^1(\mathbf{R})$ then for almost all $x\in\mathbf{R}$ we have

$$f(x)=\frac{1}{C_G}\int_0^\infty\left(\int_{\mathbf{R}}X\left[W_g({}^tXf)(a,\cdot)\right](b)G_{a,b}(x)\times\frac{|b|^{2\gamma+1}}{(Q(b))^2}\mathrm{d}b\right)\frac{\mathrm{d}a}{a^{2\gamma+2}}.$$

2) For $f\in L_Q^1\cap L_Q^2(\mathbf{R})$ and $0<\varepsilon<\delta<\infty$, the function

$$f^{\varepsilon,\delta}(x)=\frac{1}{C_G}\int_\varepsilon^\delta\int_{\mathbf{R}}X\left[W_g({}^tXf)(a,\cdot)\right](b)G_{a,b}(x)\times\frac{|b|^{2\gamma+1}}{(Q(b))^2}\mathrm{d}b\frac{\mathrm{d}a}{a^{2\gamma+2}}$$

satisfies

$$\lim_{\varepsilon\to0,\delta\to\infty}\left\|f^{\varepsilon,\delta}-f\right\|_{2,Q}=0.$$

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