Advances in Pure Mathematics, 2013, 3, 405-408
http://dx.doi.org/10.4236/apm.2013.34058 Published Online July 2013 (http://www.scirp.org/journal/apm)

On the Decomposition of a Bounded Closed Interval of the Real Line into Closed Sets

Edgar A. Cohen
Formerly of the Information Sciences Branch, Naval Surface Warfare Center, White Oak, Silver Spring, USA
Email: edcohenfam@yahoo.com

Received March 7, 2013; revised April 13, 2013; accepted May 14, 2013

ABSTRACT

It has been shown by Sierpinski that a compact, Hausdorff, connected topological space (otherwise known as a continuum) cannot be decomposed into either a finite number of two or more disjoint, nonempty, closed sets or a countably infinite family of such sets. In particular, for a closed interval of the real line endowed with the usual topology, we see that we cannot partition it into a countably infinite number of disjoint, nonempty closed sets. On the positive side, however, one can certainly express such an interval as a union of c disjoint closed sets, where c is the cardinality of the real line. For example, a closed interval is surely the union of its points, each set consisting of a single point being closed. Surprisingly enough, except for a set of Lebesgue measure 0, these closed sets can be chosen to be perfect sets, *i.e.*, closed sets every point of which is an accumulation point. They even turn out to be nowhere dense (containing no intervals). Such nowhere dense, perfect sets are sometimes called Cantor sets.

Keywords: Cantor Sets; Cardinality; Disjoint Closed Sets; Dyadic Representation; Interlacing Dyadic Expansions; Generators of Sets; Nowhere Dense

1. Introduction

In a paper published in 1918 (see [1]), Sierpinski showed that a compact, Hausdorff, connected topological space cannot be partitioned into either a finite number of two or more pairwise disjoint closed sets or a countably infinite number of such sets. It follows then that any bounded closed interval of the real line, in particular, [0,1], cannot be so decomposed. A fortiori neither can the open interval (0,1) be a union of such sets even though (0,1) is expressible as a countable union of overlapping closed sets. In contrast [0,1] can certainly be expressed as the union of its points, each of which is closed in the usual topology (see [2]). Furthermore, there is an exercise given by I. P. Natanson (see [3]) to show that [0,1] can be partitioned into c pairwise disjoint perfect sets. We shall show that, except for a set of Lebesgue measure 0, [0,1] is, in fact, decomposable into c pairwise disjoint perfect sets. These sets also turn out to be nowhere dense. As a corollary to this major result, we can show that there exists a countable union of pairwise disjoint nowhere dense perfect sets which is dense in [0,1] even though this union cannot be the interval [0,1] itself.

2. Results

Let us first repeat the Sierpinski result (see [1]) in the form of interest here, namely:

Theorem 1. The closed interval [0,1] (in the usual topology) cannot be partitioned into either a finite (two or more) or a countably infinite number of pairwise disjoint closed sets.

Proof. Sierpinski shows that the result is correct for any compact connected Hausdorff space. Since the usual topology is a metric topology and the closed interval [0,1] is connected and, by the Heine-Borel Theorem (see [4]), is compact, our case is simply a special instance of Sierpinski's general theorem. Thus, our theorem is valid.

Corollary. The open interval (0,1) cannot be partitioned into a countable number of pairwise disjoint closed sets.

Proof. If there were such a partition, one could adjoin to it the two-point set {0,1} and thus partition the closed interval [0,1] into a countable number of disjoint closed sets. That would contradict the result of Theorem 1. So the corollary follows.

Of course one can express [0,1] as a countable union of overlapping closed sets, e.g.,

$$(0,1)=\bigcup_{n=3}^{\infty}[1/n,1-1/n] \quad (1)$$

Let us next prove our major result, which parallels a problem given by I. P. Natanson (see [3]).

Theorem 2. The closed interval [0,1] can be decomposed into c nowhere dense perfect sets, all of which, except for a countable number, are pairwise disjoint.

Proof. Let x_0 be a real number in the closed interval [0,1], and consider its dyadic representation

$$x_0=0.i_1i_2i_3\cdots i_k\cdots=\sum_{k=1}^{\infty}\frac{i_k}{2^k} \quad (2)$$

(so that $i_k = 0$ or 1 for every k). Now for any real number y likewise in [0,1] with

$$y=0.j_1j_2\cdots j_k\cdots=\sum_{k=1}^{\infty}\frac{j_k}{2^k} \quad (3)$$

consider

$$x_0(y)=0.i_1j_1i_2j_2\cdots i_kj_k\cdots=\sum_{k=1}^{\infty}\frac{i_k}{2^{2k-1}}+\sum_{k=1}^{\infty}\frac{j_k}{2^{2k}}, \quad (4)$$

which is thus represented by interlacing two dyadic expansions.

So any x_0 with a given dyadic expansion can be construed as a generator of a set $\{x_0(y)\}$. We shall examine the nature of these sets with the understanding that x_0 as given by Equation (2) may take on two binary representations if it has a terminating dyadic expansion. In that case it will be necessary to use both expansions in the process of forming Equation (4). Unfortunately this dual representation will lead to some difficulties with regard to disjointness of the sets which we generate, but they will not be critical to the success of our analysis.

We first show that any given set $\{x_0(y)\}$ is closed. To see this suppose that $x_0 = 0$, the zero generator. We want to show that any z not belonging to the set $X_0 = \{x_0(y)\}$ cannot be an accumulation point of the set. First let us observe that the largest element of the set X_0 is

$$x_0(y_0)=0.010101\cdots01\cdots=\sum_{n=1}^{\infty}\frac{1}{2^{2n}}=\frac{1}{3}, \quad (5)$$

an interlacing of $x_0 = 0$ with $y_0 = 1$. Thus, if $z > 1/3$, its distance from the set is $z - 1/3 > 0$. So, since 0 belongs to the set (an interlacing of 0 with itself), one need only concern himself with those z which happen to lie in the open interval (0,1/3). Since z does not lie in the set of interest, clearly at least one of the digits of the generator for z is not 0. Realizing now that $i_1(z) = 0$, since $z < 1/3$, there must exist an n_1 such that $i_n(z) = 0$ for $1\le n < n_1$,but $i_{n_1}(z)=1$. There are now two cases to be considered: (1) $i_n(z) = 1$, $n \ge n_1$ and (2) there exists an $n_2 > n_1$ such that $i_{n_2}(z)=0$. Let us consider Case 1. If it so happens that $j_n(z) = 1$ for $n \ge n_1$, then z will belong to X_0, since z would have a dyadic expansion equivalent to that of a real number with a terminating expansion in which $j_{n_1-1}=1$ and both i_n and $j_n = 0$ for $n \ge n_1$. Observe that such a phenomenon can occur despite the fact that the generator of z is different from that of x_0. However, also note that such a phenomenon as this just implies that the point actually belongs to the set, and we are interested at this point in a z which is not in the set. So, in Case 1, when z does not belong to X_0, there must exist an $m_1 \ge n_1$ such that $j_{m_1}=0$. Thus we have the following situation:

$$z=0.000\cdots000111\cdots101j_{m_1+1}1j_{m_1+2}1\cdots, \quad (6)$$

where $i_n = j_n = 0$ when $1 \le n < n_1$, $i_n = j_n = 1$ when $n_1 \le n < m_1$, $i_{m_1}=1, j_{m_1}=0, i_n=1, n>m_1$. Next, if $x\in X_0$ and $x < z$, then the largest such x is (L for last and B for before)

$$x_{LB}=0.000\cdots001010101\cdots, \quad (7)$$

with the first 1 in slot j_{n_1}. Now, from Equation (6),

$$z=\sum_{i=2n_1-1}^{2m_1-1}\frac{1}{2^i}+\sum_{r=m_1+1}^{\infty}\frac{1}{2^{2r-1}}+\sum_{r=m_1+1}^{\infty}\frac{j_r}{2^{2r}}=\frac{1}{2^{2n_1-2}}-\frac{1}{3}\cdot\frac{1}{2^{2m_1-2}}+p, \quad (8)$$

where $p \ge 0$ and, from Equation (7),

$$x_{LB}=\sum_{i=n_1}^{\infty}\frac{1}{2^{2i}}=\frac{1}{3}\cdot\frac{1}{2^{2(n_1-1)}}. \quad (9)$$

Then, from Equations (8) and (9), we have

$$\begin{aligned} z-x_{LB}&=\frac{1}{2^{2n_1-2}}-\frac{1}{3}\cdot\frac{1}{2^{2m_1-2}}+p-\frac{1}{3}\cdot\frac{1}{2^{2n_1-2}}\\ &=\frac{2}{3}\cdot\frac{1}{2^{2n_1-2}}-\frac{1}{3}\cdot\frac{1}{2^{2m_1-2}}+p \quad (10)\\ &=\frac{1}{3}\cdot\frac{1}{2^{2n_1-2}}\left(2-\frac{1}{2^{2(m_1-n_1)}}\right)+p>\frac{1}{3}\cdot\frac{1}{2^{2n_1-2}}. \end{aligned}$$

We also note that

$$0\le p=\sum_{r=m_1+1}^{\infty}\frac{j_r}{2^{2r}}\le\sum_{r=m_1+1}^{\infty}\frac{1}{2^{2r}}=\frac{1}{3}\cdot\frac{1}{2^{2m_1}}. \quad (11)$$

Now, when $x > z$, the smallest x in X_0 (F for first and A for after) is

$$x_{FA}=0.000\cdots0001000\cdots000\cdots, \quad (12)$$

with a 1 in position j_{n_1-1}. So, using Equations (8), (11), and (12), one has

$$\begin{aligned} x_{FA}-z&=\frac{1}{2^{2n_1-2}}-\left(\frac{1}{2^{2n_1-2}}-\frac{1}{3}\cdot\frac{1}{2^{2m_1-2}}\right)-p\\ &=\frac{1}{3}\cdot\frac{1}{2^{2m_1-2}}-p \quad (13)\\ &\ge\frac{1}{3}\cdot\left(\frac{1}{2^{2m_1-2}}-\frac{1}{2^{2m_1}}\right)=\frac{1}{2^{2m_1}}. \end{aligned}$$

Now, using Equations (10) and (13), it obtains that the distance between z and X_0 is greater than or equal to $1/2^{2m_1}$, which is the minimum of the lower bounds provided by Expressions (10) and (13). Therefore, in Case (1), z is not a limit point of X_0.

Let us next consider Case (2). Here $i_{n_1}(z)=1$ and there exists $n_2 > n_1$ such that $i_{n_2}(z)=0$. Let us assume that n_1 and n_2 are the first indices for which this is valid. So z now has the form

$$z = 0.0j_1 0j_2 0j_3 \cdots j_{n_1-1} 1 j_{n_1} i_{n_1+1} j_{n_1+1} \cdots j_{n_2-1} 0 j_{n_2} \cdots \quad (14)$$

Here x_{LB} is the same as before, as given by Equations (7) and (9), and, from Equation (14), we have

$$z = \sum_{r=1}^{\infty} \frac{j_r(z)}{2^{2r}} + \frac{1}{2^{2n_1-1}} + \sum_{r=n_1+1}^{n_2-1} \frac{i_r(z)}{2^{2r-1}} + \sum_{r=n_2+1}^{\infty} \frac{i_r(z)}{2^{2r-1}}, \quad (15)$$

so that, using Equations (9) and (15),

$$z - x_{LB} = q + \frac{1}{2^{2n_1-1}} - \frac{1}{3}\cdot\frac{1}{2^{2n_1-2}} = q + \frac{1}{3}\cdot\frac{1}{2^{2n_1-1}} \ge \frac{1}{3}\cdot\frac{1}{2^{2n_1-1}}, \quad (16)$$

where $q \ge 0$. In a similar manner, using Equations (12) and (15), we have

$$z - x_{FA} = s + \frac{1}{2^{2n_1-1}} - \frac{1}{2^{2n_1}} = s + \frac{1}{2^{2n_1}} \ge \frac{1}{2^{2n_1}}, \quad (17)$$

where $s \ge 0$. It follows that z is at a distance of at least $\frac{1}{3}\cdot\frac{1}{2^{2n_1-1}}$ from X_0, and again z cannot be a limit point of the set. Thus X_0 is closed when $x_0 = 0$. Now observe that every set $\{x_0(y)\}$ is just a translation of the set generated by 0. In fact, given a generator

$$g = 0.i_1 i_2 i_3 \cdots, \quad (18)$$

that set is simply translated by the real number

$$t = 0.i_1 0 i_2 0 i_3 0 \cdots \quad (19)$$

It follows that all of our sets are closed, and, furthermore, that their union will be [0,1]. The latter claim stems from the fact that every real number in the closed interval must have a generator and from the fact that any real number can be obtained by interlacing some generator with another binary expansion.

Next we show that each set $\{x_0(y)\}$, whose elements are given by Equation (4), is a perfect set, *i.e.*, all points are points of accumulation. This is easy to see, for let us consider a particular point $x_0(y_0)$. If we change the value of j_n and keep all of the other entries the same, we obtain another point of the same set whose distance from $x_0(y_0)$ is $1/2^{2n}$. The result then follows, since n may be taken arbitrarily large. Of course, the same result could have been procured by proving it for the 0 generator case and then appealing to translation invariance, as we did for showing that all of our generated sets are closed.

To see that each set is actually nowhere dense, we need only show that property for the set generated by 0 and then appeal to translation invariance. When $x_0 = 0$, so that $i_r = 0$ for every r, consider any point x_1 belonging to X_0 and another point x_2 of that set arbitrarily close to x_1. We may suppose that the two points first differ for some j_n, which is 0 for x_1 and 1 for x_2. Then, if one modifies x_1 by changing any i_r for $r > n$ from 0 to 1, he obtains a point x_3 of the complement of X_0 lying between x_1 and x_2. It follows that X_0 is nowhere dense and, by translation invariance, that all of our sets are nowhere dense. So our sets are actually Cantor sets, or Cantor-like sets, as some might say.

We next note that our defining process itself shows that, in general, any two sets, say $\{x_0(y)\}$ and $\{x_1(y)\}$, will be disjoint, since they are constructed using different generators. Also, clearly each set contains c points, since it is built by interleaving all real numbers in [0,1] with a given generator. Furthermore, there will be c sets, since there are c generators.

However, there are a certain number of situations in which our sets need not be disjoint, and these cases arise only because rational numbers with terminating expansions have two dyadic representations. Let us observe the nature of these Cantor sets. Let us consider, for example, the generators

$$\begin{aligned} x_0^{(0)} &= 0.i_1 i_2 \cdots i_{n-1} 1000 \cdots 000 \cdots 000 \cdots \\ x_0^{(1)} &= 0.i_1 i_2 \cdots i_{n-1} 0111 \cdots 111 \cdots 111 \cdots \\ x_0^{(2)} &= 0.i_1 i_2 \cdots i_{n-1} 00111 \cdots 111 \cdots 111 \cdots \\ x_0^{(3)} &= 0.i_1 i_2 \cdots i_{n-1} 000111 \cdots 111 \cdots 111 \cdots \\ &\vdots \\ x_0^{(\infty)} &= 0.i_1 i_2 \cdots i_{n-1} 000 \cdots 000 \cdots 000 \cdots \end{aligned} \quad (20)$$

where $x_0^{(i)}$, $1 \le i < \infty$, has i zeroes before an infinite terminal string of 1's and $x_0^{(\infty)}$ is a finite expansion (the 0 expansion included). Then the following situations occur: $\{x_0^{(0)}(y)\}$ and $\{x_0^{(1)}(y)\}$ overlap, and $\{x_0^{(i)}(y)\}$ overlaps $\{x_0^{(\infty)}(y)\}$ for $1 \le i < \infty$. However, $\{x_0^{(i)}(y)\}$ will be disjoint from any set whose generator is not expressible as a terminating dyadic expansion. The collection of all such sets is clearly a countable one, and all remaining sets, c in number, are pairwise disjoint. Therefore, except for a countable family of Cantor sets, each clearly of the same Lebesgue measure (since they are just translates of one another), the remaining Cantor sets (constituting a disjoint family of c sets) form a decomposition of [0,1]. This proves the theorem.

It is interesting that almost all of the Cantor sets devised here consist only of irrational numbers as would, of course, naturally follow if there are going to be c of them. One will observe that, if the generator is irrational, then any number produced through interlacing will again be irrational. In fact rational numbers are only produced when a rational generator is interlaced with another rational number.

One also has the following corollary (see [5]):

Corollary 1. There is a subset X of [0,1] which is a union of a countable number of nowhere dense perfect sets and which is dense in [0,1].

Proof. Let X be the union of all Cantor sets whose generators are of the form given in Equations (20). Suppose that z is any real number which is not an element of X. Let us form its dyadic representation, which, of course, is unique, since it does not belong to X. Consider a fundamental neighborhood, or ball, about z. Then it is clear that some truncation of its expansion will reside both within the ball and within the set X. Since X is a countable union of nowhere dense, perfect sets, this proves the corollary.

With just a little extra effort, one can even assert that such sets can be chosen to be pairwise disjoint. In order to achieve that, we can use the sets whose generators are rational but not realizable as terminating expansions. The union of all such sets (countable in number) is also dense in [0,1] as the reader will easily see. In addition one has the following result:

Corollary 2. All of our Cantor sets are of Lebesgue measure 0. Therefore, X, of Corollary 1, is also of measure zero.

Proof. Our set X and the remaining Cantor sets devised in the proof of Theorem 2 constitute c pairwise disjoint sets whose union is the closed interval [0,1]. Now we can show that at least one of these sets must be of zero Lebesgue measure. Suppose that this is not the case. Let A_n be the collection of those sets whose measures exceed or equal $1/n$. Since all sets are disjoint from one another, clearly there are no more than n of them in A_n. Of course, there are a countable number of such sets, each of which can contain no more than a finite number of sets of the collection. However, then we would have a countable collection of such sets covering [0,1], and we know that we have c of these sets whose union is [0,1]. Therefore, there must exist a set of measure zero among them. If this set happens to be X, then any subset of it also has measure 0, and, in particular, this is true of any one of the Cantor sets. Since all of the Cantor sets are translates of one another, we would conclude that all of them have zero measure, On the other hand, if one of the other sets has measure zero, then again all Cantor sets are translates of it, and all our sets are of zero measure. This establishes the corollary.

Corollary 3. Any nowhere dense perfect set C can itself be decomposed into c pairwise disjoint nowhere dense perfect sets.

Proof. Since the set C is assumed to be a Cantor set, we can use a well-known result originally due to L. E. J. Brouwer, namely, that all Cantor sets are homeomorphic to one another. This means that we can invoke the classical Cantor ternary set (see [3]) as a prototype. Since only 0's and 2's are necessary to describe that set, this means that there is a unique encoding of each point of our given Cantor set C. It follows that our generator scheme can now be utilized (using 0's and 2's rather than 0's and 1's) to obtain c pairwise disjoint nowhere dense perfect subsets of C. In other words, we can obtain first of all a partition of the classical Cantor set into c pairwise disjoint Cantor subsets each of measure 0, and their images under a homeomorphism will also be Cantor sets disjoint from one another. Therefore, we have a partition of our original Cantor set C into c pairwise disjoint Cantor subsets. However, under a homeomorphism, measure need not be preserved, so that these subsets need not all be of measure 0.

3. Conclusion

We have established two important facts concerning the decomposition of a closed interval of the real line into disjoint closed sets: 1) Such a partition cannot be obtained using either a finite (more than 1) or a countably infinite number of closed sets, and, a fortiori, neither can an open interval be so decomposed. The latter fact does not seem to be generally emphasized in the literature. On the other hand, it is easy to show that an open interval is an F_σ set (a countable union of closed sets) if we drop the requirement that the sets be pairwise disjoint; 2) In contrast a partition into c pairwise disjoint closed sets (with c the power of the continuum) is always possible and, in fact, except for a set of Lebesgue measure 0, that can be accomplished with perfect sets (closed sets without isolated points). It turns out that, in the theory that we have developed here, the sets involved are also nowhere dense, *i.e.*, have a vacuous interior. Perfect, nowhere dense sets are sometimes called Cantor sets.

REFERENCES

[1] W. Sierpinski, "Un Theoreme sur les Continus," *Tohoku Mathematical Journal*, Vol. 13, 1918, pp. 300-305.

[2] G. F. Simmons, "Introduction to Topology and Modern Analysis," McGraw Hill Book Company, Inc., New York, 1963, p. 92.

[3] I. P. Natanson, "Theory of Functions of a Real Variable," Frederick Ungar Publishing Co., Inc., New York, 1961, pp. 49-50.

[4] D. W. Hall and G. L. Spencer II, "Elementary Topology," John Wiley and Sons, Inc., New York and Chapman & Hall, Limited, London, 1955, p. 44.

[5] M. E. Munroe, "Introduction to Measure and Integration," Addison-Wesley Publishing Co., Inc., Reading, 1953, p. 70.

