

Bilinear Mappings and the Frame Operator

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ABSTRACT

The theory of frames has been actively developed by many authors over the past two decades, both for its applications to signal processing, and for its deep connections to other areas of mathematics such as operator theory. Central to the study of frames is the frame operator. We initiate an investigation that extends the frame operator to the bilinear setting.

Keywords: Harmonic Analysis; Frames

1. Introduction

The theory of frames was initiated by Duffin and Schaeffer [1] to study some deep problems in non-harmonic Fourier series. For more than three decades, their ideas did not seem to generate much interest outside of non-harmonic Fourier series. Finally in 1986, Daubechies, Grossman, and Meyer [2] in their groundbreaking paper observed that frames can be used for painless nonorthogonal expansions for functions. Since then, frames have been used in signal processing, image processing, and data compression, as well as being studied for their deep connections to operator theory [3]. Frames are important in signal processing because they can be used to provide stable reconstruction of signals. For background in the theory of frames, see [4-6]. Central to the study of frames is the frame operator.

We initiate an investigation that extends the frame operator to the bilinear setting. Bilinear operators in harmonic analysis have been studied by many authors, see, for example, [7-10]. The conjecture that the bilinear Hilbert transform can be extended to a bounded operator has remained open for some 30 years before it was settled in the celebrated work of Lacey and Thiele [11]. The results in our current work extend the results concerning a class of bilinear operators known as paraproducts; these operators are better behaved than the usual products of functions, see [12]. The results in this article indicate that there is a rich underlying theory that awaits to be developed. The present work only touches on certain aspects of that theory.

Let H be a separable Hilbert space. A sequence

$\{x_n\}_{n=1}^\infty$ of elements in H is a *frame* for H if there exist positive constants A and B such that

$$\forall f \in H, \quad A\|f\|^2 \leq \sum_{n=1}^\infty |\langle f, x_n \rangle|^2 \leq B\|f\|^2.$$

For the rest of this article, the Hilbert space H is taken to be $L^2(\mathbb{R}^d)$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on \mathbb{R}^d . The Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ is defined by $\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \zeta \cdot x} dx$.

2. Main Results

We begin with a useful lemma that will simplify our calculations later.

Lemma 2.1 (Convolution with a radial function is a self-adjoint operator)

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a radial function, i.e. $\forall x \in \mathbb{R}^d$, $\psi(x) = \psi(|x|)$.

Define an operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$\forall f \in L^2(\mathbb{R}^d),$$

$$(Tf)(x) = \int_{\mathbb{R}^d} f(x-y)\psi(x)dy = (f * \psi)(x).$$

Then T is a self-adjoint operator. That means,

$$\forall f, g \in L^2(\mathbb{R}^d), \langle Tf, g \rangle = \langle f, Tg \rangle.$$

Proof. Let $f, g \in L^2(\mathbb{R}^d)$.

We first note that $Tf \in L^2(\mathbb{R}^d)$ since

$$\|f * \psi\|_2 \leq \|f\|_2 \|\psi\|_1.$$

$$\begin{aligned}
 \langle Tf, g \rangle &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)\psi(x)dy \right) \overline{g(x)}dx \\
 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y)\psi(x-y)dy \right) \overline{g(x)}dx \\
 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)}dy \right) \overline{g(x)}dx, \quad \psi(x) = \overline{\psi(|x|)} \\
 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\overline{\psi(y-(x+y))}g(x+y)dxdy, \quad x \mapsto x+y \\
 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\overline{\psi(-x)}g(x+y)dxdy \\
 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\overline{\psi(x)}g(y-x)dxdy, \quad x \mapsto -x \\
 &= \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} \overline{\psi(x)}g(y-x)dxdy \\
 &= \int_{\mathbb{R}^d} f(y)(g * \psi)(y)dy \\
 &= \langle f, Tg \rangle.
 \end{aligned}$$

Consider a bilinear operator $B : H \times H \rightarrow H$.

$\forall f, g \in H, B(f, g) \in H$, and B is linear in each of the two variables separately.

Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$ with compact support, and

$$\int_{\mathbb{R}^d} \psi(x)dx = 0, \quad \int_{\mathbb{R}^d} \phi(x)dx = 1. \quad \psi \text{ is radial, i.e. } \psi(x) = \overline{\psi(|x|)}.$$

We normalize ψ so that $\int_0^\infty |\hat{\psi}(t\zeta)|^2 \frac{dt}{t} = 1$. Write $\psi_t(x) = \frac{1}{t^d} \psi\left(\frac{x}{t}\right)$.

Define $B(f, g)$ by:

$$B(f, g)(x) = \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g))(x) \frac{dt}{t}. \tag{1}$$

Consider $\langle B(f, g), e_n \rangle = \int_{\mathbb{R}^d} \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g)) \cdot e_n \frac{dt}{t} dx$, where $e_n \in H$.

$$\begin{aligned}
 \langle B(f, g), e_n \rangle &= \int_{\mathbb{R}^d} \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g)) \cdot e_n \frac{dt}{t} dx \\
 &= \int_{\mathbb{R}^d} (\psi_t * f) \cdot (\phi_t * g)(x) \cdot (\psi_t * e_n)(x) dx, \quad \text{by Lemma 2.1} \\
 &= \int_{\mathbb{R}^d} (\psi_t * f) \cdot ((\phi_t * g)(x) \cdot (\psi_t * e_n)(x)) dx \\
 &= \int_{\mathbb{R}^d} f(x) \cdot \psi * ((\phi_t * g) \cdot (\psi_t * e_n))(x) dx,
 \end{aligned}$$

where the last line follows from

$$\langle f, Th \rangle = \langle Tf, h \rangle, \text{ with } h = (\psi_t * e_n) \cdot (\phi_t * g).$$

We are now ready to construct the frame operator. Let

$\{e_n\}_{n=1}^\infty$ be a sequence in H .

Fix $g \in H$. Define the analysis operator $M : H \rightarrow l_2$ by

$$\forall f \in H,$$

$$Mf = \left(\langle B(f, g), e_1 \rangle, \langle B(f, g), e_2 \rangle, \langle B(f, g), e_3 \rangle, \dots \right).$$

The frame operator $S : H \rightarrow H$ is given by

$$S = M^*M.$$

First, calculate the adjoint operator of M . $M : H \rightarrow l_2$.

$$M^* : l_2 \rightarrow H.$$

Let $c = (c_1, c_2, c_3, \dots)$ be a sequence in l_2 . Then,

$$\langle M^*c, f \rangle_H = \langle c, Mf \rangle_{l_2} = \sum_n c_n \langle B(f, g), e_n \rangle_H$$

Using $S = M^*M$, we obtain

$$\begin{aligned}
 Sf &= M^*Mf \\
 &= M^* \left(\langle B(f, g), e_1 \rangle, \langle B(f, g), e_2 \rangle, \langle B(f, g), e_3 \rangle, \dots \right)
 \end{aligned}$$

By the above calculations on $B(f, g)$, we see that:

$$\begin{aligned}
 &\langle B(f, g), e_n \rangle \\
 &= \int_{\mathbb{R}^d} \int_0^\infty f(x) \cdot (\psi_t * ((\phi_t * g) \cdot (\psi_t * e_n)))(x) \frac{dt}{t} dx \\
 &= \int_{\mathbb{R}^d} f(x) B(e_n, g)(x) dx,
 \end{aligned}$$

where

$$B(e_n, g)(x) = \int_0^\infty \psi_t * ((\psi_t * e_n) \cdot (\phi_t * g))(x) \frac{dt}{t}.$$

Therefore,

$$\begin{aligned} \langle M^*c, f \rangle_H &= \langle c, Mf \rangle_{l_2} \\ &= \sum_{n=1}^\infty c_n \cdot \langle B(e_n, g), f \rangle \\ &= \langle \sum_{n=1}^\infty c_n \cdot \langle B(e_n, g), f \rangle, \text{ where } B(e_n, g) \in H \end{aligned}$$

So, $M^*c = \sum_{n=1}^\infty c_n \cdot B(e_n, g)$,

$$c = (c_1, c_2, c_3, \dots), M^* : l_2 \rightarrow H.$$

Hence, $Sf = \sum_{n=1}^\infty (\langle B(f, g), e_n \rangle_H \cdot B(e_n, g))$, $S : H \rightarrow H$.

i.e. $Sf = \sum_{n=1}^\infty \langle f, B(e_n, g) \rangle_H \cdot B(e_n, g)$. This is our frame operator.

We have constructed a frame operator with a bilinear mapping. Let us summarize all our calculations in the following lemma.

Lemma 2.2 (Frame operator with bilinear mapping) *Let H be the Hilbert space $L^2(\mathbb{R}^d)$.*

Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$ with compact support, and $\int_{\mathbb{R}^d} \psi(x) dx = 0, \int_{\mathbb{R}^d} \phi(x) dx = 1$. ψ is radial, i.e.

$$\psi(x) = \overline{\psi(|x|)}$$

We normalize ψ so that $\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1$. Write

$$\psi_t(x) = \frac{1}{t^d} \psi\left(\frac{x}{t}\right).$$

Consider a bilinear operator $B : H \times H \rightarrow H$ defined by Equation (1).

Then the frame operator $S : H \rightarrow H$ is given by

$$\begin{aligned} Sf &= \sum_{n=1}^\infty \langle B(f, g), e_n \rangle B(e_n, g) \\ &= \sum_{n=1}^\infty \langle f, B(e_n, g) \rangle B(e_n, g). \end{aligned}$$

Proof. Our calculations established the lemma.

To prove that this bilinear operator is bounded, we need some preparation.

Lemma 2.3 *Let H be a separable Hilbert space (or a separable Banach space). Let M be a dense subspace of H . Let $B : M \times H \rightarrow H$ be a bilinear operator such that*

$$\forall f \in M, \forall g \in H, \|B(f, g)\|_H \leq C \cdot \|f\|_H \|g\|_H. \quad (2)$$

Then the above inequality holds for all f in H , for

all g in H , and B extends to a bounded bilinear operator from $H \times H$ to H .

Proof. For each f in H , there exists a sequence $\{f_n\}_{n=1}^\infty \in M$ such that

$$\|f - f_n\|_H \rightarrow 0, \text{ as } n \rightarrow \infty. \text{ Since for each } g \in H,$$

$$\begin{aligned} &\|B(f_m, g) - B(f_n, g)\|_H \\ &= \|B(f_m - f_n, g)\|_H \\ &\leq C \cdot \|f_m - f_n\|_H \cdot \|g\|_H \quad \text{by (2)} \end{aligned}$$

So, for each $g \in H, \{B(f_n, g)\}_{n=1}^\infty$ is a Cauchy sequence in H .

Hence, $B(f_n, g)$ converges in H to an element in H , and we can define a bounded bilinear operator $B(f, g) : H \times H \rightarrow H$ by $B(f, g) = \lim_{n \rightarrow \infty} B(f_n, g)$.

Definition 2.4 (BMO) *If f is a locally integrable function in \mathbb{R}^d , we say that $f \in BMO$ if there exists a constant $A < \infty$, such that for any cube $Q \subseteq \mathbb{R}^d$,*

$$\frac{1}{Q} \int_Q |f(x) - f_Q| dx \leq A.$$

Here, f_Q is the average of f over the cube. The integration is over the cube. The smallest bound A for which the above inequality is satisfied is taken to be the norm of f , and is denoted by $\|f\|_{BMO}$.

For background on BMO functions, see Chapter 4 of [13], as well as the seminal paper by C. Fefferman and Stein [14]. The next theorem on BMO functions, together with Lemma 2.3, will allow us to establish the boundedness of the bilinear operator.

Theorem 2.5 *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \psi(x) dx = 0$. Let ϕ be a bounded, integrable function that is positive, radial, and decreasing. Write*

$$\phi_t(x) = \frac{1}{t^d} \phi\left(\frac{x}{t}\right). \text{ Let } b \in BMO. \text{ Then}$$

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^\infty |\phi_t * f(x)|^2 |\psi_t * b(x)|^2 \frac{dx dt}{t} \\ &\leq C \|b\|_{BMO}^2 \int_{\mathbb{R}^d} |f(x)|^2 dx. \end{aligned}$$

Definition 2.6 *If $f \in BMO \cap L^2(\mathbb{R}^d)$, then $f \in L^2_{BMO}(\mathbb{R}^d)$.*

Remark 2.7 *The space $L^2_{BMO}(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$ in the topology of $L^2(\mathbb{R}^d)$.*

Theorem 2.8 (Boundedness of the bilinear operator) *Assume the hypothesis of Lemma 2.2. Define $B(f, g)$ by:*

$$\forall f, g \in L^2(\mathbb{R}^d),$$

$$B(f, g)(x) = \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g))(x) \frac{dt}{t}.$$

Then there exists $C > 0$ such that for each $e_n \in H$,

with $\|e_n\| = 1$,

$$\begin{aligned} \forall f \in L^2_{BMO}(\mathbb{R}^d), g \in L^2(\mathbb{R}^d), \\ \langle B(f, g), e_n \rangle \leq C \cdot \|f\|_2 \cdot \|g\|_2. \end{aligned}$$

Proof. Let $f \in L^2_{BMO}(\mathbb{R}^d)$. Let $g \in L^2(\mathbb{R}^d)$. Let $e_n \in L^2(\mathbb{R}^d)$, with $\|e_n\|_2 = 1$.

$$\begin{aligned} & \langle B(f, g), e_n \rangle \\ &= \int_{\mathbb{R}^d} \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g))(x) \cdot e_n(x) \frac{dt}{t} dx \\ &= \int_{\mathbb{R}^d} \int_0^\infty (\psi_t * f) \cdot (\phi_t * g)(x) \cdot (\psi_t * e_n)(x) \frac{dt}{t} dx \\ &\leq \left(\int_{\mathbb{R}^d} \int_0^\infty |\psi_t * f(x)|^2 \cdot |\phi_t * g(x)|^2 \frac{dt}{t} dx \right)^{1/2} \\ &\quad \cdot \left(\int_{\mathbb{R}^d} \int_0^\infty |\psi_t * e_n(x)|^2 \frac{dt}{t} dx \right)^{1/2} \\ &\equiv I_1 \cdot I_2 \end{aligned}$$

Let $G(x) = \int_0^\infty |\phi_t * g(x)|^2 \frac{dt}{t}$. Then $\|G\|_2^2 = C_1 \cdot \|g\|_2^2$

by Plancherel Theorem. Hence,

$$\begin{aligned} I_1^2 &= \int_{\mathbb{R}^d} \int_0^\infty |\psi_t * f(x)|^2 \cdot |\phi_t * g(x)|^2 \frac{dt}{t} dx \\ &\leq C_1 \cdot \|f\|_{BMO}^2 \int_{\mathbb{R}^d} |g(x)|^2 dx. \end{aligned}$$

In the last inequality, we used Theorem 2.5. Another application of Plancherel Theorem gives the following:

$$I_2^2 = \int_{\mathbb{R}^d} \int_0^\infty |\psi_t * e_n(x)|^2 \frac{dt}{t} dx \leq C_2 \cdot \|e_n\|_2.$$

Hence,

$$\langle B(f, g), e_n \rangle \leq C \cdot \|f\|_{BMO} \cdot \|g\|_2 \cdot \|e_n\|_2.$$

Therefore B is a bounded operator on $L^2_{BMO}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Recall that the space $L^2_{BMO}(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$. By Lemma 2.3, B extends to a bounded operator on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. This concludes the proof of the theorem.

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